Single Crossing Lorenz Curves and Inequality Comparisons

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Abstract

Since the order generated by the Lorenz criterion is partial, it is a natural question to wonder how to extend this order. Most of the literature that is concerned with that question focuses on *local* changes in the income distribution. We follow a different approach, and define uniform $\alpha-$spreads, which are *global* changes in the income distribution. We give necessary and sufficient conditions for an Expected Utility or Rank-Dependent Expected Utility maximizer to respect the principle of transfers and to be favorable to uniform $\alpha-$spreads. Finally, we apply these results to inequality indices.

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1 Introduction

The Pigou-Dalton principle of transfers plays a central role in the normative measurement of inequality. This criterion simply says that a (rank preserving) income transfer from a richer to a poorer person reduces inequality. This principle is equivalent to the Lorenz criterion, applied to distributions with the same total income and population size: If the Lorenz curve associated to an income distribution Y is nowhere below the one associated to the distribution X, and X has the same total income and population size than Y, then Y can be obtained from X by a finite sequence of Pigou-Dalton transfers, and therefore

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Y is less unequal than X. Furthermore, the Lorenz criterion is also equivalent to second-degree stochastic dominance for distributions with equal means. Finally, a social welfare function is compatible with the principle of transfers if, and only if, this function is $S-$concave (see e.g. Atkinson (1970), Dasgupta, Sen and Starrett (1973)).

Obviously, the weak order generated by the Lorenz criterion is partial. It is therefore a natural question to wonder how to extend the set of distributions that can be ordered. Most of the literature that is concerned with that question focuses on the principle of composite transfers, i.e., on the combination of a progressive transfer and a regressive transfer. Such an approach focuses on *local* changes, since these transfers concern (at most) four individuals. We follow here a different approach, since we restrict our attention to some *global* changes in the income distribution. Although this is certainly less general, it turns out to be enough to derive some neat characterizations of social welfare functions and inequality indices.

More precisely, we introduce the notion of *uniform* $\alpha-$*spreads*. Consider an income distribution among n agents. Now, assume that agent with rank $(k+1)$ in the income distribution pays a tax that is uniformly distributed among the remaining agents (including himself), without perturbing individuals' rank in the distribution. The resulting distribution is obtained from the initial one through a uniform $\alpha-$spread, with $\alpha = \frac{k}{n}$. Obviously, these two distributions cannot be ordered with the Lorenz criterion, since a uniform spread is a combination of progressive and regressive transfers. Moreover, the Lorenz curves associated to these two distributions cross once.

It turns out that a decision maker who behaves in accordance with the Expected Utility model is favorable to uniform $\alpha-$spreads if, and only if, his utility index is linear, whatever the value of α is. On the other hand, we find some necessary and sufficient conditions for a decision maker who behaves in accordance with the Rank-Dependent Expected Utility model to respect the principle of transfers and to be favorable to uniform $\alpha-$spreads.

Finally, since normative inequality indices rely on social welfare functions, it is then possible to apply these characterizations for inequality indices (more precisely the Atkinson-Kolm-Sen indices correspond to an utilitarian social welfare functions, whereas the Gini index and its generalizations correspond to rank-dependent social welfare functions).

The organization of the paper is as follows. In section 2 we define the notion of uniform $\alpha-$spreads, and discuss some of its properties. In section 3, we give necessary and sufficient conditions for a a decision maker who behaves in accordance with the Expected Utility model or with the Rank-Dependent Expected Utility model to respect the principle of transfer and to be favorable to uniform $\alpha-$spreads. Finally, a last section is devoted to the application of the preceding results to the problem of inequality measurement.

2 Uniform Spreads

Let D be an arbitrary interval of $\mathbb{R}$, and D° be the interior of D. We denote by $\mathcal{D}_n$ the set of rank-ordered discrete income distributions of size $n \in \mathbb{N}^*$ (where $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$) with values in D. An income distribution $X \in \mathcal{D}_n$ is defined by:

$$X = \left(x_1, \frac{1}{n}; x_2, \frac{1}{n}; \ldots; x_n, \frac{1}{n}\right),$$

with $x_1 \leq x_2 \leq \ldots \leq x_n$. Therefore, X denotes the income distribution where a fraction $\frac{1}{n}$ of the total population has an income equal to x_i, for all $i \in \{1, ..., n\}$. Note that for any income distribution $Y = (y_1, p_1; y_2, p_2; ...; y_k, p_k)$, where the p_i are rational numbers and $\sum_{i=1}^{k} p_i = 1$, there exists $m \geq 2$ such that $Y = \left(y_1, \frac{1}{m}; y_2, \frac{1}{m}; ...; y_m, \frac{1}{m}\right)$. For simplicity, we let $X = (x_1, ..., x_n)$. Furthermore, we will denote $\mathbb{D} = \cup_{n \in \mathbb{N}^*} \mathcal{D}_n$.

We denote by F_X the probability distribution function associated to X, and by F_X^{-1} the inverse distribution function defined by $F_X^{-1}(p) = inf\{x : F(x) \geq p\}$. Finally, $\bar{X} = \sum_{i=1}^{n} \frac{1}{n} x_i$ denotes the mean of $X \in \mathcal{D}_n$.

Let $\succeq$ be the decision maker's preference relation over $\mathbb{D}$. We say that a decision maker behaves in accordance with the Expected Utility model (see von Neumann and Morgenstern (1947)) if there exists a continuous and strictly increasing utility function $u : D \to \mathbb{R}$, bounded[1] on D, such that $\succeq$ is represented by:

$$U(X) = \sum_{i=1}^{n} \frac{1}{n} u(x_i).$$

[1]This assumption is required in order to avoid a super St. Petersburg paradox of the Menger type. The same restriction applies for the Rank-Dependent Expected Utility model.

