Indeterminacy under input-specific externalities and implications for fiscal policy

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Indeterminacy and Endogenous Fluctuations under Input-Specific Externalities

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Abstract

A lot of papers have analyzed the role of productive externalities on the occurrence of indeterminacy and endogenous cycles, assuming that the total productivity of factors increases with respect to average capital and labor. In this paper, we extend such type of analysis introducing a more general formulation of externalities, i.e. input-specific externalities. Indeed, we assume that different externalities affect each input in the production function. Considering a Woodford (1986) framework, we show that this generalized form of externalities allows us to obtain new dynamic results concerning the occurrence of local indeterminacy and endogenous cycles. Moreover, we exhibit some configurations where the emergence of endogenous fluctuations requires a weaker degree of increasing returns than in the usual case where externalities are represented by the total productivity of factors.

Keywords: Indeterminacy, endogenous fluctuations, externalities, increasing returns, capital-labor substitution.

Résumé

A grand nombre de papiers ont analysé le rôle des externalités dans la production sur l'indétermination et l'apparition de cycles endogènes, en supposant que la productivité totale des facteurs augmente avec le capital et le travail moyens. Dans ce papier, nous étendons ce type d'analyse en considérant une forme plus générale d'externalités, c'est-à-dire spécifiques aux facteurs de production. En effet, nous supposons que des externalités différentes affectent chaque facteur dans la fonction de production. En considérant un modèle à la Woodford (1986), nous montrons que cette généralisation des externalités nous permet d'obtenir de nouveau résultats concernant la dynamique (indétermination et cycles endogènes). De plus, nous présentons des configurations dans lesquelles l'émergence de fluctuations endogènes requière des rendements d'échelle plus faiblement croissants que lorsque les externalités apparaissent à travers la productivité totale des facteurs.

Mots-clés: Indétermination, fluctuations endogènes, externalités, rendements croissants, substitution capital-travail.

JEL classification: C62, E32.

1 Introduction

Considering dynamic general equilibrium one-sector models, a lot of papers have introduced productive externalities to analyze the role of increasing returns on local indeterminacy and the occurrence of endogenous cycles.[2] In most of them, production benefits from externalities because the total productivity of factors increases with respect to average capital and labor. In other words, these contributions departs from the perfectly competitive model by introducing a multiplicative term, that increases with respect to capital and labor, to the constant returns to scale production function ($Y = A(\overline{K}, \overline{L})F(K, L)$). Among others, one can refer to Barinci and Chéron (2001), Benhabib and Farmer (1994), Boldrin (1992), Cazzavillan (2001), Cazzavillan, Lloyd-Braga and Pintus (1998), Farmer and Guo (1994), Harrison and Weder (2002), Hintermaier (2003) or Pintus (2006).[3]

In this paper, we extend this type of analysis introducing a more general formulation of externalities in a one-sector model and we will study their implications on the dynamic stability properties and the occurrence of cycles. Instead of considering that externalities appear through the total productivity of factors, we introduce input-specific externalities. The production benefits from externalities because two different multiplicative terms, increasing with respect to average capital and labor, enter the production function, each one for a specific production factor, capital or labor ($Y = F(C(\overline{K}, \overline{L})K, D(\overline{K}, \overline{L})L)$). These two multiplicative terms, which represent externalities and ensure that returns to scale are increasing at the social level, can also be interpreted as capital and labor efficiencies. Evidently, when these two multiplicative terms are equal, we recover as a particular case externalities represented by the total productivity of factors ($C(\overline{K}, \overline{L}) = D(\overline{K}, \overline{L}) = A(\overline{K}, \overline{L})$).

We introduce such a production sector in a finance constrained economy with heterogeneous households as initially developped by Woodford (1986). With this framework, we can analyze the stability properties of the steady state, i.e. the local indeterminacy and the occurrence of endogenous cycles, using the geometrical method developped by Grandmont, Pintus, and de Vilder (1998). Moreover, we can easily compare our results with the case where productive externalities are introduced through the total productivity of factors as analyzed by Cazzavillan, Lloyd-Braga, and Pintus (1998).

Our main results concern the occurrence of indeterminacy and endoge-

[2]In two-sector models, externalities have also been introduced assuming constant returns to scale at the social level. For a seminal contribution, one can refer to Benhabib and Nishimura (1998).

[3]For a survey, see Benhabib and Farmer (1999).

nous fluctuations when capital and labor are not weak substitutes. To fix ideas, we call the direct effects of externalities the contributions of capital (labor) to externalities that affect capital (labor) as an input, whereas we call the crossing effects the contributions of capital (labor) to externalities that affect labor (capital) as an input.

We show that, when the product of crossing effects are greater than the product of direct effects on externalities, we obtain results closely related to Cazzavillan, Lloyd-Braga, and Pintus (1998). Endogenous fluctuations can occur for all elasticities of capital-labor substitution greater than a lower bound and a sufficiently elastic labor supply. On the contrary, when the product of direct effects dominates the product of crossing effects, endogenous fluctuations can no more occur for a level of capital-labor substitution sufficiently high, in particular when the labor supply is highly elastic. We further exhibit some configurations where the greater the increasing returns are, the more relevant this conclusion is. Moreover, we prove that in contrast to several existing works, the steady state can be locally determinate under the wrong slopes on the labor market, i.e. the slope of labor demand is greater than the slope of labor supply.

Comparing our formulation of externalities with the one usually adopted through the total productivity of factors, we also exhibit some configurations where the emergence of endogenous fluctuations requires weaker increasing returns under the specification of externalities that we propose. In this sense, the characterization of externalities developped in this paper promotes indeterminacy and endogenous cycles.

We finally remark that the Cobb-Douglas technology seems to be a particular case. Indeed, under a unit elasticity of capital-labor substitution, the formulation of externalities that we present and the usual one are similar, whereas we have seen that, in many cases, the dynamic stability properties can be quite different.

This paper is organized as follows. In the next section, we present the model and define the intertemporal equilibrium. In Section 3, we establish the existence of a steady state. In Section 4, we analyze the occurrence of local indeterminacy and endogenous cycles, and discuss our results. In Section 5, we summarize our main findings, while some technical details can be found in the Appendix.

2 The Model

We consider a monetary economy with discrete time, $t = 1, 2, ..., \infty$, and two types of infinitely lived households, workers and capitalists.[4] Only workers supply labor, whereas both workers and capitalists consume the final good. Moreover, workers are more impatient than capitalists, i.e. they discount the future more than the latter. Following Woodford (1986), we assume that there is a financial market imperfection that prevent workers from borrowing against their wage earnings. Therefore, in a neighborhood of a monetary steady state, capitalists hold the whole capital stock and no money, and workers save their wage income in the form of money balances. In the production sector, firms produce the final good. The production benefits from productive externalities specific for each input. As we will see, this type of externalities is more general than most of the formulations used in the literature and allows us to obtain new dynamic results.

2.1 Workers

Each worker chooses his labor supply and his consumption to maximize his utility function:

$$\sum_{t=1}^{\infty} \left[\lambda^{t-1} U(C_t^w/B) - \lambda^t V(L_t)\right] \qquad (1)$$

where C_t^w is the consumption in period t, L_t the labor supply, $B > 0$ a scaling parameter and $\lambda \in (0, 1)$ the discount factor. The utility functions U and V are characterized by the following assumptions:

Assumption 1 *The functions $U(C^w/B)$ and $V(L)$ are continuous for all $C^w \geq 0$ and $0 \leq L \leq \overline{L}$, where the labor endowment $\overline{L} > 1$ may be finite or infinite.*[5] *They are $\mathcal{C}^n$ for $C^w > 0$, $0 < L < \overline{L}$ and n large enough, with $U'(x) > 0$, $U''(x) < 0$, $V'(L) > 0$ and $V''(L) > 0$. Moreover, $lim_{L \to \overline{L}} V'(L) = +\infty$ and consumption and leisure are gross substitutes, i.e., $-xU''(x)/U'(x) < 1$.*

In what follows, we respectively note M_t^w and K_t^w the money balances and the capital stock held by workers, $\delta \in (0, 1)$ the depreciation rate of

[4]For more details on this model with heterogeneous households, see Woodford (1986). One can also refer to Barinci and Chéron (2001) or Grandmont, Pintus, and de Vilder (1998).

[5]We assume that the labor endowment is strictly greater than 1, because as we will see in the next section, the labor supply will be normalized to 1 at the steady state.

capital, r_t the nominal interest rate, w_t the nominal wage and P_t the price of the final good. Each worker maximizes his utility function (1) under the constraints:

$$P_t\left(C_t^w + K_t^w - (1-\delta)K_{t-1}^w\right) + M_t^w = M_{t-1}^w + r_t K_{t-1}^w + w_t L_t \tag{2}$$

$$P_t\left(C_t^w + K_t^w - (1-\delta)K_{t-1}^w\right) \leq M_{t-1}^w + r_t K_{t-1}^w \tag{3}$$

Equation (2) represents the usual budget constraint, while (3) is the liquidity constraint. The equilibria considered here are defined by:

$$U'(C_t^w/B) > \lambda U'(C_{t+1}^w/B)\left[(1-\delta) + r_{t+1}/P_{t+1}\right] \tag{4}$$

$$(1-\delta)P_{t+1} + r_{t+1} > P_t \tag{5}$$

So, workers always choose $K_t^w = 0$ and the financial constraint is binding, i.e. $P_t C_t^w = M_{t-1}^w$. Hence, we obtain the two following relations:

$$u\left(C_{t+1}^w/B\right) = v(L_t) \tag{6}$$

$$P_{t+1}C_{t+1} = w_t L_t \tag{7}$$

where $u(x) = xU'(x)$ and $v(L) = LV'(L)$. Under Assumption 1, there exists a function $\gamma \equiv u^{-1} \circ v$, such that $C_{t+1}/B = \gamma(L_t)$. Since consumption and leisure are gross substitutes, $\varepsilon_\gamma(L) \equiv \gamma'(L)L/\gamma(L) = [1+LV''(L)/V'(L)]/[1+xU''(x)/U'(x)] > 1$, with $x = C/B$. We deduce that the labor supply increases with respect to the real wage, with an elasticity $1/(\varepsilon_\gamma(L)-1) > 0$.

