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ABSTRACT.   In this paper, we establish theorems of the alternative for a system described by inequalities, equalities and an inclusion, which are generalizations of Tucker's classical theorem of the alternative, and develop Kuhn-Tucker necessary conditions for efficiency to mathematical programs in normed spaces involving inequality, equality and set constraints with positive Lagrange multipliers of all the components of objective functions.

1.INTRODUCTION

Theorems of the alternative play an important role in establishing necessary optimality conditions to scalar and multiobjective optimization problems. Under suitable constraint qualifications we could obtain Kuhn-Tucker necessary conditions for efficiency in terms of derivatives or directional derivatives of objective and constraint functions in some sense. Many authors have derived necessary optimality conditions under different constraint qualifications with generalized convexity (see, e.g., [1], [2], [5]-[12], [14], [16], and references therein). Note that if a Lagrange multiplier corresponding to some component of the objective function is equal to zero, then that component has no role in necessary conditions for efficiency. Hence, it is necessary to get Kuhn-Tucker necessary conditions for efficiency with Lagrange multipliers associated with all the components of objective functions to be positive.

Maeda [13] generalizes the Guignard constraints qualification for Fréchet differentiable multiobjective optimization problems consisting of only inequality constraints in finite dimensions, and establishes Kuhn-Tucker necessary

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conditions for efficiency with positive Lagrange multipliers corresponding to all the components of objective functions. A Maeda type constraint qualification in the semidifferentiable case is considered by Preda-Chitescu [15] for multiobjective mathematical programming involving inequality contraints in finite dimensions. The Kuhn-Tucker necessary conditions for efficiency obtained by Preda-Chitescu [15] are a development further to those due to Maeda [13]. Giorgi-Jiménez-Novo [4] introduce constraint qualifications, which generalize Maeda's constraint qualification [13], and prove an alternative theorem for a system comprising inequalities and equalities together with Kuhn-Tucker necessary conditions for efficiency in finite dimensions in which the positivity of Lagrange multiplies corresponding to all the components of objective functions are ensured.

The purpose of this paper is to develop Kuhn-Tucker necessary conditions for efficiency in multiobjective optimization problems involving inequality, equality and set constraints in normed spaces with positive Lagrange multipliers associated with all the components of objective functions. The remainder of the paper is organized as follows. After some preliminaries, Section 3 presents therems of the alternative for a system involving inequalities, equalities and an inclusion in a normed space, which are generalizations of Tucker's classical theorem of the alternative. In Section 4, some constraint qualifications of Abadie type are proposed and several necessary conditions for efficiency in mathematical programming with inequality, equality and set constraints are established in terms of Dini and Hadamard directional derivatives. Section 5 gives Kuhn-Tucker necessary conditions for efficiency in the problem mentioned above with positive Lagrange multipliers corresponding to all the components of the objective.

2. Preliminaries

Let X be a normed space, and let C be a nonempty subset of X. Let f, g and h be mappings from X into $\mathbb{R}^p$, $\mathbb{R}^q$ and $\mathbb{R}^r$, respectively. Note that f, g, h can be naturally expressed as follows: $f = (f_1, \ldots, f_p)$, $g = (g_1, \ldots, g_q)$, $h = (h_1, \ldots, h_r)$, where $f_k, g_j, h_\ell : X \to \mathbb{R}$ $(k = 1, \ldots, p; j = 1, \ldots, q; \ell = 1, \ldots, r)$. In this paper, we shall be concerned with the following

multiobjective programming problem:

$$\min f(x),$$

$$\text{subject to}$$

(VP)
$$g_j(x) \leqslant 0, \quad j = 1, \ldots, q;$$
$$h_\ell(x) = 0, \quad \ell = 1, \ldots, r;$$
$$x \in C.$$

Denote by M the feasible set of (VP)

$$M = \Big\{ x \in C : g_j(x) \leqslant 0, h_\ell(x) = 0, j = 1, \ldots, q; \ell = 1, \ldots, r \Big\}.$$

Recall that a point $\overline{x} \in M$ is said to be a local efficient solution to Problem (VP) if there exists a number $\delta > 0$ such that for all $x \in M \cap B(\overline{x}; \delta)$,

$$f(x) - f(\overline{x}) \notin -\mathbb{R}_+^p \setminus \{0\},$$

where R_+^p is the nonnegative orthant of $\mathbb{R}^p$, $B(\overline{x}; \delta)$ denotes the open ball of radius δ around $\overline{x}$. This means that $\overline{x} \in M$ is a local efficient solution to (VP) iff there exists $\delta > 0$ such that there is no $x \in M \cap B(\overline{x}; \delta)$ satisfying

$$f_k(x) \leqslant f_k(\overline{x}), \quad k = 1, \ldots, p,$$
$$f_i(x) < f_i(\overline{x}) \quad \text{for some } i \in \{1, \ldots, p\}.$$

Recall that the tangent cone (or contingent cone) to C at $\overline{x} \in C$ is the following set:

$$T(C; \overline{x}) = \Big\{ v \in X : \exists\, v_n \to v, \exists\, t_n \downarrow 0 \text{ such that } \overline{x} + t_n v_n \in C, \forall\, n \Big\}.$$

The cone of sequential linear directions (or sequential radial cone) to C at $\overline{x} \in C$ is the following set:

$$Z(C; \overline{x}) = \Big\{ v \in X : \exists\, t_n \downarrow 0 \text{ such that } \overline{x} + t_n v \in C, \forall\, n \Big\},$$

where $t_n \downarrow 0$ means that $t_n \longrightarrow 0^+$. Note that both these cones are nonempty. $T(C; \overline{x})$ is a closed cone, and it may be not convex; $Z(X; \overline{x}) \subset T(C; \overline{x})$.

Let $\widetilde{f}$ be a real-valued function defined on X. The following directional derivatives will be used in the sequel.