A decision maker behaves in accordance with Yaari's Dual model (see Yaari (1987)) if there exists a strictly increasing continuous frequency transformation function $f : [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$, such that $\succeq$ is represented by:

$$V(X) = \sum_{i=1}^{n} \left[f\left(\frac{n-i+1}{n}\right) - f\left(\frac{n-i}{n}\right) \right] x_i.$$

Finally, a decision maker behaves in accordance with Quiggin's Rank-Dependent Expected Utility model (see Quiggin (1982)) if there exist a continuous and strictly increasing utility function $u : D \to \mathbb{R}$, bounded on D, and a strictly increasing continuous frequency transformation function $f : [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$, such that $\succeq$ is represented by:

$$V(u(X)) = \sum_{i=1}^{n} \left[f\left(\frac{n-i+1}{n}\right) - f\left(\frac{n-i}{n}\right) \right] u(x_i).$$

We will denote for any $i \in \{1, ..., n\}$: $\Psi\left(\frac{i}{n}\right) = f\left(\frac{n-i+1}{n}\right) - f\left(\frac{n-i}{n}\right)$. In the sequel, we interpret U and V and $V(u(.))$ as social welfare functions. Obviously, U corresponds to an utilitarian social welfare function, whereas V corresponds to what we call a linear rank-dependent social welfare function, and $V(u(.))$ corresponds to a rank-dependent social welfare function.

Now, let us recall the well-known notion of Lorenz order.

Definition 1. *Let X, Y belong to $\mathbb{D}$. Y is less unequal than X in the sense of the Lorenz order, denoted $Y \succeq_L X$ iff:*

$$L(F_Y, \xi) = \frac{\int_0^\xi F_Y^{-1}(t)\,dt}{\bar{Y}} \geq \frac{\int_0^\xi F_X^{-1}(t)\,dt}{\bar{X}} = L(F_X, \xi), \ \forall \xi \in [0,1],$$

i.e., if the Lorenz function $L(F_X, \xi)$ of X is nowhere below the Lorenz function $L(F_Y, \xi)$ of Y.

We say that a decision maker respects the Lorenz order iff for all X, Y in $\mathbb{D}$,

$$Y \succeq_L X \Rightarrow Y \succeq X.$$

The Lorenz order (which is a partial order) plays a central role in the field of inequality measurement. Indeed, it had been proved (see Hardy, Littlewood and Pólya (1934)) that if $\bar{X} = \bar{Y}$, and X and Y have the same population size, then $Y \succeq_L X$ if and only if Y can be derived from X through a finite sequence of rank-preserving income transfers from richer to poorer individuals (Pigou-Dalton transfers). Although the Lorenz criterion is normatively very appealing, it suffers from a serious drawback, since the weak order generated by this criterion is obviously partial. It is therefore a natural question to wonder how to extend the set of distributions that can be ordered. Most of the literature that is concerned with that question focuses on the principle of composite transfers, i.e., on the combination of a progressive transfer and a regressive transfer.

More precisely, two kinds of composite transfers are considered: The composite transfers that preserve the variance and the mean of the initial distribution, and the ones that preserve the mean and the value of the Gini index of the initial distribution. The first one is associated with third-degree stochastic dominance (see, e.g., Shorrocks and Foster (1987), Foster and Shorrocks (1988), Davies and Hoy (1994)), whereas the second one is associated with inverse third-degree stochastic dominance (see, e.g., Muliere and Scarsini (1989), Moyes (1990), Chateauneuf and Wilthien (1998), Zoli (1999)). In both cases, necessary and sufficient conditions for a social welfare function to respect both the principle of transfers and the principle of composite transfer under consideration have been identified.

Both approaches focus on local spreads, i.e., spreads concerning only four (at most) individuals. Our approach is somewhat different, since we restrict our attention to global changes in the distribution. The main idea is the following. Consider a distribution X with n individuals, and assume that $x_k < x_{k+1}$. What would be the consequence of taxing the individual occupying the $(k+1)^{th}$ position in the ladder, without perturbing the ordering, and then redistributing the collected tax uniformly among the remaining agents? We will call such a change in a distribution an *uniform $\alpha-$spread*, with $\alpha = \frac{k}{n}$. More formally, we have the following definition.

Definition 2. *Let X, Y belong to $\mathcal{D}_n$. Y is obtained from X through a uniform $\alpha-$spread, denoted $Y \succeq_\alpha^u X$, if there exist $1 \leq k \leq n-1$, with $k \in \mathbb{N}$ and $\alpha = \frac{k}{n}$, $0 < \varepsilon \leq \frac{x_{k+1}-x_k}{n}$*

such that:

$$\begin{cases} y_i = x_i + \varepsilon, & \forall i \neq (k+1) \\ y_{k+1} = x_{k+1} - (n-1)\,\varepsilon. \end{cases}$$

Obviously, if $Y \succeq^u_\alpha X$, these two distributions cannot be ordered by the Lorenz criterion. Furthermore, a simple inspection of Definition 2 shows that the Lorenz curves associated with Y and X cross only once, and that the curve associated with Y is above the one associated with X for $\xi \leq \frac{k}{n}$, and below for $\xi > \frac{k}{n}$. It then follows that if X and Y are two distributions with the same total income and population size, a necessary condition for Y to be obtained from X by a sequence of Pigou-Dalton transfers and/or uniform $\alpha-$spreads, with $\alpha \geq \tilde{\alpha}$, is that $L(F_Y, \xi) \geq L(F_X, \xi)$ for all $\xi < \tilde{\alpha}$. In other words, the partial order associated with finite sequences of Pigou-Dalton transfers and/or uniform $\alpha-$spreads with $\alpha \geq \tilde{\alpha}$, does not allow one to rank income distributions with intersecting Lorenz curves if an intersection occurs at $\xi < \tilde{\alpha}$.