2.2 Capitalists

Capitalists behavior is represented by a representative agent who maximizes his lifetime utility function:

$$\sum_{t=1}^{\infty} \beta^t \ln C_t^c \tag{8}$$

where $\beta \in (\lambda, 1)$ is his discount factor and C_t^c his consumption. At period t, the representative agent faces the following budget constraint:

$$P_t\left(C_t^c + K_t^c - (1-\delta)K_{t-1}^c\right) + M_t^c = M_{t-1}^c + r_t K_{t-1}^c \tag{9}$$

where M_t^c is the money balances at period t and K_t^c the capital stock. Since we focus on equilibria satisfying $(1-\delta)P_{t+1} + r_{t+1} > P_t$, capitalists do not hold money ($M_t^c = 0$) because it has a lower return than capital. We deduce the following optimal solution:

$$C_t^c = (1-\beta)R_t K_{t-1} \tag{10}$$

$$K_t = \beta R_t K_{t-1} \tag{11}$$

where $R_t = 1 - \delta + r_t/P_t$ is the real gross return on capital.[6]

2.3 Production Sector

All markets are perfectly competitive and a continuum of firms, of unit mass, produce the final good using labor and capital with an internal constant returns to scale technology. However, production benefiting from externalities, returns to scale are increasing at the social level. In most of the existing papers, externalities are introduced assuming that the total productivity of factors increases with respect to average capital and labor.[7] Here, we introduce a more general formulation of externalities assuming that each input, labor and capital, is affected by a specific externality, increasing with respect to average capital and labor.

Consider that $F(K_{t-1}, L_t)$ is a well-defined strictly concave production function, homogeneous of degree one, increasing with each argument, $a_t \equiv K_{t-1}/L_t$ the capital-labor ratio and $f(x_t)$ the intensive production function that satisfies the following assumption:

Assumption 2 *The intensive production function $f(x)$ is continuous for $x \geq 0$, positively valued and differentiable as many times as needed for $x > 0$, with $f'(x) > 0$ and $f''(x) < 0$.*

At each period, the quantity of final good produced is given by:

$$\begin{aligned} Y_t &= AF(C(\overline{K}_{t-1}, \overline{L}_t)K_{t-1}, D(\overline{K}_{t-1}, \overline{L}_t)L_t) \\ &= Af\left(\frac{C(\overline{K}_{t-1}, \overline{L}_t)}{D(\overline{K}_{t-1}, \overline{L}_t)}\, a_t\right) D(\overline{K}_{t-1}, \overline{L}_t)L_t \end{aligned} \tag{12}$$

[6]The superscript on K_t^c is dropped because workers hold no capital.

[7]Such externalities have often been used in macroeconomic dynamics models. See among others Barinci and Chéron (2001), Benhabib and Farmer (1994), Boldrin (1992), Cazzavillan (2001), Cazzavillan, Lloyd-Braga, and Pintus (1998), Farmer and Guo (1994), Harrison and Weder (2002), Hintermaier (2003), Pintus (2006). For a survey, one can refer to Benhabib and Farmer (1999).

where $A > 0$ is a scaling parameter, $\overline{K}_{t-1}$ average capital and $\overline{L}_t$ average labor. $C(\overline{K}, \overline{L})$ represents externalities specific to capital, whereas $D(\overline{K}, \overline{L})$ summarizes externalities specific to the second input, labor. Note that $C(\overline{K}, \overline{L})K$ (respectively $D(\overline{K}, \overline{L})L$) can also be interpreted as capital (respectively labor) measured in efficient units. Evidently, in the particular case where $C(\overline{K}, \overline{L}) = D(\overline{K}, \overline{L})$, we recover the usual form where externalities are represented by the total productivity of factors. We further assume:

Assumption 3 *The functions $C(K, L)$ and $D(K, L)$ are continuous for all $K, L \geq 0$, positively valued and differentiable as many times as needed for $K, L > 0$. Moreover, we assume that $\varepsilon_{C,K}(K, L)$, $\varepsilon_{C,L}(K, L)$, $\varepsilon_{D,K}(K, L)$ and $\varepsilon_{D,L}(K, L) \geq 0$, where we note $\varepsilon_{h,x}(K, L) \equiv \frac{\partial h(K,L)}{\partial x} \frac{x}{h(K,L)}$, with $h(K, L) = \{C(K, L), D(K, L)\}$ and $x = \{K, L\}$.*

In particular, this assumption means that the contributions of capital and labor to the externalities $C(K, L)$ and $D(K, L)$ are always positive.

Maximizing their profits, the producers take as given the level of externalities. If ϱ_t and Ω_t respectively denote the real interest rate and the real wage, we obtain:

$$\varrho_t = C(\overline{K}_{t-1}, \overline{L}_t)A\rho\left(x_t\right) \tag{13}$$

$$\Omega_t = D(\overline{K}_{t-1}, \overline{L}_t)A\omega\left(x_t\right) \tag{14}$$

with

$$\rho(x_t) \equiv f'(x_t)\ ,\ \omega(x_t) \equiv f(x_t) - x_t f'(x_t) \ \text{ and } \ x_t \equiv \frac{C(\overline{K}_{t-1}, \overline{L}_t)}{D(\overline{K}_{t-1}, \overline{L}_t)} a_t$$

Remark that $\widetilde{\varrho}_t \equiv \varrho_t / C(\overline{K}_{t-1}, \overline{L}_t)$ and $\widetilde{\Omega}_t \equiv \Omega_t / D(\overline{K}_{t-1}, \overline{L}_t)$ represent the marginal productivities of capital and labor in efficient units.

Before determining the equilibrium, it is useful to define the following relationships. First, we note $s(x) \equiv \rho(x)x/f(x) \in (0, 1)$, with $x \equiv C(\overline{K}, \overline{L})a/D(\overline{K}, \overline{L})$, the capital share in income. Moreover, the elasticity of capital-labor substitution in efficient units is defined by $\sigma(x) = dlnx/dln(\widetilde{\Omega}/\widetilde{\varrho}) \geq 0$.[8] This implies that $1/\sigma(x) = d\ln\omega(x)/d\ln x - d\ln\rho(x)/d\ln x$. Since, $\omega'(x) = -x\rho'(x)$, we deduce that:

[8] A quite similar definition of the elasticity of capital-labor substitution is used by Pintus (2004).

$$\varepsilon_\omega(x) \equiv \frac{\omega'(x)x}{\omega(x)} = \frac{s(x)}{\sigma(x)} \text{ and } \varepsilon_\rho(x) \equiv \frac{\rho'(x)x}{\rho(x)} = -\frac{1-s(x)}{\sigma(x)} \tag{15}$$

Finally, note that the degree of returns to scale is determined by:

$$1 + s(x)[\varepsilon_{C,K}(\overline{K},\overline{L}) + \varepsilon_{C,L}(\overline{K},\overline{L})] + (1-s(x))[\varepsilon_{D,K}(\overline{K},\overline{L}) + \varepsilon_{D,L}(\overline{K},\overline{L})]$$

2.4 Intertemporal Equilibrium

Equilibrium on the labor market requires that $\overline{L}_t = L_t$ and $w_t/P_t = \Omega_t \equiv A\Omega(K_{t-1}, L_t)$, equilibrium on the capital market $\overline{K}_{t-1} = K_{t-1}$ and $r_t/P_t = \varrho_t \equiv A\varrho(K_{t-1}, L_t)$. Let $M > 0$ be the constant money supply. Since workers save their wage income in the form of money and capitalists do not hold money, the equilibrium condition on the money market can be written:

$$C_t^w = M/P_t = \Omega_t L_t \tag{16}$$

Finally, the good market equilibrium is ensured by Walras law. Then, using (6) and (11), we obtain the two following dynamic equations:

$$(1/B)\Omega_{t+1}L_{t+1} = \gamma(L_t) \tag{17}$$

$$K_t = \beta R_t K_{t-1} \tag{18}$$

where $R_t = 1 - \delta + \varrho_t$. Substituting expressions (13) and (14) into (17) and (18), we can define an intertemporal equilibrium:

Definition 1 *An intertemporal equilibrium with perfect foresight is a sequence* $(K_{t-1}, L_t) \in \mathbb{R}^2_{++}$, $t = 1, 2, ..., \infty$, *such that (17) and (18) are satisfied, where* ϱ_t *and* Ω_t *are given by (13) and (14).*

3 Existence of a Steady State

A steady state of the dynamic system (17)-(18) is a solution $(K, L) = (K_{t-1}, L_t)$ for all t, such that:

$$\frac{\theta\Omega(K,L)}{B\beta\varrho(K,L)} = \frac{\gamma(L)}{L} \tag{19}$$

$$A\varrho(K,L) = \theta/\beta \tag{20}$$

where $\theta \equiv 1 - \beta(1-\delta)$.