The lower Dini derivative of $\widetilde{f}$ at $\overline{x} \in X$ in a direction $v \in X$ is

$$\underline{D}\widetilde{f}(\overline{x}; v) = \liminf_{t \downarrow 0} \frac{\widetilde{f}(\overline{x} + tv) - \widetilde{f}(\overline{x})}{t} \; ;$$

The upper Dini derivative of $\widetilde{f}$ at $\overline{x}$ in the direction v is

$$\overline{D}\widetilde{f}(\overline{x}; v) = \limsup_{t \downarrow 0} \frac{\widetilde{f}(\overline{x} + tv) - \widetilde{f}(\overline{x})}{t} \; ;$$

The lower Hadamard derivative of $\widetilde{f}$ at $\overline{x}$ in the direction v is

$$\underline{d}\widetilde{f}(\overline{x}; v) = \liminf_{\substack{t \downarrow 0 \\ u \to v}} \frac{\widetilde{f}(\overline{x} + tu) - \widetilde{f}(\overline{x})}{t} \; ;$$

The upper Hadamard derivative of $\widetilde{f}$ at $\overline{x}$ in the direction v is

$$\overline{d}\widetilde{f}(\overline{x}; v) = \limsup_{\substack{t \downarrow 0 \\ u \to v}} \frac{\widetilde{f}(\overline{x} + tu) - \widetilde{f}(\overline{x})}{t} \; .$$

If $\underline{D}\widetilde{f}(\overline{x}; v) = \overline{D}\widetilde{f}(\overline{x}; v)$, we shall denote their common value by $D\widetilde{f}(\overline{x}; v)$. It is usual directional derivative of $\widetilde{f}$ at $\overline{x}$ in the direction v. In case $D\widetilde{f}(\overline{x}; .)$ is a continuous linear mapping, $\widetilde{f}$ is said to be Gâteaux differentiable at $\overline{x}$, and

$$D\widetilde{f}(\overline{x}; v) = \langle \nabla_G \widetilde{f}(\overline{x}), v \rangle,$$

where $\nabla_G \widetilde{f}(\overline{x})$ denotes the Gâteaux derivative of $\widetilde{f}$ at $\overline{x}$, and $\langle \nabla_G \widetilde{f}(\overline{x}), v \rangle$ is the value of the linear functional $\nabla_G \widetilde{f}(\overline{x})$ at the point v. Thus if $\widetilde{f}$ is Fréchet differentiable at $\overline{x}$ with Fréchet derivative $\nabla f(\overline{x})$, then

$$D\widetilde{f}(\overline{x}; v) = \langle \nabla \widetilde{f}(\overline{x}), v \rangle.$$

Similarly, if $\underline{d}\widetilde{f}(\overline{x}; v) = \overline{d}\widetilde{f}(\overline{x}; v)$, we also denote their common value by $d\widetilde{f}(\overline{x}; v)$. This is the Hadamard derivative of $\widetilde{f}$ at $\overline{x}$ in the direction v. Note that if $d\widetilde{f}(\overline{x}; v)$ exists, then also $D\widetilde{f}(\overline{x}; v)$ exists, and they are equal.

We set

$$I(\overline{x}) = \{j \in \{1,\ldots,q\} : g_j(\overline{x}) = 0\};$$
$$Q = \{x \in C : f_k(x) \leqslant f_k(\overline{x}), g_j(x) \leqslant 0, h_\ell(x) = 0,$$
$$k = 1,\ldots,p; j = 1,\ldots,q; \ell = 1,\ldots,r\};$$
$$Q^i = \{x \in C : f_k(x) \leqslant f_k(\overline{x}), g_j(x) \leqslant 0, h_\ell(x) = 0,$$
$$k = 1,\ldots,p, k \neq i; j = 1,\ldots,q; \ell = 1,\ldots,r\} \quad (i \in \{1,\ldots,p\}).$$

If for each $v \in Z(C;\overline{x})$, $Dh_\ell(\overline{x};v)$ $(\ell = 1,\ldots,r)$ exist, we put

$$C_D(Q;\overline{x}) = \Big\{ v \in Z(C;\overline{x}) : \underline{D}f_k(\overline{x};v) \leqslant 0, \ k = 1,\ldots,p,$$
$$\underline{D}g_j(\overline{x};v) \leqslant 0, \ j \in I(\overline{x}),$$
$$Dh_\ell(\overline{x};v) = 0, \ \ell = 1,\ldots,r \Big\}.$$

If for each $v \in T(C;\overline{x})$, $dh_\ell(\overline{x};v)$ $(\ell = 1,\ldots,r)$ exists, we put

$$C_d(Q;\overline{x}) = \Big\{ v \in T(C;\overline{x}) : \underline{d}f_k(\overline{x};v) \leqslant 0, \ k = 1,\ldots,p,$$
$$\underline{d}g_j(\overline{x};v) \leqslant 0, \ j \in I(\overline{x}),$$
$$dh_\ell(\overline{x};v) = 0, \ \ell = 1,\ldots,r \Big\}.$$

In view of the positive homogeneity of lower Dini and Hadamard directional derivatives, $C_D(Q;\overline{x})$ and $C_d(Q;\overline{x})$ are cones with vertices at the origin.

Let K be a cone in X with vertex at the origin. Denote by K^* the dual cone of K

$$K^* = \{\xi \in X^* : \langle \xi, v \rangle \geqslant 0, \forall\, v \in K\},$$

where X^* is the topological dual of X. Note that K^* is a weakly* closed convex cone.

In what follows we recall three results in [3] (Lemma 5.8, Lemma 5.11 and Theorem 10.4), which will be used in the proofs of Theorems 3.1 and 3.2 below.

Proposition 2.1 [3]. *Let K_α $(\alpha \in I)$ be weakly closed convex cones in X. Assume that $\sum\limits_{\alpha \in I} K_\alpha^*$ is weakly closed. Then,*

$$\Big(\bigcap_{\alpha \in I} K_\alpha \Big)^* = \sum_{\alpha \in I} K_\alpha^*.$$

Proposition 2.2 [3] (Dubovitskii-Milyutin Theorem). *Assume that* $K_1, \ldots,$ K_n, K_{n+1} *are convex cones with vertices at the origin in* X, *and* $K_1, \ldots, K_n$ *are open. Then,* $\bigcap\limits_{i=1}^{n+1} K_i = \emptyset$ *if and only if there exist* $\xi_i \in K_i^*$ $(i = 1, \ldots, n+1)$, *not all zero, such that*

$$\xi_1 + \cdots + \xi_n + \xi_{n+1} = 0.$$

Proposition 2.3 [3] (Farkas-Minkowski Theorem). *Let*

$$K = \{x \in \mathbb{R}^m : \langle a_i, x \rangle \geqslant 0, a_i \in \mathbb{R}^m, i = 1, \ldots, n\}.$$

Then,

$$K^* = \Big\{ \sum_{i=1}^n a_i y_i : y_i \geqslant 0, \ i = 1, \ldots, n \Big\}.$$

3. Theorems of the alternative

In order to derive Kuhn-Tucker necessary conditions for efficiency, in this section we investigate theorems of the alternative for a system consisting of inequalities, equalities and an inclusion.