Definition 3. *A decision maker is favorable to uniform $\alpha-$spreads if $Y \succeq X$ whenever $Y \succeq^u_\alpha X$.*

We say that a decision maker satisfies the principle of uniform $\alpha-$spreads if he is favorable to uniform $\alpha-$spreads. Observe that, if Y is obtained from X by a uniform $\alpha-$spread, and Y' is obtained from X by a uniform $\alpha'-$spread of the same amount, with $\alpha' > \alpha$, then Y' is obtained from Y by a Pigou-Dalton transfer from the agent in position α' to that in position α. This leads us to the following Proposition.

Proposition 1. *If a decision maker respects the principle of transfer and is favorable to uniform $\alpha-$spreads, then he is favorable to uniform $\alpha'-$spreads, for all $\alpha' \geq \alpha$.*

Proposition 1 leads us to a natural definition of a decision maker's sensitivity to uniform spreads.

Definition 4. *The degree of sensitivity to uniform spreads of a decision maker who respects the Lorenz order is defined by:*

$$\alpha^u = 1 - \inf\left\{\alpha : \textit{the decision maker is favourable to uniform } \alpha - \textit{spreads}\right\}.$$

Because it is not assumed that the size of the population is fixed, and because it can be arbitrarily large, the degree of sensitivity to uniform spreads can take any value in the interval $[0, 1]$. Assume, for instance, that the decision maker is favorable to uniform $\frac{1}{n}-$spreads. Then, when n tends to ∞, the infimum of α such that the decision maker is favorable to uniform $\alpha-$spreads is equal to $\lim_{n\to\infty} \frac{1}{n} = 0$, and therefore the decision maker's degree of sensitivity to uniform spreads is equal to 1. Observe that, in this case, the decision maker is favorable to *any* uniform spread. On the other hand, a decision maker who respects the principle of transfers, must at least be favorable to uniform $\frac{n-1}{n}-$spreads, since these spreads are actually a sequence of Pigou-Dalton transfers. However, assume that the decision maker is favorable *only* to uniform $\frac{n-1}{n}-$spreads and to Pigou-Dalton transfers. Then, his degree of sensitivity to uniform spreads is equal to $1 - \lim_{n\to\infty} \frac{n-1}{n} = 0$.

Observe that a uniform $\frac{k}{n}-$spread can be seen as the combination of a sequence of progressive transfers from the individual occupying the $(k+1)^{th}$ position in the ladder to the k poorest individuals, and a sequence of regressive transfers from the same individual to the $(n-k-1)$ richest individuals. The results of these transfers are a reduction of inequality among the $(k+1)$ poorest individuals, and an increase of inequality among the $(n-k)$ richest individuals. Therefore, a uniform $\frac{k}{n}-$spread seems appealing for large values of k when the size of the population is large, and the decision maker respects the Lorenz order: It means that the decision maker is ready to accept an increase of inequality among the very rich persons, provided that it is accompanied by a decrease of inequality among the rest of the population. Roughly speaking, the decision maker's degree of sensitivity to uniform spreads measures the size of the population among which the reduction of inequality is not seen as a priority by the decision maker. The extreme case is that of a Rawlsian decision maker, who is mainly concerned by the poorest individual: His degree of sensitivity to uniform spreads is then equal to 1. This does not mean, however, that such a decision maker is not favorable to Pigou-Dalton transfers among richer individuals. But a policy that increases the poorest individual's income is then seen as favorable, even if the cost of such a policy is an increase of inequality among the rest of the population. A natural interpretation is that, if the decision maker is favorable to uniform $\frac{k}{n}$-spreads, he considers the k poorest individuals as "poor" individuals. However, it does not imply, unlike to the "focusing axiom" used in poverty measurement, that the decision maker is not concerned with richer individuals. Hence, the principle of uniform $\alpha-$spreads lies somewhere between the principle of transfers and the focusing principle.

3 Uniform Spreads and Social Welfare Functions

We give here necessary and sufficient conditions for a decision maker who respects the Lorenz order to be favorable to uniform $\alpha-$spreads. We successively focus on decision makers who behave in accordance with the Expected Utility model, with the Rank-Dependent Expected Utility model, and with Yaari's dual model, which is a particular case of the Rank-Dependent Expected Utility model.

3.1 Uniform Spreads and the Expected Utility model

Our first result is, at first sight, striking: A decision maker who behaves in accordance with the Expected Utility model is favorable to uniform $\alpha-$spreads if, and only if, his Social Welfare Function reduces to the mathematical expectation of the income distribution, whatever the value of α is.

Theorem 1. *Let $\alpha \in \,]0,1[\cap \mathbb{Q}$. For a decision maker who behaves in accordance with the Expected Utility model, with a utility function two times continuously differentiable on D°, the following two propositions are equivalent:*
(i) The decision maker is favorable to uniform $\alpha-$spreads.
(ii) $u(x) = x$, $\forall x \in D$ (up to an increasing affine transformation).