Following Cazzavillan, Lloyd-Braga, and Pintus (1998), the existence of a steady state is established by choosing appropriately the two scaling parameters $A > 0$ and $B > 0$ so as to ensure that one steady state coincides with $(K, L) = (1, 1)$. From equation (20), we obtain a unique solution:

$$A = \frac{\theta}{\beta \varrho(1,1)} > 0 \tag{21}$$

Using relation (19), we get:

$$u\left(\frac{\theta \Omega(1,1)}{B\beta \varrho(1,1)}\right) = v(1) \tag{22}$$

From Assumption 1, u is decreasing in B. Therefore, there exists a unique $B > 0$ satisfying this last equation if and only if $lim_{x \to 0} u(x) < v(1) < lim_{x \to +\infty} u(x)$.[9]

Proposition 1 *Under $lim_{x\to 0} u(x) < v(1) < lim_{x\to +\infty} u(x)$ and Assumptions 1-3, $(K, L) = (1, 1)$ is a stationary solution of the dynamic system (17)-(18) if and only if A and B are the unique solutions of (21) and (22).*

4 Local Dynamics and Bifurcation Analysis

In this section, we analyze the local stability of the steady state and the occurrence of bifurcations. We will see that the key parameters of this analysis will be the elasticities of labor supply, of capital-labor substitution, and of externalities with respect to capital and labor. To ease the presentation of the results, we use the geometrical method developped by Grandmont, Pintus, and de Vilder (1998), which applies to discrete time nonlinear two-dimensional dynamic systems. Therefore, we first differentiate the dynamic system (17)-(18). If we note ε_γ the elasticity of $\gamma(L)$, and $\varepsilon_{\varrho,K}$, $\varepsilon_{\varrho,L}$, $\varepsilon_{\Omega,K}$ and $\varepsilon_{\Omega,L}$ the elasticities of $\varrho(K, L)$ and $\Omega(K, L)$ with respect to K and L, evaluated at the steady state defined in Proposition 1, we get:

$$\frac{dK_t}{K} = (\theta \varepsilon_{\varrho,K} + 1)\frac{dK_{t-1}}{K} + \theta \varepsilon_{\varrho,L}\frac{dL_t}{L} \tag{23}$$

$$\frac{dL_{t+1}}{L} = -\frac{\varepsilon_{\Omega,K}(1+\theta\varepsilon_{\varrho,K})}{1+\varepsilon_{\Omega,L}}\frac{dK_{t-1}}{K} + \frac{\varepsilon_\gamma - \theta\varepsilon_{\Omega,K}\varepsilon_{\varrho,L}}{1+\varepsilon_{\Omega,L}}\frac{dL_t}{L} \tag{24}$$

[9] In order to be as short as possible, we do not analyze uniqueness or multiplicity of steady states.

Using (15) and noting $\varepsilon_{h,x} = \varepsilon_{h,x}(1,1)$, with $h = \{C, D\}$ and $x = \{K, L\}$, $s \equiv s(1)$ and $\sigma \equiv \sigma(1)$, we obtain the following elasticities evaluated at the steady state:

$$\varepsilon_{\varrho,K} = \varepsilon_{C,K} - \frac{(1-s)(\varepsilon_{C,K} - \varepsilon_{D,K} + 1)}{\sigma} \tag{25}$$

$$\varepsilon_{\varrho,L} = \varepsilon_{C,L} - \frac{(1-s)(\varepsilon_{C,L} - \varepsilon_{D,L} - 1)}{\sigma} \tag{26}$$

$$\varepsilon_{\Omega,K} = \varepsilon_{D,K} + \frac{s(\varepsilon_{C,K} - \varepsilon_{D,K} + 1)}{\sigma} \tag{27}$$

$$\varepsilon_{\Omega,L} = \varepsilon_{D,L} + \frac{s(\varepsilon_{C,L} - \varepsilon_{D,L} - 1)}{\sigma} \tag{28}$$

Using these expressions, the trace T and the determinant D of the associated Jacobian matrix, which represent respectively the sum and the product of the two eigenvalues, i.e. the roots of the characteristic polynomial $Q(\lambda) \equiv \lambda^2 - T\lambda + D = 0$, can be written:

$$\begin{aligned} T &= \frac{\sigma}{\sigma(1+\varepsilon_{D,L}) - s(1+\varepsilon_{D,L} - \varepsilon_{C,L})}(\varepsilon_\gamma - 1) + T_1(\sigma), \quad \text{with} \\ T_1(\sigma) &= 1 + \frac{\sigma[1 + \theta(\varepsilon_{C,K}(1+\varepsilon_{D,L}) - \varepsilon_{D,K}\varepsilon_{C,L})]}{\sigma(1+\varepsilon_{D,L}) - s(1+\varepsilon_{D,L} - \varepsilon_{C,L})} \\ &- \frac{\theta[\varepsilon_{C,K} + s\varepsilon_{C,L} + (1-s)(1+\varepsilon_{D,L}) + \varepsilon_{D,L}\varepsilon_{C,K} - \varepsilon_{D,K}\varepsilon_{C,L}]}{\sigma(1+\varepsilon_{D,L}) - s(1+\varepsilon_{D,L} - \varepsilon_{C,L})} \end{aligned} \tag{29}$$

$$D = \varepsilon_\gamma D_1(\sigma), \quad \text{with} \quad D_1(\sigma) = \frac{\sigma(1+\theta\varepsilon_{C,K}) - \theta(1-s)(1+\varepsilon_{C,K} - \varepsilon_{D,K})}{\sigma(1+\varepsilon_{D,L}) - s(1+\varepsilon_{D,L} - \varepsilon_{C,L})} \tag{30}$$

Following Grandmont, Pintus, and de Vilder (1998), we study the local stability (indeterminacy) of the steady state and the occurrence of bifurcations by analyzing the trace T and the determinant D in the plane (see Figures 1-4). On the line (AC), one eigenvalue is equal to 1. On the line (AB), one eigenvalue is equal to -1. On the segment $[BC]$, the two eigenvalues are complex conjugates and have an unit modulus. Furthermore, if (T, D) is inside (ABC), the steady state is a sink, i.e. locally indeterminate since one variable is predetermined. When $1 - T + D > (<)0$ and $1 + T + D < (>)0$, the steady state is a saddle. Otherwise, it is a source. In the last two cases, the steady state is locally determinate.

Suppose now that T and D change when a parameter, called the bifurcation parameter, varies continuously. When (T, D) crosses the line (AC), a transcritical or a pitchfork bifurcation generically occurs.[10] When (T, D) crosses the line (AB), one gets a flip bifurcation, i.e. a cycle of period 2 appears around the steady state. When (T, D) crosses the segment $[BC]$, one gets a Hopf bifurcation, i.e. an invariant closed curve appears around the steady state.[11] Moreover, sunspot equilibria can appear around a steady state if it is locally indeterminate. They can also occur in the neighborhood of a cycle of period two if it is locally stable and in a neighborhood of an invariant closed curve if the Hopf bifurcation is supercritical.[12]

In order to keep the analysis as simple as possible, we assume in what follows:

Assumption 4 *(i) θ is sufficiently small, such that the following inequalities are satisfied:*

- $s > \theta(1-s)$;
- $s(1+\varepsilon_{D,L}-\varepsilon_{C,L})(1+\theta\varepsilon_{C,K}) > \theta(1+\varepsilon_{D,L})[\varepsilon_{C,K}+s\varepsilon_{C,L}+(1-s)(1+\varepsilon_{D,L})]+\theta(\varepsilon_{C,K}\varepsilon_{D,L}-\varepsilon_{D,K}\varepsilon_{C,L})[(1-s)(1+\varepsilon_{D,L})+s\varepsilon_{C,L}]$;
- $s(1+\varepsilon_{D,L}-\varepsilon_{C,L}) > \theta[(1+\varepsilon_{D,L})(1-s+\varepsilon_{C,K})+\varepsilon_{C,L}(s-\varepsilon_{D,K})]$;

(ii) $\varepsilon_{h,x} < 1-s$, *with* $h = \{C, D\}$ *and* $x = \{K, L\}$.

Note that, in this type of model, the length of period is small. Then, the assumption that θ is small is not very restrictive, because δ is usually close to 0 and β close to 1. The second part of Assumption 4 means that externalities cannot be too strong. This last remark is in accordance with empirical studies establishing that if returns to scale are increasing, they are not too strong.

We choose $\varepsilon_\gamma \in (1, +\infty)$ as the bifurcation parameter. It is closely related to the elasticity of labor supply $1/(\varepsilon_\gamma - 1)$. From equations (29) and (30), we notice that (T, D) describes a half-line Δ when ε_γ varies in $(1, +\infty)$. This half-line Δ starts from $(T_1(\sigma), D_1(\sigma))$ when ε_γ tends to 1 and has a slope $S(\sigma)$ equal to:

[10]When one eigenvalue crosses the value 1, a saddle, a transcritical or a pitchfork bifurcation generically occurs. However, since there exists at least one steady state (see Proposition 1), it excludes saddle bifurcations.

[11]For further information about local bifurcation theory, see for example Grandmont (1988) and Wiggins (1990).

[12]For more details, see Guesnerie and Woodford (1992) and Grandmont, Pintus, and de Vilder (1998).

$$S(\sigma) = 1 + \theta\varepsilon_{C,K} - \theta(1-s)\frac{1+\varepsilon_{C,K}-\varepsilon_{D,K}}{\sigma} \tag{31}$$

Consequently, studying local indeterminacy and the occurrence of endogenous cycles only requires to analyze $(T_1(\sigma), D_1(\sigma))$, the slope $S(\sigma)$ and how they change when the elasticity of capital-labor substitution σ varies.