Let X be a normed space with the topological dual X^*. Let a_k, b_j, c_ℓ be vectors in X^* $(k = 1, \ldots, p; j = 1, \ldots, q; \ell = 1, \ldots, r)$, and let C be a nonempty subset of X. For $i \in \{1, \ldots, p\}$, we set

$$\begin{aligned} &A_k = \{v \in X : \langle a_k, v \rangle \leqslant 0\} \quad (k = 1, \ldots, p; k \neq i),\\ &\widetilde{A}_i = \{v \in X : \langle a_i, v \rangle < 0\},\\ &B_j = \{v \in X : \langle b_j, v \rangle \leqslant 0\} \quad (j = 1, \ldots, s),\\ &C_\ell = \{v \in X : \langle c_\ell, v \rangle = 0\} \quad (\ell = 1, \ldots, r). \end{aligned}$$

Note that A_k and B_j $(k = 1, \ldots, p, k \neq i; j = 1, \ldots, s)$ are closed convex cones with vertices at the origin, $\widetilde{A}_i$ is an open convex cone with vertex at the origin, and C_ℓ $(\ell = 1, \ldots, r)$ are closed linear subspaces of X.

Theorem 3.1. *Assume that*

(a) K *is an arbitrary nonempty convex subcone of* $T(C; \overline{x})$ *with vertex at the origin, and* K *is closed;*

(b) *For each $i \in \{1, \ldots, p\}$, the set*

$$\sum_{\substack{k=1 \\ k \neq i}}^{p} A_k^* + \sum_{j=1}^{s} B_j^* + \sum_{\ell=1}^{r} C_\ell^* + K^*$$

is weakly* closed in X^*.

Then exactly one of the following two conclusions holds:
  (i) *For each $i \in \{1, \ldots, p\}$, the system*

$$\langle a_k, v \rangle \leqslant 0, \quad k = 1, \ldots, p; k \neq i \tag{1}$$
$$\langle a_i, v \rangle < 0, \tag{2}$$
$$\langle b_j, v \rangle \leqslant 0, \quad j = 1, \ldots, s, \tag{3}$$
$$\langle c_\ell, v \rangle = 0, \quad \ell = 1, \ldots, r, \tag{4}$$
$$v \in K, \tag{5}$$

has no solution $v \in X$.
  (ii) *There exist $\overline{\lambda}_k > 0$ $(k = 1, \ldots, p)$, $\overline{\mu}_j \geqslant 0$ $(j = 1, \ldots, s)$ and $\overline{\nu}_\ell \in \mathbb{R}$ $(\ell = 1, \ldots, r)$ such that*

$$\sum_{k=1}^{p} \overline{\lambda}_k \langle a_k, v \rangle + \sum_{j=1}^{s} \overline{\mu}_j \langle b_j, v \rangle + \sum_{\ell=1}^{r} \overline{\nu}_\ell \langle c_\ell, v \rangle \geqslant 0 \quad (\forall \, v \in K). \tag{6}$$

Remark 3.1. If assumption (a) is replaced by that K is a convex nonempty subcone of $Z(C; \overline{x})$ and K is closed, then Theorem 3.1 is still valid, since $Z(C; \overline{x}) \subset T(C; \overline{x})$.

Proof of Theorem 3.1.
  (i) $\Rightarrow$ (ii): We need only consider the case of all $a_k \neq 0$ $(k = 1, \ldots, p)$, since in case there exists $a_{k_0} = 0$, we shall take $\overline{\lambda}_{k_0} = 1$. For each $i \in \{1, \ldots, p\}$, suppose that the system (1)-(5) has no solution $v \in X$. Putting

$$D_i = \left(\bigcap_{\substack{k=1 \\ k \neq i}}^{p} A_k \right) \cap \left(\bigcap_{j=1}^{s} B_j \right) \cap \left(\bigcap_{\ell=1}^{r} C_\ell \right) \cap K,$$

we can see that D_i is a nonempty closed convex cone in X with vertex at the origin, and

$$\widetilde{A}_i \cap D_i = \emptyset.$$

Note that $\widetilde{A}_i$ is a nonempty convex cone with vertex at the origin, as $a_i \neq 0$.

We invoke Proposition 2.2 to deduce that there exist $\xi_i \in \widetilde{A}_i^*$ and $\eta_i \in D_i^*$, not all zero, such that

$$\xi_i + \eta_i = 0. \tag{7}$$

If follows readily from (7) that $\xi_i \neq 0$ (also $\eta_i \neq 0$). Since the convex cones A_k, B_j, C_ℓ ($k = 1, \ldots, p, k \neq i; j = 1, \ldots, s; \ell = 1, \ldots, r$) and K are closed, they are weakly closed. Thus all the hypotheses of Proposition 2.1 are fulfilled. Taking account of Proposition 2.1, we get

$$D_i^* = \sum_{\substack{k=1 \\ k \neq i}}^{p} A_k^* + \sum_{j=1}^{s} B_j^* + \sum_{\ell=1}^{r} C_\ell^* + K^*. \tag{8}$$

On the other hand, in view of Theorem 10.2 in [3] on dual cones, we have

$$A_k^* = \{\lambda a_k : \lambda \leqslant 0\}, \quad k = 1, \ldots, p; k \neq i;$$
$$\widetilde{A}_i^* = \{\lambda a_i : \lambda \leqslant 0\} \quad (\text{as } a_i \neq 0);$$
$$B_j^* = \{\mu b_j : \mu \leqslant 0\}, \quad j = 1, \ldots, s;$$
$$C_\ell^* = \{\nu c_\ell : \nu \in \mathbb{R}\}, \quad \ell = 1, \ldots, r.$$

Since $\xi_i \in \widetilde{A}_i^*$, $\xi_i \neq 0$, it follows that $\xi_i = \lambda_i a_i$ with $\lambda_i < 0$. By virtue of (8), there exists $\lambda_{ik} \leqslant 0$ ($k = 1, \ldots, p; k \neq i$), $\mu_{ij} \leq 0$ ($j = 1, \ldots, s$), $\nu_{i\ell} \in \mathbb{R}$ ($\ell = 1, \ldots, r$) and $\zeta_i \in K^*$ such that

$$\eta_i = \sum_{\substack{k=1 \\ k \neq i}}^{p} \lambda_{ik} a_k + \sum_{j=1}^{s} \mu_{ij} b_j + \sum_{\ell=1}^{r} \nu_{i\ell} c_\ell + \zeta_i.$$