Proof. Fix k and $n > 2$ such that $1 \leq k \leq n-1$ and $\alpha = \frac{k}{n}$. The decision maker is favorable to uniform $\alpha-$spreads if for any $X = (x_1, ..., x_n)$ in $\mathbb{D}$ such that $x_k < x_{k+1}$ and ε such that $0 < \varepsilon \leq \frac{x_{k+1}-x_k}{n}$ and $x_n + \varepsilon \in D$,

$$\sum_{i \neq k+1} u(x_i + \varepsilon) + u(x_{k+1} - (n-1)\varepsilon) \geq \sum_i u(x_i),$$

which is equivalent to:

$$\sum_{i<k+1}\left[u\left(x_i+\varepsilon\right)-u\left(x_i\right)\right]+\sum_{i>k+1}\left[u\left(x_i+\varepsilon\right)-u\left(x_i\right)\right]\geq u\left(x_{k+1}\right)-u\left(x_{k+1}-\left(n-1\right)\varepsilon\right). \quad (1)$$

<u>We first prove that $(i)\Rightarrow u''(x)\leq 0$ for all $x\in D^\circ$.</u>

Let y and x, in D° with $y<x$ be arbitrarily chosen, and let $x_1=x_2=...=x_k=y$, $x_{k+1}=x_{k+2}=...=x_n=x$. Then (1) implies, for all $\varepsilon\in(0,\frac{x-y}{n}]$ such that $x+\varepsilon\in D$:

$$k\left[u\left(y+\varepsilon\right)-u\left(y\right)\right]+\left(n-k-1\right)\left[u\left(x+\varepsilon\right)-u\left(x\right)\right]\geq u\left(x\right)-u\left(x-\left(n-1\right)\varepsilon\right).$$

Divide this expression by $(n-1)\varepsilon$:

$$\frac{k}{n-1}\left(\frac{u\left(y+\varepsilon\right)-u\left(y\right)}{\varepsilon}\right)+\frac{n-k-1}{n-1}\left(\frac{u\left(x+\varepsilon\right)-u\left(x\right)}{\varepsilon}\right)\geq\frac{u\left(x\right)-u\left(x-\left(n-1\right)\varepsilon\right)}{(n-1)\varepsilon}.$$

Now let ε tend to 0. One obtains:

$$\frac{k}{n-1}u'\left(y\right)+\frac{n-k-1}{n-1}u'\left(x\right)\geq u'\left(x\right),$$

and therefore, $u'(y)\geq u'(x)$ for all y and x in D° such that $y<x$. Hence $u''(x)\leq 0$ for all x in D°.

<u>We now prove that $(i)\Rightarrow u''(x)\geq 0$ for all $x\in D^\circ$.</u>

Let x, y and β be arbitrarily chosen such that $x<y$, $\beta>0$, and $x-\beta$ and y belong to D°. Let $x_1=...=x_k=x-\beta$, $x_{k+2}=...=x_n=y$, and $x_{k+1}=x$. If the decision maker is favorable to uniform $\alpha-$spreads, then, for $\varepsilon\in(0,\frac{\beta}{n}]$ such that $y+\varepsilon\in D$:

$$k\left[u\left(x-\beta+\varepsilon\right)-u\left(x-\beta\right)\right]+\left(n-k-1\right)\left[u\left(y+\varepsilon\right)-u\left(y\right)\right]\geq u\left(x\right)-u\left(x-\left(n-1\right)\varepsilon\right).$$

Divide this expression by $(n-1)\varepsilon$:

$$\frac{k}{n-1}\left(\frac{u\left(x-\beta+\varepsilon\right)-u\left(x-\beta\right)}{\varepsilon}\right)+\frac{n-k-1}{n-1}\left(\frac{u\left(y+\varepsilon\right)-u\left(y\right)}{\varepsilon}\right)\geq\frac{u\left(x\right)-u\left(x-\left(n-1\right)\varepsilon\right)}{(n-1)\varepsilon}.$$

Now let ε tend to 0. We obtain:

$$\frac{k}{n-1}u'\left(x-\beta\right)+\frac{n-k-1}{n-1}u'\left(y\right)\geq u'\left(x\right).$$

Now let β tend to 0. We get: $u'(y)\geq u'(x)$ for any x and y in D° such that $x<y$. Hence $u''(x)\geq 0$ for any x in D°.

Since $u''(x) \geq 0$ and $u''(x) \leq 0$ for any x in D°, and since u is continuous on D, $u(x) = x$, up to an increasing affine transformation, for all x in D. We have hence proved that (i) implies (ii).

That (ii) implies (i) is trivial, and the proof is completed. $\square$

Note that Theorem 1 doesn't depend on any assumption about the decision maker's attitude toward Pigou-Dalton transfers.

Actually, this result doesn't really come as a surprise. Indeed, a uniform spread is a combination of progressive and regressive transfers. In the Expected Utility model, the size of the impact of a transfer depends on the income distance between the individuals concerned by this transfer, and on the size of this transfer: The size of the impact of a small transfer $\varepsilon > 0$ from an individual with income x to an individual with income y is given by $[u'(y) - u'(x)]\,\varepsilon$. Assume that u is strictly concave on some interval $[a, b]$. Consider then, for an arbitrarily chosen $1 \leq k < n$ the distribution in which (i) x_k is arbitrarily close to a with $x_k > a$, (ii) $x_i \in (a, x)$ for all $i < k$, so that the impact of each progressive transfer is as close to 0 as one would like, and (iii) x_i is arbitrarily close to b with $x_i < b$ for all $i > k$, so that the impact of each regressive transfer is as close to $[u'(b) - u'(a)]\varepsilon$ as one would like. Since by strict concavity of u, $[u'(b) - u'(a)] < 0$, it then follows that the net impact of these transfers can be negative, and therefore the decision maker cannot be favorable to uniform $\frac{k}{n}-$spreads. A similar argument applies in the strictly convex case. Therefore, it must be the case that u is linear.

3.2 Uniform Spreads and Rank-Dependent Expected Utility model

Now, let us consider a decision maker who behaves in accordance with the Rank-Dependent Expected Utility model. First, we recall the following result (see Chew, Karni and Safra (1987)).