Under Assumptions 3-4, we first notice that the slope of the half-line Δ increases with respect to σ, from $-\infty$ when σ tends to 0, to $1+\theta\varepsilon_{C,K} \geq 1$ when σ tends to $+\infty$. In particular, $S(\sigma_{F_1}) = -1$ with $\sigma_{F_1} \equiv \theta(1-s)(1+\varepsilon_{C,K}-\varepsilon_{D,K})/(2+\theta\varepsilon_{C,K})$, $S(\sigma_0)=0$ with $\sigma_0 \equiv \theta(1-s)(1+\varepsilon_{C,K}-\varepsilon_{D,K})/(1+\theta\varepsilon_{C,K})$, and $S(\sigma_T)=1$ with $\sigma_T \equiv (1-s)(1+\varepsilon_{C,K}-\varepsilon_{D,K})/\varepsilon_{C,K}$.[13]

Secondly, when σ varies, $(T_1(\sigma),\ D_1(\sigma))$ describes a line Δ_1 whose the slope S_1 is defined by:

$$\begin{aligned}
&S_1 = \frac{D_1'(\sigma)}{T_1'(\sigma)} = \frac{N}{M}, \quad \text{with} \\
&N \equiv \theta(1+\varepsilon_{D,L})(1-s)(1+\varepsilon_{C,K}-\varepsilon_{D,K}) - s(1+\varepsilon_{D,L}-\varepsilon_{C,L})(1+\theta\varepsilon_{C,K}) \\
&M \equiv \theta(1+\varepsilon_{D,L})[\varepsilon_{C,K}+s\varepsilon_{C,L}+(1-s)(1+\varepsilon_{D,L})] + \theta(\varepsilon_{C,K}\varepsilon_{D,L} \\
&\quad - \varepsilon_{D,K}\varepsilon_{C,L})[(1-s)(1+\varepsilon_{D,L})+s\varepsilon_{C,L}] - s(1+\varepsilon_{D,L}-\varepsilon_{C,L})(1+\theta\varepsilon_{C,K})
\end{aligned} \tag{32}$$

Using Assumptions 3-4, we deduce that $D_1(\sigma)$ and $T_1(\sigma)$ are both decreasing, and $S_1 \geq 1$.[14] On Δ_1, the two extreme points are $(T_1(0), D_1(0))$ and $(T_1(+\infty), D_1(+\infty))$. The first one is determined by:

$$\begin{aligned}
&T_1(0) = 1 + \theta\frac{\varepsilon_{C,K}+s\varepsilon_{C,L}+(1-s)(1+\varepsilon_{D,L})+\varepsilon_{D,L}\varepsilon_{C,K}-\varepsilon_{D,K}\varepsilon_{C,L}}{s(1+\varepsilon_{D,L}-\varepsilon_{C,L})}, \\
&D_1(0) = \frac{\theta(1-s)(1+\varepsilon_{C,K}-\varepsilon_{D,K})}{s(1+\varepsilon_{D,L}-\varepsilon_{C,L})} > 0 \ , \text{ and } 1 - T_1(0) + D_1(0) = \\
&-\theta\frac{s(\varepsilon_{C,K}+\varepsilon_{C,L})+\varepsilon_{D,K}(1-s-\varepsilon_{C,L})+\varepsilon_{D,L}(1-s+\varepsilon_{C,K})}{s(1+\varepsilon_{D,L}-\varepsilon_{C,L})} \leq 0
\end{aligned} \tag{33}$$

Note that $(T_1(0), D_1(0))$ is on the right side of (AC), above the horizontal axis. $(T_1(+\infty), D_1(+\infty))$ is defined as follows:

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[13]Evidently, the existence of $\sigma_T > 0$ requires $\varepsilon_{C,K} > 0$.

[14]Indeed, Assumption 3 ensures that $M < 0$. Then, one can prove that $S_1 \geq 1$ is equivalent to $(1-s)(1+\varepsilon_{D,L})[\varepsilon_{D,L}(1+\varepsilon_{C,K})+\varepsilon_{D,K}(1-\varepsilon_{C,L})] + (1+\varepsilon_{D,L})s\varepsilon_{C,K} + s\varepsilon_{C,L}[1+\varepsilon_{D,L}(1+\varepsilon_{C,K})-\varepsilon_{D,K}\varepsilon_{C,L}] \geq 0$, which is always satisfied.

$$T_1(+\infty) = 1 + \frac{1+\theta(\varepsilon_{C,K}(1+\varepsilon_{D,L}) - \varepsilon_{D,K}\varepsilon_{C,L})}{1+\varepsilon_{D,L}}\ ,\ D_1(+\infty) = \frac{1+\theta\varepsilon_{C,K}}{1+\varepsilon_{D,L}} > 0$$

$$\text{and } 1 - T_1(+\infty) + D_1(+\infty) = \theta\frac{\varepsilon_{D,K}\varepsilon_{C,L} - \varepsilon_{C,K}\varepsilon_{D,L}}{1+\varepsilon_{D,L}} \tag{34}$$

$(T_1(\sigma), D_1(\sigma))$ decreases from $(T_1(0), D_1(0))$ to $(-\infty, -\infty)$ when $\sigma = \sigma_\infty \equiv s(1+\varepsilon_{D,L}-\varepsilon_{C,L})/(1+\varepsilon_{D,L})$, and from $(+\infty, +\infty)$ to $(T_1(+\infty), D_1(+\infty))$ when σ is greater than σ_∞ and increases to $+\infty$. Therefore, $D_1(+\infty) < 1$ is a necessary condition to have indeterminacy when capital and labor are not too weak substitutes. In the rest of the paper, we restrict our attention to this more interesting case, assuming:

Assumption 5 $\varepsilon_{D,L} > \theta\varepsilon_{C,K}$.

Furthermore, we can notice that the position of $(T_1(+\infty), D_1(+\infty))$ with respect to the line (AC) depends on the value of $\varepsilon_{D,K}\varepsilon_{C,L} - \varepsilon_{C,K}\varepsilon_{D,L}$. We have two configurations:

(i) if $\varepsilon_{D,K}\varepsilon_{C,L} \geq \varepsilon_{C,K}\varepsilon_{D,L}$, $(T_1(+\infty),\ D_1(+\infty))$ is above or on (AC) (see Figure 1);

(ii) if $\varepsilon_{D,K}\varepsilon_{C,L} < \varepsilon_{C,K}\varepsilon_{D,L}$, $(T_1(+\infty), D_1(+\infty))$ is below (AC) (see Figure 2).

Under Assumption 5, we define σ_{H_2} such that $D_1(\sigma_{H_2}) = 1$. We obtain:

$$\sigma_{H_2} \equiv \frac{s(1+\varepsilon_{D,L}-\varepsilon_{C,L}) - \theta(1-s)(1+\varepsilon_{C,K}-\varepsilon_{D,K})}{\varepsilon_{D,L} - \theta\varepsilon_{C,K}} \tag{35}$$

If $\varepsilon_{D,K}\varepsilon_{C,L} \geq \varepsilon_{C,K}\varepsilon_{D,L}$, $(T_1(\sigma),\ D_1(\sigma))$ crosses the segment $[BC]$ for $\sigma = \sigma_{H_2}$ and is inside (ABC) for all $\sigma \geq \sigma_{H_2}$. If $\varepsilon_{D,K}\varepsilon_{C,L} < \varepsilon_{C,K}\varepsilon_{D,L}$, $(T_1(\sigma), D_1(\sigma))$ can cross the segment $[BC]$ for $\sigma = \sigma_{H_2}$, can be inside (ABC) for $\sigma \in [\sigma_{H_2}, \sigma_{I_2}]$ and is below the (AC) line for all $\sigma > \sigma_{I_2}$,[15] where σ_{I_2} is defined by $1 - T_1(\sigma_{I_2}) + D_1(\sigma_{I_2}) = 0$, i.e.

$$\sigma_{I_2} \equiv 1 + \frac{s(\varepsilon_{C,K}+\varepsilon_{C,L}) + (1-s)(\varepsilon_{D,K}+\varepsilon_{D,L})}{\varepsilon_{C,K}\varepsilon_{D,L} - \varepsilon_{D,K}\varepsilon_{C,L}} \tag{36}$$

[15] We will see later that, in the case studied in this paper, Δ_1 is on the left of point C and goes inside (ABC).

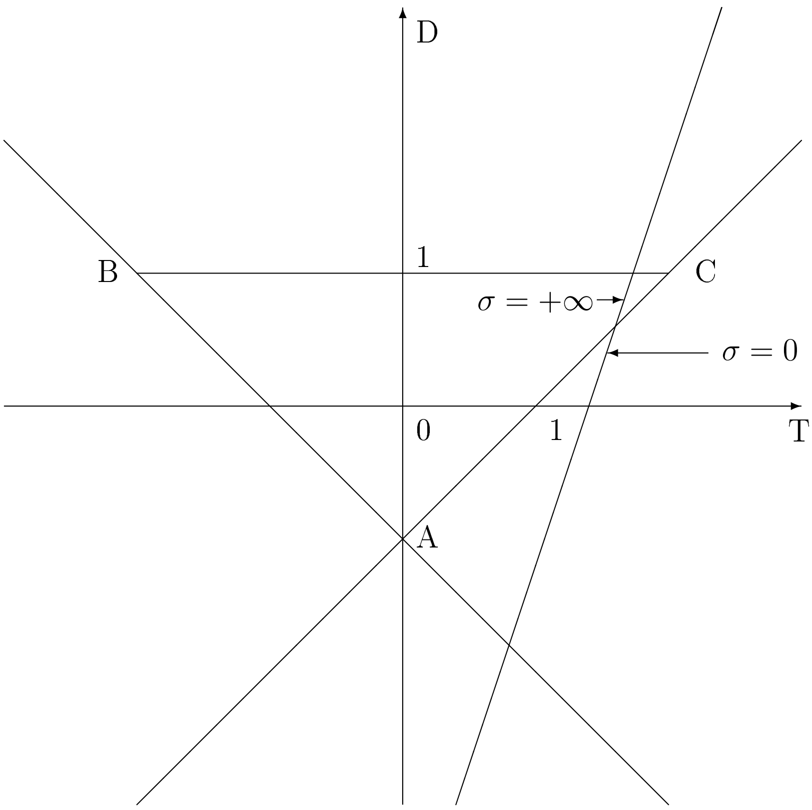

Figure 1: Representation of Δ_1 when $\varepsilon_{D,K}\varepsilon_{C,L} \geq \varepsilon_{C,K}\varepsilon_{D,L}$

Using these notations and geometrical results, we first study the local stability of the steady state and the occurrence of bifurcations for a weak substitution between capital and labor ($\sigma < \sigma_\infty$).