Setting $\overline{\lambda}_{ik} = -\lambda_{ik}$ ($k = 1, \ldots, p; k \neq i$), $\overline{\lambda}_{ii} = -\lambda_i$, $\overline{\mu}_{ij} = -\mu_{ij}$ ($j = 1, \ldots, s$), $\overline{\nu}_{i\ell} = -\nu_{i\ell}$ ($\ell = 1, \ldots, r$), one gets that $\overline{\lambda}_{ik} \geqslant 0$ ($k = 1, \ldots, p; k \neq i$), $\overline{\lambda}_{i,i} > 0$, $\overline{\mu}_{ij} \geqslant 0$ ($j = 1, \ldots, s$), and $\overline{\nu}_{i\ell} \in \mathbb{R}$ ($\ell = 1, \ldots, r$). It follows from (7) that

$$\sum_{k=1}^{p} \overline{\lambda}_{ik} a_k + \sum_{j=1}^{s} \overline{\mu}_{ij} b_j + \sum_{\ell=1}^{r} \overline{\nu}_{i\ell} c_j = \zeta_i \in K^*,$$

which implies that

$$\sum_{k=1}^{p} \overline{\lambda}_{ik} \langle a_k, v \rangle + \sum_{j=1}^{s} \overline{\mu}_{ij} \langle b_j, v \rangle + \sum_{\ell=1}^{r} \overline{\nu}_{i\ell} \langle c_j, v \rangle \geqslant 0 \quad (\forall v \in K). \tag{9i}$$

Note that for each $i \in \{1, \ldots, p\}$, we obtain the inequality (9i). Adding up both sides of (9i), $i = 1, \ldots, p$, and putting $\overline{\lambda}_k = \sum\limits_{i=1}^{p} \overline{\lambda}_{ik}$, $\overline{\mu}_j = \sum\limits_{i=1}^{p} \overline{\mu}_{ij}$ and $\overline{\nu}_\ell = \sum\limits_{i=1}^{p} \nu_{i\ell}$, we obtain that $\overline{\lambda}_k > 0$, $\overline{\mu}_j \geqslant 0$, $\overline{\nu}_\ell \in \mathbb{R}$ ($k = 1, \ldots, p; j = 1, \ldots, s; \ell = 1, \ldots, r$), and

$$\sum_{k=1}^{p} \overline{\lambda}_k \langle a_k, v \rangle + \sum_{j=1}^{s} \overline{\mu}_j \langle b_j, v \rangle + \sum_{\ell=1}^{r} \overline{\nu}_\ell \langle c_\ell, v \rangle \geqslant 0 \quad (\forall\, v \in K).$$

(ii) $\Rightarrow$ (i): Suppose that there exist $\overline{\lambda}_k > 0$, $\overline{\mu}_j \geqslant 0$ and $\overline{\nu}_\ell \in \mathbb{R}$ ($k = 1, \ldots, p; j = 1, \ldots, s; \ell = 1, \ldots, r$) satisfying (6). If (i) were false, there would exist $i \in \{1, \ldots, p\}$ such that the system (1)-(5) has a solution $v_0 \in X$. It follows from this that

$$\sum_{k=1}^{p} \overline{\lambda}_k \langle a_k, v_0 \rangle + \sum_{j=1}^{s} \overline{\mu}_j \langle b_j, v_0 \rangle + \sum_{\ell=1}^{r} \overline{\nu}_\ell \langle c_\ell, v_0 \rangle < 0,$$

which contradicts (6). The proof is complete. $\qquad\square$

Corollary 3.1. *Suppose that C is convex, and for each $i \in \{1, \ldots, p\}$ the set*

$$\sum_{\substack{k=1 \\ k \neq i}}^{p} A_k^* + \sum_{j=1}^{s} B_j^* + \sum_{\ell=1}^{r} C_\ell^* + T(X; \overline{x})^*$$

is weakly* closed in X^*. Then exactly one of the following two conclusions holds:

(i') For each $i \in \{1, \ldots, p\}$, the system (1)-(5), in which K is replaced by $T(C; \overline{x})$, has no solution $v \in X$.

(ii') There exist $\overline{\lambda}_k > 0$, $\overline{\mu}_j \geqslant 0$, $\overline{\nu}_\ell \in \mathbb{R}$ ($k = 1, \ldots, p; j = 1, \ldots, s; \ell = 1, \ldots, r$) such that

$$\sum_{k=1}^{p} \overline{\lambda}_k \langle a_k, v \rangle + \sum_{j=1}^{s} \overline{\mu}_j \langle b_j, v \rangle + \sum_{\ell=1}^{r} \overline{\nu}_\ell \langle c_\ell, v \rangle \geqslant 0 \quad (\forall\, v \in T(C; \overline{x})).$$

Proof. Since C is nonempty convex, $T(C; \overline{x})$ is a nonempty closed convex cone. Applying Theorem 3.1 to $K = T(C; \overline{x})$, we obtain the desired assertion of Corollary 3.1. $\qquad\square$

For each $i \in \{1, \ldots, p\}$, we set

$$E_i = \Big(\bigcap_{\substack{k=1 \\ k \neq i}}^{p} A_k \Big) \cap \Big(\bigcap_{j=1}^{s} B_j \Big) \cap \Big(\bigcap_{\ell=1}^{r} C_\ell \Big).$$

It is obvious that E_i is a nonempty closed cone with vertex at the origin.

In case $\dim X < +\infty$, with the help of the Farkas-Minkowski theorem, condition (b) in Theorem 3.1 will be replaced by a weakened condition as in the following theorem.