Theorem 2. *For a decision maker who behaves in accordance with the Rank-Dependent Expected Utility model, with a frequency transformation f differentiable on* $[0, 1]$*, the following two propositions are equivalent:*

(i) *The decision maker respects the Lorenz order.*

(ii) u *is concave and* f *is convex.*

We also need to define the index of thriftiness of a utility function, introduced by Chateauneuf, Cohen and Meilijson (1997). This index is defined by:

$$T_u = \sup_{\{x,y \in D | x<y\}} \frac{u'(x)}{u'(y)}.$$

The following theorem gives necessary and sufficient conditions for a decision maker who behaves in accordance with the Rank-Dependent Expected Utility model to respect the Lorenz order and to be favorable to uniform $\alpha-$spreads.

Theorem 3. *Let* $\alpha \in \,]0,1[\, \cap \mathbb{Q}$*. For a decision maker who behaves in accordance with the Rank-Dependent Expected Utility model, with a frequency transformation function differentiable on* $[0,1]$*, and with a utility function* u *continuously differentiable on* D*, the following two propositions are equivalent:*

(i) *The decision maker respects the Lorenz order and is favorable to uniform* $\alpha-$*spreads.*

(ii) f *is convex,* u *is concave on* D *and* $\frac{f'(1-\alpha)-1}{f(1-\alpha)} \leq \frac{1-T_u}{T_u}$.

Proof. $\underline{(i) \Rightarrow (ii)}$

We know from Theorem 2 that a decision maker who behaves in accordance with the Rank-Dependent Expected Utility model respects the Lorenz order if and only if u is concave and f is convex.

Let $\alpha = \frac{l}{r} \in \,]0,1[\, \cap \mathbb{Q}$ be fixed. Let $n = rm$ and $k = lm$, where $m \in \mathbb{N}^*$ is arbitrarily chosen. Assume that the decision maker is favorable to uniform $\alpha-$spreads, and let $X = (x_1, ..., x_n) \in \mathbb{D}$ with $x_k < x_{k+1}$. Then, for any $0 < \varepsilon \leq \frac{x_{k+1}-x_k}{n}$ such that $x_n + \varepsilon \in D$:

$$\sum_{i \neq k+1} \Psi\left(\frac{i}{n}\right) u(x_i + \varepsilon) + \Psi\left(\frac{k+1}{n}\right) u(x_{k+1} - (n-1)\varepsilon) \geq \sum_{i=1}^{n} \Psi\left(\frac{i}{n}\right) u(x_i),$$

which is equivalent to:

$$\begin{aligned} & \sum_{i<k+1} \Psi\left(\frac{i}{n}\right) [u(x_i+\varepsilon) - u(x_i)] + \sum_{i>k+1} \Psi\left(\frac{i}{n}\right) [u(x_i+\varepsilon) - u(x_i)] \qquad (2) \\ \geq \; & \Psi\left(\frac{k+1}{n}\right) [u(x_{k+1}) - u(x_{k+1} - (n-1)\varepsilon)]. \end{aligned}$$

Let x, y and β be arbitrarily chosen such that $x < y$, $\beta > 0$, $x, y \in D^{\circ}$ and $x - \beta \in D$. Let $x_{k+1} = x$, $x_1 = x_2 = ... = x_k = x - \beta$ and $x_{k+2} = ... = x_n = y$. Then (2) implies for $\varepsilon \in (0, \frac{\beta}{n}]$ such that $y + \varepsilon \in D$:

$$\begin{aligned} & \left[1 - f\left(\frac{n-k}{n}\right)\right]\left[u\left(x - \beta + \varepsilon\right) - u\left(x - \beta\right)\right] + f\left(\frac{n-k-1}{n}\right)\left[u\left(y + \varepsilon\right) - u\left(y\right)\right] \\ \geq\ & \left[f\left(\frac{n-k}{n}\right) - f\left(\frac{n-k-1}{n}\right)\right]\left[u\left(x\right) - u\left(x - \left(n-1\right)\varepsilon\right)\right]. \end{aligned}$$

Divide this expression by $(n-1)\varepsilon$:

$$\begin{aligned} & \frac{1 - f\left(\frac{n-k}{n}\right)}{n-1}\left(\frac{u\left(x - \beta + \varepsilon\right) - u\left(x - \beta\right)}{\varepsilon}\right) + \frac{f\left(\frac{n-k-1}{n}\right)}{n-1}\left(\frac{u\left(y + \varepsilon\right) - u\left(y\right)}{\varepsilon}\right) \\ \geq\ & \left[f\left(\frac{n-k}{n}\right) - f\left(\frac{n-k-1}{n}\right)\right]\frac{u\left(x\right) - u\left(x - \left(n-1\right)\varepsilon\right)}{(n-1)\varepsilon}. \end{aligned}$$

Now let ε tend to 0, and multiply both sides by $(n-1)$. This leads to:

$$\left[1 - f\left(\frac{n-k}{n}\right)\right]u'\left(x - \beta\right) + f\left(\frac{n-k-1}{n}\right)u\left(y\right) \geq \left(n-1\right)\left[f\left(\frac{n-k}{n}\right) - f\left(\frac{n-k-1}{n}\right)\right]u'\left(x\right).$$

Now let β tend to 0. One gets:

$$f\left(\frac{n-k-1}{n}\right)u'\left(y\right) \geq n\left[f\left(\frac{n-k}{n}\right) - f\left(\frac{n-k-1}{n}\right)\right]u'\left(x\right) + f\left(\frac{n-k-1}{n}\right)u'\left(x\right) - u'\left(x\right).$$