Since the locus $(T_1(\sigma), D_1(\sigma))$ decreases along the line Δ_1 and $S(\sigma)$ increases with respect to σ, Δ makes a counterclockwise rotation around Δ_1. In particular when $\sigma = 0$, the half-line Δ is vertical, crossing (AC) ($\varepsilon_\gamma = \varepsilon_{\gamma_T}$) between the horizontal axis and the point C, and the segment $[BC]$ ($\varepsilon_\gamma = \varepsilon_{\gamma_H}$).[16] This first case applies for all $\sigma \leq \sigma_{F_1}$. In what follows, note that σ_{H_1} is such that the half-line Δ goes through the point B, σ_{I_1} is such that Δ goes through A and σ_{F_2} is such that the line Δ_1 crosses (AB).[17] When σ belongs to $(\sigma_{F_1}, \sigma_{H_1})$, Δ does not only cross (AC) and $[BC]$, but also the line (AB) ($\varepsilon_\gamma = \varepsilon_{\gamma_F}$) above B. When σ becomes greater than σ_{H_1} and is smaller than σ_{I_1}, the half-line Δ only crosses (AC), and (AB) between

[16]Note that under Assumption 4, $T_1(0)$ is strictly smaller than 2.

[17]Unfortunately, we cannot explicitly determine σ_{H_1} and σ_{I_1}, while σ_{F_2} is given in the Appendix.

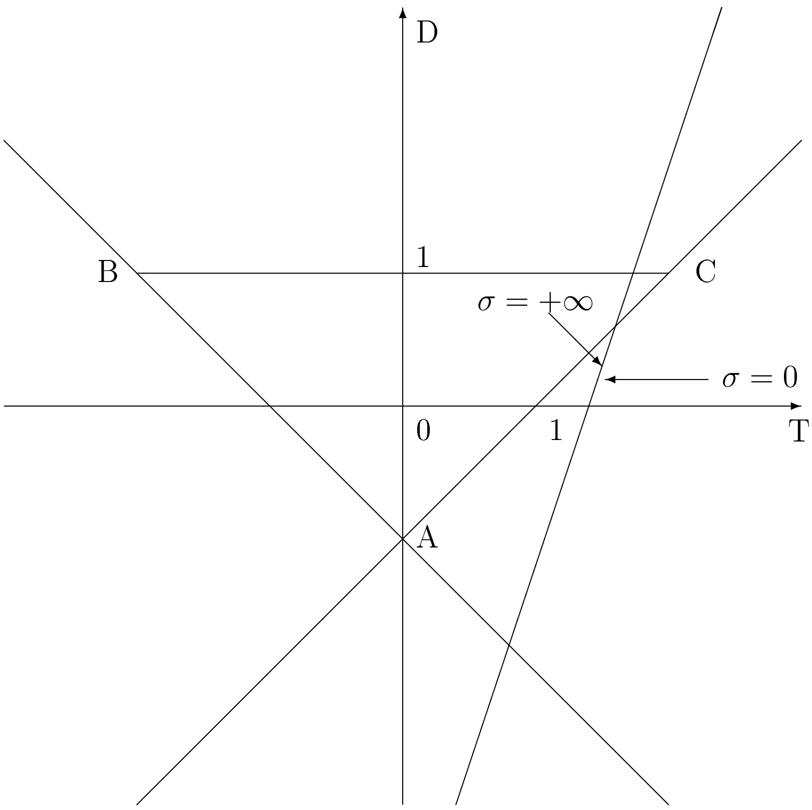

Figure 2: Representation of Δ_1 when $\varepsilon_{D,K}\varepsilon_{C,L} < \varepsilon_{C,K}\varepsilon_{D,L}$

A and B. When σ becomes greater than σ_{I_1}, the half-line Δ goes below the point A. We obtain two different cases: When $\sigma_{I_1} < \sigma < \sigma_{F_2}$, Δ crosses (AB) and (AC) and when $\sigma_{F_2} \leq \sigma < \sigma_{\infty}$, Δ only crosses (AC).

These results which are common to all the configurations can be summarized as follows (see also Figure 3):

Proposition 2 *(Local dynamics for a weak capital-labor substitution) Assuming that there is a steady state (Proposition 1) and that Assumptions 1-4 are satisfied, the following generically holds.*[18]

(i) When $0 < \sigma \leq \sigma_{F_1}$, the steady state is a saddle for $1 < \varepsilon_\gamma < \varepsilon_{\gamma_T}$, undergoes a transcritical or a pitchfork bifurcation for $\varepsilon_\gamma = \varepsilon_{\gamma_T}$, is a sink for $\varepsilon_{\gamma_T} < \varepsilon_\gamma < \varepsilon_{\gamma_H}$, undergoes a Hopf bifurcation for $\varepsilon_\gamma = \varepsilon_{\gamma_H}$, is a source for $\varepsilon_\gamma > \varepsilon_{\gamma_H}$.

[18]The values ε_{γ_T}, ε_{γ_H} and ε_{γ_F} are given in the Appendix.

(ii) *When* $\sigma_{F_1} < \sigma < \sigma_{H_1}$*, the steady state is a saddle for* $1 < \varepsilon_\gamma < \varepsilon_{\gamma_T}$*, undergoes a transcritical or a pitchfork bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_T}$*, is a sink for* $\varepsilon_{\gamma_T} < \varepsilon_\gamma < \varepsilon_{\gamma_H}$*, undergoes a Hopf bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_H}$*, is a source for* $\varepsilon_{\gamma_H} < \varepsilon_\gamma < \varepsilon_{\gamma_F}$*, undergoes a flip bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_F}$*, is a saddle for* $\varepsilon_\gamma > \varepsilon_{\gamma_F}$*.*

(iii) *When* $\sigma_{H_1} < \sigma < \sigma_{I_1}$*, the steady state is a saddle for* $1 < \varepsilon_\gamma < \varepsilon_{\gamma_T}$*, undergoes a transcritical or a pitchfork bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_T}$*, is a sink for* $\varepsilon_{\gamma_T} < \varepsilon_\gamma < \varepsilon_{\gamma_F}$*, undergoes a flip bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_F}$*, is a saddle for* $\varepsilon_\gamma > \varepsilon_{\gamma_F}$*.*

(iv) *When* $\sigma_{I_1} < \sigma < \sigma_{F_2}$*, the steady state is a saddle for* $1 < \varepsilon_\gamma < \varepsilon_{\gamma_F}$*, undergoes a flip bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_F}$*, is a source for* $\varepsilon_{\gamma_F} < \varepsilon_\gamma < \varepsilon_{\gamma_T}$*, undergoes a transcritical or a pitchfork bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_T}$*, is a saddle for* $\varepsilon_\gamma > \varepsilon_{\gamma_T}$*.*

(v) *When* $\sigma_{F_2} \leq \sigma < \sigma_\infty$*, the steady state is a source for* $1 < \varepsilon_\gamma < \varepsilon_{\gamma_T}$*,undergoes a transcritical or a pitchfork bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_T}$*, is a saddle for* $\varepsilon_\gamma > \varepsilon_{\gamma_T}$*.*

This proposition establishes that endogenous fluctuations can occur for sufficiently weak elasticities of capital-labor substitution. Indeed, local indeterminacy can emerge for $0 < \sigma < \sigma_{I_1}$ and deterministic cycles can appear through the occurrence of a Hopf bifurcation for $0 < \sigma < \sigma_{H_1}$ and through the occurrence of a flip bifurcation for $\sigma_{F_1} < \sigma < \sigma_{F_2}$. One can remark that such results are not so different than those obtained under perfect competition and constant returns to scale by Grandmont, Pintus, and de Vilder (1998). The main difference concerns the occurrence of a transcritical or pitchfork bifurcation, which is not possible in their framework.

We now analyze what happens for $\sigma > \sigma_\infty$. As we have already noticed, when σ increases from σ_∞ to $+\infty$, $D_1(\sigma)$ decreases from $+\infty$ to $D_1(+\infty) \in (0,1)$ along the line Δ_1. Two different configurations can arise depending on the value of $\varepsilon_{C,K}\varepsilon_{D,L} - \varepsilon_{D,K}\varepsilon_{C,L}$.

To begin, assume that the first configuration applies, i.e. $\varepsilon_{D,K}\varepsilon_{C,L} \geq \varepsilon_{C,K}\varepsilon_{D,L}$, which means that $(T_1(\sigma), D_1(\sigma))$ is inside (ABC) for all $\sigma > \sigma_{H_2}$. In order to simplify the presentation, we restrict our attention to the following case:

Assumption 6 $\frac{(1-s)\varepsilon_{D,L}}{1+\varepsilon_{D,L}-\varepsilon_{C,L}} > \frac{s\varepsilon_{C,K}}{1+\varepsilon_{C,K}-\varepsilon_{D,K}}$.