Theorem 3.2. *Let $\dim X < +\infty$, and let K be a nonempty convex subcone of $T(C; \overline{x})$ with vertex at the origin, and K closed. Assume that for each $i \in \{1, \ldots, p\}$, the set $E_i^* + K^*$ is closed. Then exactly one of the following two conclusions holds:*

(!) *For each $i \in \{1, \ldots, p\}$, the system (1)-(5) has no solution $v \in X$.*

(!!) *There exists $\overline{\lambda}_k > 0$, $\overline{\mu}_j \geqslant 0$ and $\overline{\nu}_\ell \in \mathbb{R}$ $(k = 1, \ldots, p; j = 1, \ldots, s; \ell = 1, \ldots, r)$ such that (6) holds.*

Proof. Since $\dim X < +\infty$, it holds that $\dim X^* = \dim X$, and so all the topologies on X^* conside. Making use of Proposition 2.3, we deduce that for each $i \in \{1, \ldots, p\}$,

$$E_i^* = \Big\{ \sum_{\substack{k=1 \\ k \neq i}}^{p} \lambda_{ik} a_k + \sum_{j=1}^{s} \mu_{ij} b_j + \sum_{\ell=1}^{r} \nu_{i\ell} c_\ell : \lambda_{ik} \leqslant 0, \mu_{ij} \leqslant 0, \nu_{i\ell} \in \mathbb{R},$$

$$k = 1, \ldots, p, k \neq i; j = 1, \ldots, s; \ell = 1, \ldots, r \Big\},$$

which leads to the following

$$E_i^* = \sum_{\substack{k=1 \\ k \neq i}}^{p} A_k^* + \sum_{j=1}^{s} B_j^* + \sum_{\ell=1}^{r} C_\ell^* \quad (i = 1, \ldots, p).$$

So, by assumption, the set

$$\sum_{\substack{k=1 \\ k \neq i}}^{p} A_k^* + \sum_{j=1}^{s} B_j^* + \sum_{\ell=1}^{r} C_\ell^* + K^*$$

is closed. Thus all the hypotheses of Theorem 3.1 are fulfilled, and hence from Theorem 3.1 the conclusion follows. □

In case $\dim X < +\infty$ and $C = X$, from Theorem 3.2 we can obtain Tucker's classical theorem of the alternative (see, e.g., [14]) as a special case.

Corollary 3.2. *Let* $\dim X < +\infty$. *Then exactly one of the following two assertions holds:*

(!') *For each* $i \in \{1, \ldots, p\}$, *the system* (1)-(4) *has no solution* $v \in X$.

(!!') *There exists* $\overline{\lambda}_k > 0$, $\overline{\mu}_j \geqslant 0$, $\overline{\nu}_\ell \in \mathbb{R}$ $(k = 1, \ldots, p; j = 1, \ldots, s; \ell = 1, \ldots, r)$ *such that*

$$\sum_{k=1}^{p} \overline{\lambda}_k a_k + \sum_{j=1}^{s} \overline{\mu}_j b_j + \sum_{\ell=1}^{r} \overline{\nu}_\ell c_\ell = 0. \tag{10}$$

Proof. For $C = X$, it results that $T(C; \overline{x}) = X$, and hence $T(C; \overline{x})^* = \{0\}$. Moreover, since $\dim X < +\infty$, for each $i \in \{1, \ldots, p\}$, E_i^* is a nonempty closed convex cone in X^*, and $0 \in E_i^*$. Hence, $E_i^* + T(C; \overline{x})^* = E_i^*$, and so $E_i^* + T(C; \overline{x})^*$ is closed in X^*. We now apply Theorem 3.2 to $C = X$ and deduce that (!') is equivalent to that there exists $\overline{\lambda}_k > 0$, $\overline{\mu}_j \geqslant 0$, $\overline{\nu}_\ell \in \mathbb{R}$ $(k = 1, \ldots, p; j = 1, \ldots, s; \ell = 1, \ldots, r)$ such that

$$\sum_{k=1}^{p} \overline{\lambda}_k \langle a_k, v \rangle + \sum_{j=1}^{s} \overline{\mu}_j \langle b_j, v \rangle + \sum_{\ell=1}^{r} \overline{\nu}_\ell \langle c_\ell, v \rangle \geqslant 0 \quad (\forall\, v \in T(X; \overline{x}) = X),$$

which is equivalent to (10). $\qquad \square$

4. Constraint qualifications and necessary conditions for efficiency

We now turn to Problem (P). Hereafter we shall introduce two constraint qualifications of Abadie type in terms of Dini and Hadamard directional derivatives and derive necessary conditions for efficiency.

Proposition 4.1. *Let* $\overline{x} \in M$.

a) *If for each* $v \in T(C; \overline{x})$, *the Hadamard directional derivatives* $dh_1(\overline{x}; v)$, $\ldots, dh_r(\overline{x}; v)$ *exist, then*

$$\bigcap_{i=1}^{p} T(Q^i; \overline{x}) \subset C_d(Q; \overline{x}). \tag{11}$$

b) *If for each* $v \in Z(C; \overline{x})$, *the Dini directional derivatives* $Dh_1(\overline{x}; v), \ldots,$ $Dh_r(\overline{x}; v)$ *exists, then*

$$\bigcap_{i=1}^{p} Z(Q^i; \overline{x}) \subset C_D(Q; \overline{x}). \tag{12}$$

Proof. We shall only prove (11), while (12) is analogously treated. We begin with showing that for $i = 1, \ldots, p$,

$$T(Q^i; \overline{x}) \subset C_d(Q^i; \overline{x}), \tag{13}$$

where

$$
\begin{aligned}
C_d(Q^i, \overline{x}) = \Big\{ v \in T(C; \overline{x}) : & \underline{d}f_k(\overline{x}; v) \leqslant 0, \ k = 1, \ldots, p; k \neq i, \\
& \underline{d}g_j(\overline{x}; v) \leqslant 0, \ j \in I(\overline{x}), \\
& dh_\ell(\overline{x}; v) = 0, \ \ell = 1, \ldots, r \Big\}.
\end{aligned}
$$

For $i \in \{1, \ldots, p\}$, taking $v \in T(Q^i; \overline{x})$, there exist $t_n \downarrow 0$ and $v_n \to v$ such that $\overline{x} + t_n v_n \in Q^i \ (\forall n)$. Then $\overline{x} + t_n v_n \in C \ (\forall n)$, and so $v \in T(C; \overline{x})$.

Moreover, for $i \in \{1, \ldots, p\}$, since $\overline{x} + t_n v_n \in Q^i$, it holds that

$$
\begin{aligned}
f_k(\overline{x} + t_n v_n) &\leqslant f_k(\overline{x}), \quad k = 1, \ldots, p; k \neq i; \\
g_j(\overline{x} + t_n v_n) &\leqslant 0 = g_j(\overline{x}), \quad j \in I(\overline{x}); \\
h_\ell(\overline{x} + t_n v_n) &= 0 = h_\ell(\overline{x}), \quad \ell = 1, \ldots, r.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\underline{d}f_k(\overline{x}; v) &\leqslant \liminf_{n \to \infty} \frac{f_k(\overline{x} + t_n v_n) - f_k(\overline{x})}{t_n} \leqslant 0, \quad k = 1, \ldots, p; k \neq i; \\
\underline{d}g_j(\overline{x}; v) &\leqslant \liminf_{n \to \infty} \frac{g_j(\overline{x} + t_n v_n) - g_j(\overline{x})}{t_n} \leqslant 0, \quad j \in I(\overline{x}); \\
dh_\ell(\overline{x}; v) &= \lim_{n \to \infty} \frac{h_\ell(\overline{x} + t_n v_n) - h_\ell(\overline{x})}{t_n} = 0, \quad \ell = 1, \ldots, r.
\end{aligned}
$$