Divide both terms by $u'\left(y\right)$:

$$f\left(\frac{n-k-1}{n}\right) \geq n\left[f\left(\frac{n-k}{n}\right) - f\left(\frac{n-k-1}{n}\right)\right]\frac{u'\left(x\right)}{u'\left(y\right)} + f\left(\frac{n-k-1}{n}\right)\frac{u'\left(x\right)}{u'\left(y\right)} - \frac{u'\left(x\right)}{u'\left(y\right)}.$$

Hence:

$$f\left(1 - \frac{l}{r} - \frac{1}{rm}\right) \geq rm\left[f\left(1 - \frac{l}{r}\right) - f\left(1 - \frac{l}{r} - \frac{1}{rm}\right)\right]\frac{u'\left(x\right)}{u'\left(y\right)} + f\left(1 - \frac{l}{r} - \frac{1}{rm}\right)\frac{u'\left(x\right)}{u'\left(y\right)} - \frac{u'\left(x\right)}{u'\left(y\right)}.$$

Let m tend to $+\infty$. We then have:

$$f\left(1 - \frac{l}{r}\right) \geq f'\left(1 - \frac{l}{r}\right)\frac{u'\left(x\right)}{u'\left(y\right)} + f\left(1 - \frac{l}{r}\right)\frac{u'\left(x\right)}{u'\left(y\right)} - \frac{u'\left(x\right)}{u'\left(y\right)}.$$

Hence:

$$f\left(1 - \alpha\right)\left(1 - \frac{u'\left(x\right)}{u'\left(y\right)}\right) \geq \left[f'\left(1 - \alpha\right) - 1\right]\frac{u'\left(x\right)}{u'\left(y\right)}. \tag{3}$$

Therefore:

$$\frac{f'(1-\alpha)-1}{f(1-\alpha)} \leq \frac{1-\frac{u'(x)}{u'(y)}}{\frac{u'(x)}{u'(y)}}.$$

Since the right hand side of this inequality decreases when $\frac{u'(x)}{u'(y)}$ increases, this inequality is satisfied for all $x<y$ if and only if:

$$\frac{f'(1-\alpha)-1}{f(1-\alpha)} \leq \frac{1-T_u}{T_u}.$$

$\underline{(ii) \Rightarrow (i)}$

By Theorem 2, it is sufficient to show that the decision maker is favorable to uniform $\alpha-$spreads. Since u is concave it is enough to prove that for any x and y in D such that $x<y$, and any $\varepsilon>0$ such that $\varepsilon \leq \frac{y-x}{n}$ and $y+\varepsilon \in D$,

$$\begin{aligned}
&\sum_{i<k+1} \Psi\left(\frac{i}{n}\right)[u(x+\varepsilon)-u(x)]+\sum_{i>k+1} \Psi\left(\frac{i}{n}\right)[u(y+\varepsilon)-u(y)] \\
\geq\ & \Psi\left(\frac{k+1}{n}\right)[u(x+n\varepsilon)-u(x+\varepsilon)],
\end{aligned}$$

for all (k,n) for which $\alpha=\frac{k}{n}$. Consider any such (k,n).

The concavity of u implies, for any $\varepsilon \in (0,\frac{y-x}{n}]$ such that $y+\varepsilon \in D$:

$$\begin{aligned}
\frac{u(x+\varepsilon)-u(x)}{\varepsilon} &\geq u'(x+\varepsilon), \\
\frac{u(y+\varepsilon)-u(y)}{\varepsilon} &\geq u'(y+\varepsilon), \\
\frac{u(x+n\varepsilon)-u(x+\varepsilon)}{(n-1)\varepsilon} &\leq u'(x+\varepsilon).
\end{aligned}$$

Hence, it is enough to prove that, for any $y>x$ in D and any $\varepsilon \in (0,\frac{y-x}{n}]$ such that $y+\varepsilon \in D$:

$$\sum_{i<k+1} \Psi\left(\frac{i}{n}\right) u'(x+\varepsilon)+\sum_{i>k+1} \Psi\left(\frac{i}{n}\right) u'(y+\varepsilon) \geq (n-1)\Psi\left(\frac{k+1}{n}\right) u'(x+\varepsilon).$$

which may be written as follows:

$$\begin{aligned}
&\left[1-f\left(1-\frac{k}{n}\right)\right] u'(x+\varepsilon)+f\left(1-\frac{k+1}{n}\right) u'(y+\varepsilon) \\
&\geq (n-1)\left[f\left(1-\frac{k}{n}\right)-f\left(1-\frac{k+1}{n}\right)\right] u'(x+\varepsilon).
\end{aligned}$$

Dividing both terms by $u'(y+\varepsilon)$ leads to:

$$\begin{aligned} & f\left(1-\frac{k+1}{n}\right) \\ \geq\ & n\left[f\left(1-\frac{k}{n}\right)-f\left(1-\frac{k+1}{n}\right)\right]\frac{u'(x+\varepsilon)}{u'(y+\varepsilon)}+f\left(1-\frac{k+1}{n}\right)\frac{u'(x+\varepsilon)}{u'(y+\varepsilon)}-\frac{u'(x+\varepsilon)}{u'(y+\varepsilon)}. \end{aligned}$$

Since f is convex, we have:

$$n\left[f\left(1-\frac{k}{n}\right)-f\left(1-\frac{k+1}{n}\right)\right]\leq f'\left(1-\frac{k}{n}\right).$$

It is hence enough to prove:

$$f\left(1-\frac{k+1}{n}\right)\left[1-\frac{u'(x+\varepsilon)}{u'(y+\varepsilon)}\right]\geq\left[f'\left(1-\frac{k}{n}\right)-1\right]\frac{u'(x+\varepsilon)}{u'(y+\varepsilon)}.$$