This means that $\sigma_T > \sigma_{H_2}$. We deduce that when $\sigma_\infty < \sigma \leq \sigma_{H_2}$, the half-line Δ only crosses (AC). When $\sigma_{H_2} < \sigma < \sigma_T$, Δ crosses $[BC]$

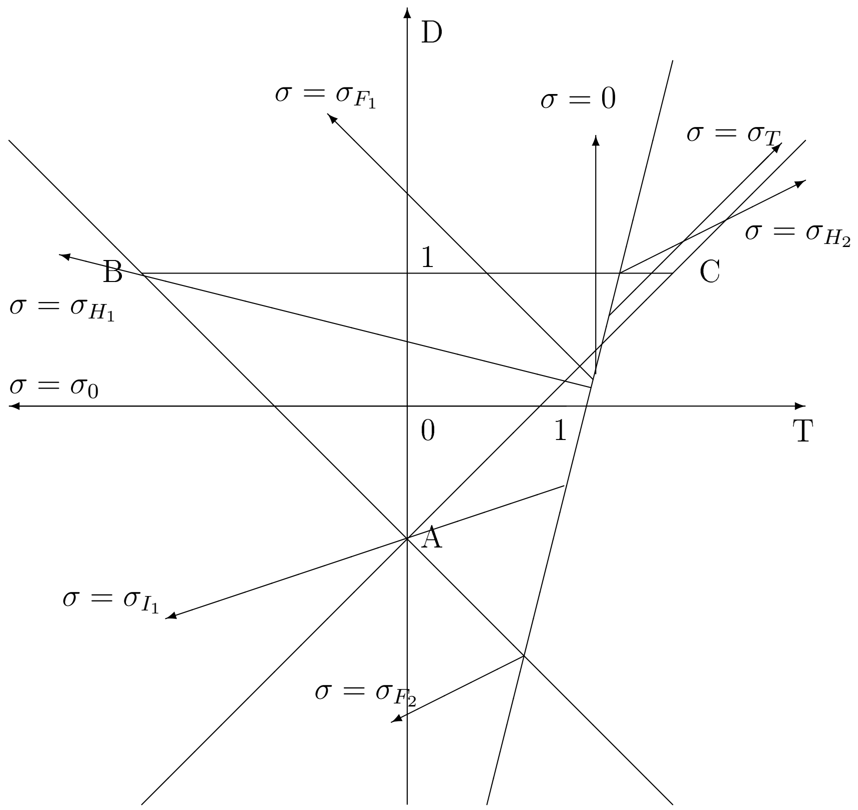

Figure 3: Representation of Δ when $\varepsilon_{D,K}\varepsilon_{C,L} \geq \varepsilon_{C,K}\varepsilon_{D,L}$

and (AC) above C. Finally, when $\sigma \geq \sigma_T$, the half-line Δ only crosses the segment $[BC]$.[19]

Consider now that the second configuration applies, i.e. $\varepsilon_{D,K}\varepsilon_{C,L} < \varepsilon_{C,K}\varepsilon_{D,L}$. Recall that now, $(T_1(\sigma), D_1(\sigma))$ crosses the line (AC) for $\sigma = \sigma_{I_2}$ and is below (AC) for $\sigma > \sigma_{I_2}$. We further note that, under Assumption 6, we have $\sigma_{I_2} > \sigma_T > \sigma_{H_2}$. This last inequality implies that the line Δ_1 crosses (ABC) and is on the left side of C. Then, Δ only crosses (AC) for $\sigma_\infty < \sigma \leq \sigma_{H_2}$, crosses the segment $[BC]$ and the line (AC) above C for $\sigma_{H_2} < \sigma < \sigma_T$ and only crosses $[BC]$ for $\sigma_T < \sigma \leq \sigma_{I_2}$. For greater values of the elasticity of capital-labor substitution ($\sigma > \sigma_{I_2}$), there exist some differences between the two following cases[20]:

(i) if $1 + \varepsilon_{D,L} \leq (1 + \theta\varepsilon_{C,K})\varepsilon_{C,L}\varepsilon_{D,K}/(\theta\varepsilon_{C,K}^2)$, Δ crosses (AC) below C and

[19]Indeed, as it is shown in the Appendix, in the configuration where $\varepsilon_{D,K}\varepsilon_{C,L} \geq \varepsilon_{C,K}\varepsilon_{D,L}$, Δ always crosses $[BC]$ when $\sigma > \sigma_{H_2}$.

[20]See the Appendix for more details.

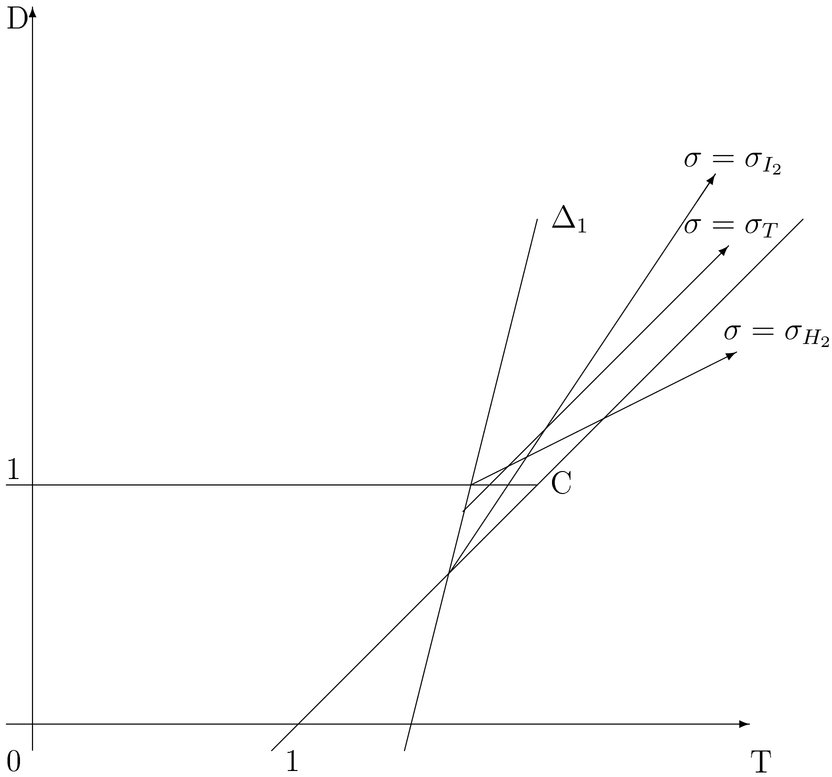

Figure 4: Representation of Δ when $\varepsilon_{D,K}\varepsilon_{C,L} < \varepsilon_{C,K}\varepsilon_{D,L}$

$[BC]$ for all $\sigma > \sigma_{I_2}$;

(ii) if $1 + \varepsilon_{D,L} > (1 + \theta\varepsilon_{C,K})\varepsilon_{C,L}\varepsilon_{D,K}/(\theta\varepsilon_{C,K}^2)$, Δ crosses (AC) below C and $[BC]$ for $\sigma_{I_2} < \sigma < \sigma_{I_3}$ and only crosses (AC) above C for $\sigma > \sigma_{I_3}$.

All these results can be summarized as follows (see also Figures 3 and 4):

Proposition 3 *(Local dynamics when capital-labor substitution is not too weak) Assuming that there is a steady state (Proposition 1) and that Assumptions 1-6 are satisfied, the following generically holds.*[21]

1. $\varepsilon_{D,K}\varepsilon_{C,L} \geq \varepsilon_{C,K}\varepsilon_{D,L}$*:*

(i) When $\sigma_\infty < \sigma \leq \sigma_{H_2}$, the steady state is a source for $1 < \varepsilon_\gamma < \varepsilon_{\gamma_T}$, undergoes a transcritical or a pitchfork bifurcation for $\varepsilon_\gamma = \varepsilon_{\gamma_T}$, is a saddle for $\varepsilon_\gamma > \varepsilon_{\gamma_T}$.

[21]The values ε_{γ_T}, ε_{γ_H} and ε_{γ_F} and some technical details are given in the Appendix.

(ii) *When* $\sigma_{H_2} < \sigma < \sigma_T$*, the steady state is a sink for* $1 < \varepsilon_\gamma < \varepsilon_{\gamma_H}$*, undergoes a Hopf bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_H}$*, is a source for* $\varepsilon_{\gamma_H} < \varepsilon_\gamma < \varepsilon_{\gamma_T}$*, undergoes a transcritical or a pitchfork bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_T}$*, is a saddle for* $\varepsilon_\gamma > \varepsilon_{\gamma_T}$*.*

(iii) *When* $\sigma \geq \sigma_T$*, the steady state is a sink for* $1 < \varepsilon_\gamma < \varepsilon_{\gamma_H}$*, undergoes a Hopf bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_H}$*, is a source for* $\varepsilon_\gamma > \varepsilon_{\gamma_H}$*.*

2. $\varepsilon_{D,K}\varepsilon_{C,L} < \varepsilon_{C,K}\varepsilon_{D,L}$:

(a) $1 + \varepsilon_{D,L} \leq (1 + \theta\varepsilon_{C,K})\varepsilon_{C,L}\varepsilon_{D,K}/(\theta\varepsilon_{C,K}^2)$:

(i) *When* $\sigma_\infty < \sigma \leq \sigma_{H_2}$*, the steady state is a source for* $1 < \varepsilon_\gamma < \varepsilon_{\gamma_T}$*, undergoes a transcritical or a pitchfork bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_T}$*, is a saddle for* $\varepsilon_\gamma > \varepsilon_{\gamma_T}$*.*

(ii) *When* $\sigma_{H_2} < \sigma < \sigma_T$*, the steady state is a sink for* $1 < \varepsilon_\gamma < \varepsilon_{\gamma_H}$*, undergoes a Hopf bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_H}$*, is a source for* $\varepsilon_{\gamma_H} < \varepsilon_\gamma < \varepsilon_{\gamma_T}$*, undergoes a transcritical or a pitchfork bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_T}$*, is a saddle for* $\varepsilon_\gamma > \varepsilon_{\gamma_T}$*.*

(iii) *When* $\sigma_T \leq \sigma \leq \sigma_{I_2}$*, the steady state is a sink for* $1 < \varepsilon_\gamma < \varepsilon_{\gamma_H}$*, undergoes a Hopf bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_H}$*, is a source for* $\varepsilon_\gamma > \varepsilon_{\gamma_H}$*.*