Hence, $v \in C_d(Q^i; \overline{x})$. Thus we already arrive at (13). It follows from (13) that

$$\bigcap_{i=1}^p T(Q^i; \overline{x}) \subset \bigcap_{i=1}^p C_d(Q^i; \overline{x}) = C_d(Q; \overline{x}),$$

as was to be shown. $\qquad\square$

Note that the converse inclusions of (11) and (12) do not in general hold. Hence, in order to derive necessary conditions for efficiency in Problem (VP),

it is reasonable to introduce the following constraint qualifications of Abadie type at $\overline{x}$:

$$C_d(Q; \overline{x}) \subset \bigcap_{i=1}^{p} T(Q^i; \overline{x}), \tag{14}$$

$$C_D(Q; \overline{x}) \subset \bigcap_{i=1}^{p} Z(Q^i; \overline{x}). \tag{15}$$

They are generalizations of the generalized Abadie constraint qualifications in [4], [13], [15].

If for each $v \in T(C; \overline{x})$, the Hadamard directional derivatives $df_k(\overline{x}; v)$ and $dh_\ell(\overline{x}; v)$ $(k = 1, \ldots, p; \ell = 1, \ldots, r)$ exist, for $i \in \{1, \ldots, p\}$, we set

$$L_d^i(f; \overline{x}) = \{v \in T(C; \overline{x}) : df_i(\overline{x}; v) < 0, df_k(\overline{x}; v) \leqslant 0, k = 1, \ldots, p; k \neq i\},$$
$$L_d(M; \overline{x}) = \{v \in T(C; \overline{x}) : \underline{d}g_j(\overline{x}; v) \leqslant 0, j \in I(\overline{x}),$$
$$dh_\ell(\overline{x}; v) = 0, \ell = 1, \ldots, r\},$$

where M denotes the feasible set of Problem (VP).

A necessary condition for efficiency can be stated as follows.

Theorem 4.1. *Let $\overline{x}$ be a local efficient solution to Problem (VP). Assume that the function g_j $(j \notin I(\overline{x}))$ are continuous at $\overline{x}$, and for each $v \in T(C; \overline{x})$, the Hadamard directional derivatives $df_k(\overline{x}; v)$ and $dh_\ell(\overline{x}, v)$ $(k = 1, \ldots, p; \ell = 1, \ldots, r)$ exist. Suppose, in addition, that the constraint qualification (14) holds at $\overline{x}$. Then, for each $i \in \{1, \ldots, p\}$,*

$$L_d^i(f; \overline{x}) \cap L_d(M; \overline{x}) = \emptyset. \tag{16}$$

Proof. Assume the contrary, that there exists $i_0 \in \{1, \ldots, p\}$ such that

$$L_d^{i_0}(f; \overline{x}) \cap L_d(M; \overline{x}) \neq \emptyset,$$

which implies that there exists $v_0 \in L_d^{i_0}(f; \overline{x}) \cap L_d(M; \overline{x})$. Since $v_0 \in L_d^{i_0}(f; \overline{x})$, it holds that

$$df_{i_0}(\overline{x}; v_0) < 0, \tag{17}$$
$$df_k(\overline{x}; v_0) \leqslant 0, \quad k = 1, \ldots, p; k \neq i_0.$$

It is obvious that $v_0 \in C_d(Q; \overline{x})$. Making use of the constraint qualification (14), we get that $v_0 \in \bigcap_{i=1}^{p} T(Q^i; \overline{x})$, and so $v_0 \in T(Q^{i_0}; \overline{x})$. Therefore, there exist sequence $t_n \downarrow 0$ and $v_n \to v_0$ such that $\overline{x} + t_n v_n \in Q^{i_0}$ ($\forall n$). Hence, $\overline{x} + t_n v_n \in C$, and

$$
\begin{aligned}
f_k(\overline{x} + t_n v_n) &\leqslant f_k(\overline{x}), \quad k = 1, \ldots, p; k \neq i_0; \\
g_j(\overline{x} + t_n v_n) &\leqslant 0, \quad j \in I(\overline{x}); \\
h_\ell(\overline{x} + t_n v_n) &= 0, \quad \ell = 1, \ldots, r.
\end{aligned}
$$

Moreover, for $j \notin I(\overline{x})$, one has $g_j(\overline{x}) < 0$. In view of the continuity of g_j ($j \notin I(\overline{x})$), there exists a natural number N_1 such that for all $n \geqslant N_1$, $g_j(\overline{x} + t_n v_n) \leqslant 0$ ($j \notin I(\overline{x})$).

On the other hand, since $\overline{x}$ is a local efficiency solution of Problem (VP), there exists a number $\delta > 0$ such that there is no $x \in M \cap B(\overline{x}; \delta)$ satisfying

$$
\begin{aligned}
f_k(x) &\leq f_k(\overline{x}), \quad k = 1, \ldots, p, \\
f_i(x) &< f_i(\overline{x}) \quad \text{for some } i \in \{1, \ldots, p\}.
\end{aligned}
$$

From the proofs above, it follows that there is a natural number N ($\geqslant N_1$) such that for all $n \geqslant N$, $\overline{x} + t_n v_n \in M \cap B(\overline{x}; \delta)$. Consequently, for all $n \geqslant N$,

$$
f_{i_0}(\overline{x} + t_n v_n) > f_{i_0}(\overline{x}),
$$

which leads to the following

$$
df_{i_0}(\overline{x}; v_0) \geqslant 0.
$$

This conflicts with (17). Hence, for each $i \in \{1, \ldots, p\}$, (16) holds. $\qquad \square$

Remark 4.1. Theorem 4.1 is a generalization of Theorem 3.1 in [13].