Since u is concave, $\frac{u'(x+\varepsilon)}{u'(y+\varepsilon)}>1$ for all $0<x<y$ and $\varepsilon>0$. Since f is increasing, the preceding inequality is satisfied whenever:

$$f\left(1-\frac{k}{n}\right)\left[1-\frac{u'(x+\varepsilon)}{u'(y+\varepsilon)}\right]\geq\left[f'\left(1-\frac{k}{n}\right)-1\right]\frac{u'(x+\varepsilon)}{u'(y+\varepsilon)},$$

which is equivalent to:

$$\frac{f'\left(1-\frac{k}{n}\right)-1}{f\left(1-\frac{k}{n}\right)}\leq\frac{1-\frac{u'(x+\varepsilon)}{u'(y+\varepsilon)}}{\frac{u'(x+\varepsilon)}{u'(y+\varepsilon)}}.$$

Since the right hand side of this inequality decreases when $\frac{u'(x+\varepsilon)}{u'(y+\varepsilon)}$ increases, this last inequality is satisfied whenever:

$$\frac{f'\left(1-\frac{k}{n}\right)-1}{f\left(1-\frac{k}{n}\right)}\leq\frac{1-T_u}{T_u},$$

which is the desired result. □

Note that Theorem 3 implies the following result.

Corollary 1. *Let $\alpha\in\,]0,1[\,\cap\,\mathbb{Q}$. For a decision maker who behaves in accordance with Yaari's dual model with a frequency transformation function f differentiable on $[0,1]$, the following two propositions are equivalent:*

(i) The decision maker is favorable to uniform $\alpha-$spreads and respects the Lorenz order.

(ii) f is convex and $f'(1-\alpha)\leq 1$.

Proof. If $u(x) = x$, $\frac{1-T_u}{T_u} = 0$. Hence, Theorem 3 implies that a decision maker who behaves in accordance with Yaari's dual model (i.e., a decision maker who behaves in accordance with the Rank-Dependent Expected Utility model with a linear utility index) respects the Lorenz order and is favorable to uniform $\alpha-$spreads if and only if f is convex and $f'(1-\alpha) \leq 1$.

□

The conditions of Corollary 1 have a natural interpretation. Assume that we have $f'(1-\alpha) = 1$, and that f is convex. This implies that $f'(p) > 1$ for all p such that $1-\alpha < p < 1$ and $f'(p) < 1$ for all $0 \leq p \leq 1-\alpha$. Hence, the $\alpha\%$ poorest individuals are "over-weighted" (i.e., the decision maker gives them a weight greater than $\frac{1}{n}$, where n is the size of the population), and the $(1-\alpha)\%$ richest ones are "under-weighted".

Remark 1. *The condition $f'(1-\alpha) \leq 1$ is necessary for a decision maker who behaves in accordance with the Rank-Dependent Expected Utility model with a frequency transformation function f differentiable on $[0,1]$ and a utility function u continuously differentiable on D to respect the Lorenz order and to be favorable to uniform $\alpha-$spreads.*

Proof. By Theorem 3, if a decision maker who behaves in accordance with the Rank-Dependent Expected Utility model respects the Lorenz order and is favorable to uniform $\alpha-$spreads, then:

$$\frac{f'(1-\alpha)-1}{f(1-\alpha)} \leq \frac{1-T_u}{T_u}.$$

Theorem 3 also implies that u is concave. Thus, $T_u \geq 1$. Hence, the preceding inequality implies $f'(1-\alpha) \leq 1$. □

Let us now apply our different results to the problem of inequality measurement.

4 Inequality Indices and Uniform Spreads

Following Kolm (1969), Atkinson (1970) and Sen (1973), one can derive an inequality measure from a social welfare function. Let $\Xi(X)$ be the *per capita* income which, if distributed equally, is indifferent to $X \in \mathbb{D}$ according to the social welfare function W. This "equally distributed equivalent income" is implicitly defined by the relation:

$W(X) = W(\Xi(X)\ e)$, where e denotes the unit vector of $\mathbb{R}^n$. It is then possible to define an inequality index: $I(X) = 1 - \frac{\Xi(X)}{\bar{X}}$. The Atkinson index relies on an utilitarian social welfare function, whereas the Gini index and its generalizations rely on a rank-dependent social welfare function.

We say that an inequality index respects the principle of transfers if for all X and Y such that Y is obtained from X through a finite sequence of Pigou-Dalton transfers, $I(Y) \leq I(X)$. Similarly, we say that an inequality index respects the principle of $\alpha-$uniform spreads if for all X and Y as in Definition 2, $I(Y) \leq I(X)$. By a slight abuse of notation, we call degree of sensitivity to uniform spreads of an inequality index I the degree of sensitivity to uniform spreads of a decision maker endowed with the social welfare function on which relies I.

First, consider the Atkinson index defined by

$$\begin{cases} I_A(X) = 1 - \left[\sum_{i=1}^{n} \frac{1}{n}\left(\frac{x_i}{\bar{X}}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}, \quad \varepsilon \neq 1 \\ I_A(X) = 1 - \prod_{i=1}^{n} \left(\frac{x_i}{\bar{X}}\right)^{\frac{1}{n}}, \quad \varepsilon = 1. \end{cases}$$

This index relies on the following Expected Utility social welfare functions:

$$\begin{cases} U_A(X) = \sum_{i=1}^{n} \frac{1}{n} \frac{x_i^{1-\varepsilon}}{1-\varepsilon}, \quad \varepsilon \neq 1 \\ U_A(X) = \sum_{i=1}^{n} \frac{1}{n} \ln(x_i), \quad \varepsilon = 1. \end{cases}$$

The following proposition immediately follows from Theorem 1:

Proposition 2. *Let α belong to $]0,1[\cap \mathbb{Q}$. The Atkinson index respects the principle of uniform $\alpha-$spreads if and only if $\varepsilon = 0$.*

Hence, the Atkinson index cannot respect the principle of uniform $\alpha-$spreads, whatever α is, unless the index reduces to a constant. But it seems difficult to raise any objection to the principle of uniform $\alpha-$spreads, at least for very high values of α. This may be seen as a limit of the Atkinson index from a normative point of view.