(iv) *When* $\sigma > \sigma_{I_2}$*, the steady state is a saddle for* $1 < \varepsilon_\gamma < \varepsilon_{\gamma_T}$*, undergoes a transcritical or a pitchfork bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_T}$*, is a sink for* $\varepsilon_{\gamma_T} < \varepsilon_\gamma < \varepsilon_{\gamma_H}$*, undergoes a Hopf bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_H}$*, is a source for* $\varepsilon_\gamma > \varepsilon_{\gamma_H}$*.*

(b) $1 + \varepsilon_{D,L} > (1 + \theta\varepsilon_{C,K})\varepsilon_{C,L}\varepsilon_{D,K}/(\theta\varepsilon_{C,K}^2)$:

(i) *When* $\sigma_\infty < \sigma \leq \sigma_{H_2}$*, the steady state is a source for* $1 < \varepsilon_\gamma < \varepsilon_{\gamma_T}$*, undergoes a transcritical or a pitchfork bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_T}$*, is a saddle for* $\varepsilon_\gamma > \varepsilon_{\gamma_T}$*.*

(ii) *When* $\sigma_{H_2} < \sigma < \sigma_T$*, the steady state is a sink for* $1 < \varepsilon_\gamma < \varepsilon_{\gamma_H}$*, undergoes a Hopf bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_H}$*, is a source for* $\varepsilon_{\gamma_H} < \varepsilon_\gamma < \varepsilon_{\gamma_T}$*, undergoes a transcritical or a pitchfork bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_T}$*, is a saddle for* $\varepsilon_\gamma > \varepsilon_{\gamma_T}$*.*

(iii) *When* $\sigma_T \leq \sigma \leq \sigma_{I_2}$*, the steady state is a sink for* $1 < \varepsilon_\gamma < \varepsilon_{\gamma_H}$*, undergoes a Hopf bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_H}$*, is a source for* $\varepsilon_\gamma > \varepsilon_{\gamma_H}$*.*

(iv) *When* $\sigma_{I_2} < \sigma < \sigma_{I_3}$*, the steady state is a saddle for* $1 < \varepsilon_\gamma < \varepsilon_{\gamma_T}$*, undergoes a transcritical or a pitchfork bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_T}$*, is a sink for* $\varepsilon_{\gamma_T} < \varepsilon_\gamma < \varepsilon_{\gamma_H}$*, undergoes a Hopf bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_H}$*, is a source for* $\varepsilon_\gamma > \varepsilon_{\gamma_H}$*.*

(*v*) *When* $\sigma > \sigma_{I_3}$, *the steady state is a saddle for* $1 < \varepsilon_\gamma < \varepsilon_{\gamma_T}$, *undergoes a transcritical or a pitchfork bifurcation for* $\varepsilon_\gamma = \varepsilon_{\gamma_T}$, *is a source for* $\varepsilon_\gamma > \varepsilon_{\gamma_T}$.

This proposition summarizes the conditions for the emergence of endogenous fluctuations under a not too weak capital-labor substitution ($\sigma > \sigma_\infty$). In what follows, to ease the discussion, we call $\varepsilon_{C,K}$ and $\varepsilon_{D,L}$ the direct effects of externalities and $\varepsilon_{D,K}$ and $\varepsilon_{C,L}$ the crossing effects. Indeed, the first ones correspond to the contribution of capital (labor) on externalities representing capital (labor) efficiency in the production function. On the contrary, the last ones correspond to the contribution of capital (labor) on externalities representing labor (capital) efficiency.

Recall first that endogenous fluctuations can occur only if Assumption 5 is satisfied. This means that the contribution of labor to externalities $D(K, L)$ has not to be too weak with respect to the contribution of capital to externalities $C(K, L)$. In other words, only direct effects have a role on this necessary condition for indeterminacy. This first comment can be related to the paper by Cazzavillan, Lloyd-Braga, and Pintus (1998) who analyze, in a Woodford (1986) model, the usual case where externalities are represented by the total productivity of factors, i.e. $C(K, L) = D(K, L)$. They establish that the emergence of indeterminacy and cycles for a high elasticity of capital-labor substitution requires that the contribution of labor to externalities has to be sufficiently high with respect to the contribution of capital. The more general formulation of externalities used in this paper allows to clarify that the necessary condition for indeterminacy only requires a not too weak direct effect of labor on externalities that affect labor efficiency. Moreover, Proposition 3 establishes that, due to the more general formulation of externalitites, two configurations emerge depending on the level of the direct effects with respect to the crossing effects on externalities.

If the product of crossing effects are greater than the product of direct effects ($\varepsilon_{D,K}\varepsilon_{C,L} \geq \varepsilon_{C,K}\varepsilon_{D,L}$), indeterminacy and endogenous cycles can appear for all elasticities of capital-labor substitution greater than σ_{H_2}. This result is quite similar to the one obtained by Cazzavillan, Lloyd-Braga, and Pintus (1998) where the product of crossing effects are, by assumption, equal to the product of direct effects ($\varepsilon_{D,K}\varepsilon_{C,L} = \varepsilon_{C,K}\varepsilon_{D,L}$). In other words, we provide here a generalization of their result to the case where the product of crossing effects are not only equal but also greater than the product of direct effects of externalities.

In this case, it is also interesting to relate our analysis to Benhabib and Farmer (1994). As it is well-known, these authors have shown, considering an infinitely lived agent model, that local indeterminacy requires wrong slopes on the labor market, i.e. the slope of labor demand must be greater than the slope of labor supply.[22] In our framework, when $\varepsilon_{D,K}\varepsilon_{C,L} \geq \varepsilon_{C,K}\varepsilon_{D,L}$ and capital-labor is not weak, the occurrence of indeterminacy requires that:

$$\varepsilon_{\gamma} < \varepsilon_{\gamma_H} = \frac{1 + \varepsilon_{D,L} - s(1 + \varepsilon_{D,L} - \varepsilon_{C,L})/\sigma}{1 + \theta\varepsilon_{C,K}(1 - \sigma_T/\sigma)} \tag{37}$$

Using equation (28), we remark that $\varepsilon_{D,L} - s(1+\varepsilon_{D,L} - \varepsilon_{C,L})/\sigma$ represents the slope of labor demand, whereas $\varepsilon_{\gamma} - 1$ is the slope of labor supply. Therefore, when $\sigma > \sigma_T$, local indeterminacy requires the wrong slopes, whereas when $\sigma \in (\sigma_{H_2}, \sigma_T)$, indeterminacy can occur when the slope of labor demand is smaller than the slope of labor supply. This result already obtained by Barinci and Chéron (2001) is extended here to the case of a non unitary elasticity of capital-labor substitution and a more general formulation of externalities.

Consider now that the product of crossing effects are weaker than the product of direct effects ($\varepsilon_{D,K}\varepsilon_{C,L} < \varepsilon_{C,K}\varepsilon_{D,L}$). As in the previous case, endogenous fluctuations can occur for a sufficiently elastic labor supply as soon as σ becomes greater than σ_{H_2}. However, for high enough elasticities of capital-labor substitution ($\sigma > \sigma_{I_2}$), indeterminacy and endogenous cycles can no more occur, particularly under a highly elastic labor supply is. This means that, in this configuration and in contrast to a lot of existing results,[23] endogenous fluctuations do no more emerge when the real wage is sufficiently increasing in labor, production factors are substitutes and the elasticity of labor supply is high enough. Furthermore, consider the case where the crossing effects are equal to zero, i.e. $\varepsilon_{D,K} = \varepsilon_{C,L} = 0$. Therefore, $\sigma_{I_2} = 1 + s/\varepsilon_{D,L} + (1-s)/\varepsilon_{C,K}$ decreases with the level of externalities. This means that the greater the increasing returns are, the more relevant our last conclusion is. Stronger increasing returns do not always promote local indeterminacy.

Note also that the steady state can be locally determinate under the wrong slopes. To illustrate this last remark, consider that $\varepsilon_{D,L} > \theta\varepsilon_{C,K} > 0$ and $\varepsilon_{D,K} = \varepsilon_{C,L} = 0$. In the limit case where ε_{γ} tends to 1 and σ to $+\infty$, the labor supply has a slope equal to 0 and the labor demand to $\varepsilon_{D,L}$. In this case, the steady state is a saddle even if the positive slope of labor demand is higher than the slope of labor supply.

We will now exhibit some simple configurations where, in comparison with the usual type of externalities (see Cazzavillan, Lloyd-Braga, and Pin-

[22] Note that this result can be relaxed when one does not restrict the analysis to an unit elasticity of factor substitution. See Pintus (2006).

[23] See Benhabib and Farmer (1999) for a survey.

tus (1998)), our formulation of externalities can provide results promoting indeterminacy and endogenous cycles. In what follows, consider the case where $\varepsilon_{D,K} = \varepsilon_{C,K} = 0$. Indeterminacy requires $\sigma > \sigma_{H_2}$ and $\varepsilon_\gamma < \varepsilon_{\gamma_H}$. In the case where externalities enter the total productivity of factors, we have $\varepsilon_{C,L} = \varepsilon_{D,L} \equiv \varepsilon_{E,L}$, where $\varepsilon_{E,L}$ represents the level of increasing returns. Using (35) and (42), we obtain $\sigma_{H_2} = \frac{s-\theta(1-s)}{\varepsilon_{E,L}}$ and $\varepsilon_{\gamma_H} = \frac{\sigma(1+\varepsilon_{E,L})-s}{\sigma-\theta(1-s)}$. Consider now the model developped in this paper assuming further $\varepsilon_{C,L} = 0$. Then, $(1-s)\varepsilon_{D,L}$ represents the level of increasing returns and we have $\sigma_{H_2} = \frac{s(1+\varepsilon_{D,L})-\theta(1-s)}{\varepsilon_{D,L}}$ and $\varepsilon_{\gamma_H} = \frac{(\sigma-s)(1+\varepsilon_{D,L})}{\sigma-\theta(1-s)}$.