If for each $v \in Z(C; \overline{x})$, the Dini directional derivatives $Df_k(\overline{x}; v)$ and $Dh_\ell(\overline{x}; v)$ ($k = 1, \ldots, p; \ell = 1, \ldots, r$) exist, for $i \in \{1, \ldots, p\}$, we set

$$
\begin{aligned}
L_D^i(f; \overline{x}) &= \{v \in Z(C; \overline{x}) : Df_i(\overline{x}; v) < 0, Df_k(\overline{x}; v) \leqslant 0, k = 1, \ldots, p; k \neq i\}, \\
L_D(M; \overline{x}) &= \{v \in Z(C; \overline{x}) : \underline{D}g_j(\overline{x}; v) \leqslant 0, j \in I(\overline{x}), \\
&\qquad\qquad Dh_\ell(\overline{x}; v) = 0, \ell = 1, \ldots, r\}.
\end{aligned}
$$

By an argument analogous to that used for the proof of Theorem 4.1, we obtain the following

Theorem 4.2. *Let $\overline{x}$ be a local efficient solution of Problem* (VP). *Assume that the functions g_j $(j \notin I(\overline{x}))$ are continuous at $\overline{x}$, and for each $v \in Z(C; \overline{x})$, the Dini directional derivatives $Df_k(\overline{x}; v)$ and $Dh_\ell(\overline{x}; v)$ $(k = 1, \ldots, p; \ell = 1, \ldots, r)$ exist. Suppose. furthermore, that the constraint qualification* (15) *holds at $\overline{x}$. Then, for each $i \in \{1, \ldots, p\}$,*

$$L_D^i(f; \overline{x}) \cap L_D(M; \overline{x}) = \emptyset.$$

5. Kuhn-Tucker necessary conditions for efficiency

In this section, turning back Problem (VP), we suppose that the functions f_k, g_j and h_ℓ are Gâteaux differentiable at $\overline{x}$ with Gâteaux derivatives $\nabla_G f_k(\overline{x})$, $\nabla_G g_j(\overline{x})$ and $\nabla_G h_\ell(\overline{x})$ $(k = 1, \ldots, p; j = 1, \ldots, q; \ell = 1, \ldots, r)$, and the functions g_j $(j \notin I(\overline{x}))$ are continuous. Then, for each $v \in X$

$$df_k(\overline{x}; v) = Df_k(\overline{x}; v) = \langle \nabla_G f_k(\overline{x}), v \rangle \quad (k = 1, \ldots, p),$$
$$dg_j(\overline{x}; v) = Dg_j(\overline{x}; v) = \langle \nabla_G g_j(\overline{x}), v \rangle \quad (j = 1, \ldots, q),$$
$$dh_\ell(\overline{x}; v) = Dh_\ell(\overline{x}; v) = \langle \nabla_G h_\ell(\overline{x}), v \rangle \quad (\ell = 1, \ldots, r),$$

and
$$C_d(Q; \overline{x}) = \Big\{ v \in T(C; \overline{x}) : \langle \nabla_G f_k(\overline{x}), v \rangle \leqslant 0, \ k = 1, \ldots, p,$$
$$\langle \nabla_G g_j(\overline{x}), v \rangle \leqslant 0, \ j \in I(\overline{x}),$$
$$\langle \nabla_G h_\ell(\overline{x}), v \rangle = 0, \ \ell = 1, \ldots, r \Big\}.$$

Note that if the functions g_j $(j \notin I(\overline{x}))$ only are Gâteaux differentiable at $\overline{x}$, then they are not necessarily continuous at $\overline{x}$. Using the notations A_k, $\widetilde{A}_i$, B_j, C_ℓ as in Section 3, and taking $a_k = \nabla_G f_k(\overline{x})$, $b_j = \nabla_G g_j(\overline{x})$, $c_\ell = \nabla_G h_\ell(\overline{x})$ $(k = 1, \ldots, p; j \in I(\overline{x}); \ell = 1, \ldots, r)$, we obtain the following Kuhn-Tucker necessary conditions for efficiency to (VP) in which Lagrange multipliers associated with all the components of the objective are positive.

Theorem 5.1. *Let $\overline{x}$ be a local efficient slution to Problem* (VP), *and let K be an arbitrary nonempty convex subcone of $T(C; \overline{x})$ with vertex at the origin and K closed. Assume that for each $i \in \{1, \ldots, p\}$, the set*

$$\sum_{\substack{k=1 \\ k \neq i}}^{p} A_k^* + \sum_{j \in I(\overline{x})} B_j^* + \sum_{\ell=1}^{r} C_\ell^* + K^*$$

is weakly* closed in X^*. Suppose also that the constraint qualification (14) holds at $\overline{x}$. Then there exist $\overline{\lambda}_k > 0$, $\overline{\mu}_j \geqslant 0$ and $\overline{\nu}_\ell \in \mathbb{R}$ ($k = 1, \ldots, p; j = 1, \ldots, q; \ell = 1, \ldots, r$) such that

$$\sum_{k=1}^{p} \overline{\lambda}_k \langle \nabla_G f_k(\overline{x}), v \rangle + \sum_{j=1}^{q} \overline{\mu}_j \langle \nabla_G g_j(\overline{x}), v \rangle + \sum_{\ell=1}^{r} \overline{\nu}_\ell \langle \nabla_G h_\ell(\overline{x}), v \rangle \geqslant 0 \quad (\forall\, v \in K),$$
$$\tag{18}$$
$$\overline{\mu}_j g_j(\overline{x}) = 0 \qquad (j = 1, \ldots, q). \tag{19}$$

Proof. We invoke Theorem 4.1 to deduce that for each $i \in \{1, \ldots, p\}$, the system

$$\langle \nabla_G f_k(\overline{x}), v \rangle \leqslant 0, \quad k = 1, \ldots, p; k \neq i; \tag{20}$$
$$\langle \nabla_G f_i(\overline{x}), v \rangle < 0, \tag{21}$$
$$\langle \nabla_G g_j(\overline{x}), v \rangle \leqslant 0, \quad j \in I(\overline{x}), \tag{22}$$
$$\langle \nabla_G h_\ell(\overline{x}), v \rangle = 0, \quad \ell = 1, \ldots, r, \tag{23}$$
$$v \in K, \tag{24}$$

has no solution $v \in X$.