Let us now consider the large class of Yaari indices. These indices are defined as follows (Yaari (1988), Ebert (1988)):

$$I_{GG}(X) = 1 - \frac{1}{\bar{X}}\left(\sum_{i=1}^{n}\left[f\left(\frac{n-i+1}{n}\right) - f\left(\frac{n-i}{n}\right)\right] x_i\right).$$

Applying Corollary 1, we obtain the following result:

Proposition 3. *Let α belong to $]0,1[\cap\mathbb{Q}$. A generalized Yaari index with a frequency transformation function f differentiable on $[0,1]$ respects the principle of uniform $\alpha-$spreads and the principle of transfers if and only if $f'(1-\alpha)\leq 1$ and f convex.*

Donaldson and Weymark (1980) and Bossert (1990) define the sub-class of Yaari indices which satisfy an aggregation axiom. These indices, known as S-Gini indices, are defined as follows:

$$I_{SG}(X)=1-\frac{\sum_{i=1}^{n}\left[\left(\sum_{j=i}^{n}p_j\right)^{\delta}-\left(\sum_{j=i+1}^{n}p_j\right)^{\delta}\right]x_i}{\bar{X}},$$

with $\delta>1$. These indices correspond to the following social welfare function:

$$V_{SG}(X)=\sum_{i=1}^{n}\left[f\left(\sum_{j=i}^{n}p_j\right)-f\left(\sum_{j=i+1}^{n}p_j\right)\right]x_i,$$

with $f(p)=p^{\delta}$. Note that for $\delta=2$, I_{SG} is nothing but the Gini index. The following proposition establishes a link between the degree of sensitivity to uniform spreads (see Definition 4) of a $S-$Gini index and the value of the parameter δ:

Proposition 4. *For a S-Gini with parameter $\delta>1$, the degree of sensitivity to uniform spreads of the index is equal to $\delta^{\frac{1}{1-\delta}}$. Furthermore, the degree of sensitivity to uniform spreads of the index is greater or equal to $\frac{1}{e}$ for all $\delta>1$, and is increasing with respect to δ.*

Proof. Let I_{SG} be a $S-$Gini index with parameter $\delta>1$. According to Definition 4, the degree of sensitivity to uniform spreads of I_{SG} is defined by:

$$\alpha^u(\delta)=1-\inf\{\alpha:\ \text{the decision maker is favourable to uniform }\alpha\text{−spreads }\},$$

where the decision maker's preferences are represented by

$$V_{SG}(X)=\sum_{i=1}^{n}\left[\left(\sum_{j=i}^{n}p_j\right)^{\delta}-\left(\sum_{j=i+1}^{n}p_j\right)^{\delta}\right]x_i.$$

Observe that, for $\delta>1$, $f(p)=p^{\delta}$ is convex and f is differentiable on $[0,1]$. Therefore, I_{SG} respects the principle of transfers for all $\delta>1$. By Proposition 3, I_{SG} respects the principle of uniform $\alpha-$spreads, for $\alpha \in]0,1[\cap\mathbb{Q}$ if, and only if: $f'(1-\alpha) \leq 1$, i.e., $\alpha \geq 1-\delta^{\frac{1}{1-\delta}} = 1-\phi(\delta)$, with $\phi(\delta) = \delta^{\frac{1}{1-\delta}}$. Observe that $\phi'(\delta) = \exp(\xi(\delta))\,\xi'(\delta)$, with $\xi(\delta) = \frac{\ln\delta}{1-\delta}$. Let $\psi(\delta) = \frac{1-\delta}{\delta} + \ln\delta$. Then: $\xi'(\delta) = \frac{\psi(\delta)}{(1-\delta)^2}$. Since $\psi'(\delta) = \frac{\delta-1}{\delta^2}$, we get: $\psi'(\delta) > 0$ for all $\delta > 1$. Furthermore, $\psi(1) = 0$. Therefore, $\psi(\delta) > 0$ for all $\delta > 1$. Hence, $\xi'(\delta) > 0$ for all $\delta > 1$, which entails $\phi'(\delta) > 0$ for all $\delta > 1$. Hence, the degree of sensitivity to uniform spreads of I_{SG} with parameter δ is $\phi(\delta)$, and the greater δ is, the higher is the degree of sensitivity to uniform spreads of the index. Finally, we have $\lim_{\delta\to 1}\phi(\delta) = \frac{1}{e}$ □

Finally, the very general inequality index (let us call it a super-generalized Gini index):

$$I_{SSG}(X) = 1 - \frac{1}{\bar{X}}u^{-1}\left(\sum_{i=1}^{n}\left[f\left(\frac{n-i+1}{n}\right) - f\left(\frac{n-i}{n}\right)\right]u(x_i)\right),$$

considered by Ebert (1988) and Chateauneuf (1996) corresponds to a Rank-Dependent Expected Utility-like social welfare function, with a utility function u and a frequency transformation function f. Applying Theorem 3 we obtain the following result.

Proposition 5. *Let α belong to $]0,1[\cap\mathbb{Q}$. A super-generalized Gini index with a frequency transformation function f differentiable on $[0,1]$ and a utility function u continuously differentiable on D respects the principle of transfers and the principle of uniform $\alpha-$spreads if and only if $f'' \geq 0$, $u'' \leq 0$ and $\frac{f'(1-\alpha)-1}{f(1-\alpha)} \leq \frac{1-T_u}{T_u}$.*

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