Equalizing first the two expressions of σ_{H_2}, we get $\varepsilon_{E,L} = \frac{(s-\theta(1-s))\varepsilon_{D,L}}{s(1+\varepsilon_{D,L})-\theta(1-s)}$ which is greater than $(1-s)\varepsilon_{D,L}$ for all $\varepsilon_{D,L} < \frac{s-\theta(1-s)}{1-s}$. This means that, under a not too strong degree of externalities, one needs a smaller level of increasing returns to obtain the same range of elasticities of capital-labor substitution compatible with indeterminacy. In other words, for the same level of increasing returns, the range of σ compatible with indeterminacy is greater in the model developped in this paper.

Equalizing now the two expressions of ε_{γ_H}, we obtain $\varepsilon_{E,L} = \varepsilon_{D,L}(1-s/\sigma)$, which is greater than $(1-s)\varepsilon_{D,L}$ for $\sigma > 1$ and smaller otherwise. Therefore, when $\sigma > 1$, elasticities of labor supply compatible with indeterminacy require a smaller level of increasing returns in the model developped in this paper or, in other terms, for the same level of increasing returns, less elastic labor supply are compatible with indeterminacy. Evidently, we obtain the opposite conclusion for $\sigma < 1$.

To summarize this last discussion, in some configurations, endogenous fluctuations can occur when capital and labor are substitutes under less restrictive conditions in our framework than in the model where externalities appear through the total productivity of factors, because, taken as given σ and ε_γ, indeterminacy requires weaker increasing returns.

Before providing concluding remarks, let us note the following. In this paper, we have shown that, considering a more general formulation of externalities, local indeterminacy and endogenous fluctuations can occur under different conditions than in the more usual case studied by Cazzavillan, Lloyd-Braga, and Pintus (1998) where externalities are represented by the total productivity of factors. However, in the Cobb-Douglas case ($\sigma = 1$), the two formulations of externalities are equivalent. Therefore, we can argue that the Cobb-Douglas technology is very specific. To prove this last remark, note that when $\sigma = 1$, the production is given by:

$$Y = A(C(K,L)K)^s(D(K,L)L)^{1-s} = AC(K,L)^s D(K,L)^{1-s} K^s L^{1-s} \quad (38)$$

Therefore, the level of externalities represented by $C(K,L)^s D(K,L)^{1-s}$ plays exactly the same role than externalities that should appear through the total productivity of factors. Using our notations, the contributions of labor and capital to these total externalities are respectively defined by $s\varepsilon_{C,L} + (1-s)\varepsilon_{D,L}$ and $s\varepsilon_{C,K} + (1-s)\varepsilon_{D,K}$.

5 Conclusion

In this paper, we introduce input-specific externalities in a Woodford (1986) model to analyze their role on the stability properties of the steady state. This specification of externalities is a generalization of the usual type of productive externalities characterized by a total productivity of factors that increases with respect to average capital and labor.

In this framework, there exist configurations of parameters such that, in contrast to previous results, endogenous fluctuations cannot occur for a sufficiently high capital-labor substitution, particularly when the labor supply is highly elastic. We also stress that, in some cases, this result is more relevant under a stronger level of increasing returns. Another direct implication of this result is that the steady state can be locally determinate under the wrong slopes on the labor market.

Comparing our conclusions with the case where productive externalities enter the total productivity of factors, we exhibit some configurations where, in our framework, indeterminacy and endogenous cycles can occur under a weaker degree of increasing returns when capital and labor are substitutes.

Finally, we also conclude that the Cobb-Douglas technology appears to be quite specific. Indeed, under a unit elasticity of capital-labor substitution, our form of externalities and the usual one are equivalent, whereas we have seen that, when we consider a more general production function, dynamic properties can be quite different.

Appendix

Hopf bifurcation for $\sigma > \sigma_{H_2}$ and existence of σ_{I_3}

Assuming $\sigma > \sigma_{H_2}$, we establish the conditions such that $T < 2$ when $D = 1$. $D = 1$ is equivalent to $\varepsilon_\gamma = 1/D_1(\sigma)$. Substituting this expression into $T < 2$, we obtain:

$$a\sigma^2 + b\sigma + c > 0 \tag{39}$$

with

$$
\begin{aligned}
&a = \theta\varepsilon_{C,K}(\varepsilon_{D,L} - \theta\varepsilon_{C,K}) + \theta(1+\theta\varepsilon_{C,K})(\varepsilon_{D,K}\varepsilon_{C,L} - \varepsilon_{D,L}\varepsilon_{C,K}) \\
&b = \theta\varepsilon_{D,K}[(1-s)(1+\varepsilon_{D,L}) - \varepsilon_{C,L}] + \theta\varepsilon_{C,L}[\theta(1-s)\varepsilon_{D,K}^2 \\
&+ (1+\varepsilon_{C,K})(s - \theta(1-s)\varepsilon_{D,K})] + \theta^2\varepsilon_{C,K}[(1+\varepsilon_{D,L})(1-s)(1-\varepsilon_{D,K}) \\
&+ s\varepsilon_{C,L} + (1+\varepsilon_{D,L})(1-s+(2-s)\varepsilon_{C,K}) - \varepsilon_{D,K}\varepsilon_{C,L}] \\
&c = \theta(1-s)(1+\varepsilon_{C,K} - \varepsilon_{D,K})[s(1+\varepsilon_{D,L} - \varepsilon_{C,L}) \\
&- \theta((1+\varepsilon_{D,L})(1-s+\varepsilon_{C,K}) + \varepsilon_{C,L}(s - \varepsilon_{D,K}))]
\end{aligned}
\tag{40}
$$

Under Assumption 4, we deduce that $b > 0$ and $c > 0$. Then, we have two cases:

(i) if $1+\varepsilon_{D,L} \leq (1+\theta\varepsilon_{C,K})\varepsilon_{C,L}\varepsilon_{D,K}/(\theta\varepsilon_{C,K}^2)$ (i.e. $a \geq 0$), the inequality (39) is satisfied for all $\sigma > 0$;

(ii) if $1+\varepsilon_{D,L} < (1+\theta\varepsilon_{C,K})\varepsilon_{C,L}\varepsilon_{D,K}/(\theta\varepsilon_{C,K}^2)$ (i.e. $a < 0$), there exists a unique $\sigma_{I_3} > 0$ such that the inequality (39) is satisfied for all $\sigma < \sigma_{I_3}$.

This means that in case (i), Δ always crosses $[BC]$ when $\sigma > \sigma_{H_2}$. In particular, this arises when $\varepsilon_{D,K}\varepsilon_{C,L} \geq \varepsilon_{D,L}\varepsilon_{C,K}$. In case (ii), Δ crosses $[BC]$ for $\sigma < \sigma_{I_3}$, but does not cross any more $[BC]$ when $\sigma > \sigma_{I_3}$. Note that, by continuity, σ_{I_3} is strictly greater than σ_{I_2}.

The value of σ_{F_2}

σ_{F_2} is defined by $1 + T_1(\sigma) + D_1(\sigma) = 0$. After some computations, we obtain:

$$
\begin{aligned}
\sigma_{F_2} = &\frac{2s(1+\varepsilon_{D,L} - \varepsilon_{C,L}) + \theta[(1-s)(2+\varepsilon_{D,L} - \varepsilon_{D,K}) + (2-s)\varepsilon_{C,K}}{2(2+\varepsilon_{D,L}) + \theta[\varepsilon_{C,K}(2+\varepsilon_{D,L}) - \varepsilon_{D,K}\varepsilon_{C,L}]} \\
&\frac{+s\varepsilon_{C,L} + \varepsilon_{D,L}\varepsilon_{C,K} - \varepsilon_{D,K}\varepsilon_{C,L}]}{2(2+\varepsilon_{D,L}) + \theta[\varepsilon_{C,K}(2+\varepsilon_{D,L}) - \varepsilon_{D,K}\varepsilon_{C,L}]}
\end{aligned}
\tag{41}
$$

The value of ε_{γ_H}

ε_{γ_H} is defined by $D = 1$. We obtain:

$$
\varepsilon_{\gamma_H} \equiv \frac{1}{D_1(\sigma)} = \frac{\sigma(1+\varepsilon_{D,L}) - s(1+\varepsilon_{D,L} - \varepsilon_{C,L})}{\sigma(1+\theta\varepsilon_{C,K}) - \theta(1-s)(1+\varepsilon_{C,K} - \varepsilon_{D,K})} \tag{42}
$$

The value of ε_{γ_F}

ε_{γ_F} is defined by $1 + T + D = 0$. After some computations, we obtain:

$$\varepsilon_{\gamma_F} = 1 + \frac{2(2+\varepsilon_{D,L}) + \theta[\varepsilon_{C,K}(2+\varepsilon_{D,L}) - \varepsilon_{D,K}\varepsilon_{C,L}]}{2+\theta\varepsilon_{C,K}} \frac{\sigma_{F_2} - \sigma}{\sigma - \sigma_{F_1}} \tag{43}$$

The value of ε_{γ_T}

ε_{γ_T} is defined by $1 - T + D = 0$. After some computations, we obtain:

$$\varepsilon_{\gamma_T} = 1 + \frac{(\varepsilon_{D,L}\varepsilon_{C,K} - \varepsilon_{D,K}\varepsilon_{C,L})(\sigma - \sigma_{I_2})}{\varepsilon_{C,K}(\sigma - \sigma_T)} \tag{44}$$

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