Applying Theorem 3.1 to $a_k = \nabla_G f_k(\overline{x})$, $b_j = \nabla_G g_j(\overline{x})$, $c_\ell = \nabla_G h_\ell(\overline{x})$ ($k = 1, \ldots, p; j \in I(\overline{x}); \ell = 1, \ldots, r$) yields the existence of $\overline{\lambda}_k > 0$, $\overline{\mu}_j \geqslant 0$ and $\overline{\nu}_\ell \in \mathbb{R}$ ($k = 1, \ldots, p; j \in I(\overline{x}); \ell = 1, \ldots, r$) satisfying

$$\sum_{k=1}^{p} \overline{\lambda}_k \langle \nabla_G f_k(\overline{x}), v \rangle + \sum_{j \in I(\overline{x})} \overline{\mu}_j \langle \nabla_G g_j(\overline{x}), v \rangle + \sum_{\ell=1}^{r} \overline{\nu}_\ell \langle \nabla_G h_\ell(\overline{x}), v \rangle \geqslant 0\ (\forall\, v \in K).$$

For $j \notin I(\overline{x})$, we take $\overline{\mu}_j = 0$ and obtain (18). Moreover, we also get (19), because for $j \in I(\overline{x})$, $g_j(\overline{x}) = 0$ and for $j \notin I(\overline{x})$, $\mu_j = 0$. $\qquad\square$

Corollary 5.1. *Let $\overline{x}$ be a local efficient solution of Problem* (VP), *and C convex. Assume that for each $i \in \{1, \ldots, p\}$, the set*

$$\sum_{\substack{k=1 \\ k \neq i}}^{p} A_k^* + \sum_{j \in I(\overline{x})} B_j^* + \sum_{\ell=1}^{r} C_\ell^* + T(C; \overline{x})^*$$

is weakly* closed in X^*. Suppose, in addition, that the constraint qualification (14) holds at $\overline{x}$. Then there exist $\overline{\lambda}_k > 0$, $\overline{\mu}_j \geqslant 0$ and $\overline{\nu}_\ell \in \mathbb{R}$ ($k = 1, \ldots, p; j =$

$1, \ldots, q; \ell = 1, \ldots, r)$ *such that* (18) *and* (19) *hold, in which K is replaced by* $T(C; \overline{x})$.

Proof. Since C is nonempty convex, $T(C; \overline{x})$ is a nonempty closed convex cone of X. Applying Theorem 5.1 to $K = T(C; \overline{x})$, we deduce the desired conclusion. $\square$·

In case X is finite dimensional, we obtain the following Kuhn-Tucker necessary conditions for efficiency.

Theorem 5.2. *Let* $\dim X < +\infty$, *and let* $\overline{x}$ *be a local efficient solution to Problem* (VP). *Let K be an arbitrary nonempty convex subcone of $T(C; \overline{x})$ with vertex at the origin, and K closed. Assume that for each $i \in \{1, \ldots, p\}$, the set $E_i^* + K^*$ is closed in X^*, where*

$$E_i = \Big(\bigcap_{\substack{k=1 \\ k \neq i}}^{p} A_k \Big) \cap \Big(\bigcap_{j \in I(\overline{x})} B_j \Big) \cap \Big(\bigcap_{\ell=1}^{r} C_\ell \Big)$$

with $a_k = \nabla_G f_k(\overline{x})$, $b_j = \nabla_G g_j(\overline{x})$, $c_\ell = \nabla_G h_\ell(\overline{x})$ ($k = 1, \ldots, p; j \in I(\overline{x}); \ell = 1, \ldots, r$). Suppose, furthemore, that the constraint qualification (14) *holds at $\overline{x}$. Then there exists $\overline{\lambda}_k > 0$, $\overline{\mu}_j \geqslant 0$ and $\overline{\nu}_\ell \in \mathbb{R}$ ($k = 1, \ldots, p; j = 1, \ldots, q; \ell = 1, \ldots, r$) such that* (18) *and* (19) *hold.*

Proof. Making use of Theorem 4.1, we deduce that for each $i \in \{1, \ldots, p\}$, the system (20)-(24) has no solution $v \in X$. The remainder of this proof is made in the same way as in the proof of Theorem 5.1 by using Theorem 3.2 instead of Theorem 3.1. $\square$

In case X is finite dimensional and $C = X$, the following result shows that the condition of $E_i^* + T(C; \overline{x})^*$ to be closed can be omitted.

Corollary 5.2. *Let* $\dim X < +\infty$, *$C = X$, and let $\overline{x}$ be an local efficient solution of Problem* (VP). *Suppose that the constraint qualification* (14) *holds at $\overline{x}$. Then there exists $\overline{\lambda}_k > 0$, $\overline{\mu}_j \geqslant 0$ and $\overline{\nu}_\ell \in \mathbb{R}$ ($k = 1, \ldots, p; j = 1, \ldots, q; \ell = 1, \ldots, r$) such that*

$$\sum_{k=1}^{p} \overline{\lambda}_k \nabla_G f_k(\overline{x}) + \sum_{j=1}^{q} \overline{\mu}_j \nabla_G g_j(\overline{x}) + \sum_{\ell=1}^{r} \overline{\nu}_\ell \nabla_G h(\overline{x}) = 0, \qquad (25)$$

$$\overline{\mu}_j g_j(\overline{x}) = 0 \quad (j = 1, \ldots, q). \qquad (26)$$

Proof. As also in the proof of Corollary 3.2 we have that for $C = X$, $T(C; \overline{x}) = X$ and $E_i^* + T(C; \overline{x})^* = E_i^*$. So $E_i^* + T(C; \overline{x})^*$ is closed in X^*. Applying

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Theorem 5.2 to $C = X$, we deduce that there exists $\overline{\lambda}_k > 0$, $\overline{\mu}_j \geqslant 0$ and $\overline{\nu}_\ell \in \mathbb{R}$ $(k = 1, \ldots, p; j = 1, \ldots, q; \ell = 1, \ldots, r)$ such that (26) holds, and

$$\sum_{k=1}^{p} \overline{\lambda}_k \langle \nabla_G f_k(\overline{x}), v \rangle + \sum_{j=1}^{q} \overline{\mu}_j \langle \nabla_G g_j(\overline{x}), v \rangle + \sum_{\ell=1}^{r} \overline{\nu}_\ell \langle \nabla_G h_\ell(\overline{x}), v \rangle \geqslant 0$$

$$(\forall\, v \in T(C; \overline{x}) = X),$$

which leads to the following

$$\sum_{k=1}^{p} \overline{\lambda}_k \nabla_G f_k(\overline{x}) + \sum_{j=1}^{q} \overline{\mu}_j \nabla_G g_j(\overline{x}) + \sum_{\ell=1}^{r} \overline{\nu}_\ell \nabla_G h_\ell(\overline{x}) = 0,$$

as was to be shown. $\qquad\square$

Remark 5.1. (a) From Corollary 5.2 we obtain Theorem 4.1 in [13] as a special case.

(b) If the cone $T(C; \overline{x})$ is replaced by the cone $Z(C; \overline{x})$, the Theorems 5.1 and 5.2 are still valid.

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