Human capital accumulation over the life cycle under multiple sources of uncertainty

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Abstract

In this paper, we analyse the human capital accumulation process over the life cycle of individuals under the hypothesis of earnings uncertainty. To do so, we develop a continuous time dynamic programming model which takes into account several sources of uncertainty, concerning the human capital accumulation process and the labour market. In particular, the risks over future wage rates and over-education are explicitly taken into account. The model permits an in-depth study of each source's effect of uncertainty on human capital investment. We demonstrate that investment in education may be encouraged, depending on the individual's perception of the different risks.

key words : human capital, life cycle, risk, stochastic optimal control.

JEL Classification : I20, J24, C61, D99, G11.

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1 Introduction

Since the pioneering studies of Schultz (1961) and Becker (1962) in the area of education, schooling and training have been considered as investments in human capital in the same way as physical capital. But for many years, the risky nature of this investment was overlooked in the analysis. Over the past two decades, however, European labour markets have seen the rise of an often chronic and sometimes long-term form of youth unemployment, accompanied by wage downgrading at the time of first hiring. The vulnerability of the young people and the uncertainty they face at the beginning of their working life are expressed not only by the increased difficulties in finding a job which corresponds to their level of schooling but also by employment rates which are much more sensitive to the economic situation than those found among other categories of the labour force. In the area of education, we also observe a diversification of training streams and pathways. At aggregate level, this increased heterogeneity may accentuate the uncertainty about the quality of the pathways. At individual level, it tends to blur employers' perceptions of the students' productivity potentials.

As a result, the predictions of models based on the hypothesis of perfect foresight about the future value of the human capital accumulated seem less and less adapted to the European observed data. It thus seems necessary to resituate the problem of human capital investment, as well as that of the efficiency of the educational systems stemming from it, in a more general framework which takes uncertainty into account. The question which then arises is to what extent and in what forms uncertainty influences decisions to invest in human capital.

The first rigorous theoretical analysis of this question was proposed by Levhari and Weiss (1974) in a two-period model. The uncertainty about future wages, or in other words, the return to the human capital investment, is assumed to come from two sources. The first has to do with the learning process in the educational system and covers a group of exogenous individual and collective features such as, for example, the students' scholastic aptitude, but also the quality of the courses, the schools, the teachers, the scholar paths and so on. The second arises from the labour market and covers the future conditions of labour supply and demand. These two type of souces of uncertainty, which are intuitive and fairly realistic, are not present in the model specifications. Indeed, the sources of uncertainty are not separated out; rather, they are aggregated and represented by a single random variable. Uncertainty is defined by the variance of earnings, in the sense of Rothschild-Stiglitz (1970). The effect of uncertainty is studied on the basis of the correlation between the average and future marginal returns to human capital. If this correlation is positive, or if the variance of the gains increases with the level of schooling (which amounts to the same thing), the anticipated return to human capital will be greater than that to a financial asset presumed to be certain. In this case, risk averse individuals will protect themselves by reducing their human capital investment. A negative correlation between average and marginal returns has the opposite effect: an increase in the level of schooling will reduce the variance of the future gains. In this case, investment in human capital is encouraged when risk increases. The hypothesis generally employed in the empirical literature, however, is that of an increasing risk[1] (increasing variance of earnings), which leads to the assumption of a lower education level in a situation of uncertainty.

More recently, Kodde (1986) has provided empirical results which, contradicting Levhari and Weiss's prediction, indicate on the contrary the existence of a positive relationship between uncertainty and investment in human capital. In an attempt

[1] See Low and Ormiston (1991) for empirical evidence on the NLS data.

to take this empirical observation into account, Snow and Warren (1990) developed an extension of the Levhari and Weiss model by including the hypothesis of an endogenous labour supply. This permits the introduction of an income effect which may make the relationship between risk and investment ambiguous. In other words, the investment may increase or decrease with risk.

In our view, this ambiguity of the effect of risk on investment stems from the aggregation of the different sources of uncertainty through a single random parameter capturing all the effects. It might be thought a priori that these different sources of uncertainty have an unequal, if not contradictory, effect on the investment in human capital. In the models already cited, moreover, the fact that the individual's planning horizon is reduced to two periods does not allow the intertemporal nature of the human capital investment to be brought out.

In order to get beyond these two limitations, we propose a dynamic programming model in continuous time which allows us not only to study the individual's behaviour over the whole of his or her life cycle but also to break down and separately identify the effects of the different sources of uncertainty.

In the next section, we present the general principles of the problem of human capital investment in a stochastic context. Section 3 is then devoted to the specification of our model. In section 4, we present the results of the model in the general case where individual preferences are not specified. In section 5, after identifying the individual preferences, we discuss the effect of the different sources of uncertainty on optimal investment in human capital. Our conclusions are presented in section 6.

2 The continuous-time stochastic model of human capital: preliminary elements

The study of the dynamic nature of human capital investment in a context of uncertainty can, as in the case of certainty, take two directions. In the first (e.g., Fan [1993], Hogan and Walker [2002]), the theory of real options is applied to the problem of education, with a model of schooling choice transposed to the uncertain case. In the second (Williams [1979]), the portfolio theory is applied to the problem of investment choice over the individual's entire life cycle.

The first group of studies develop a pure schooling model in which the individual has to decide on the optimal, definitive date for stopping his or her studies. Here, education is considered to be an irreversible investment[2]. The problem facing the individuals is the following: as long as they remain in the educational system, they have at each date the option to leave school and enter the labour market at a wage rate which depends stochastically on the amount of time spent in school. Once that option is exercised, they can no longer return to the educational system and will, throughout their life cycle, receive an income which depends uniquely on the accumulated schooling. In the presence of risk, individuals are encouraged to postpone their exit from the school system because of the irreversibility of their choice. Indeed, by remaining in school, they have the option of leaving during the following period with the aim of taking advantage of a 'better draw' in the distribution of returns. They also have the choice of remaining in school in order to avoid a 'bad draw'. For this reason, uncertainty increases the potential advantage of the option. But if, in anticipation of a low wage, individuals do not exercise their

[2] The stochastic versions of the optimal stopping problem are analysed in Kamien and Schwartz (1991). The application of these techniques to physical capital investment is reviewed in Dixit and Pindick (1994).

option, the loss of value of the option remains unchanged. This asymmetry of the effects of uncertainty on the value of the option incites individuals to postpone their exit from the school system even longer when uncertainty about wages is great[3].

Thus, the duration of schooling is an increasing function not only of the anticipated return but also of the risk associated with education. This model introduces micro-economic elements in the analysis of the phenomenon of continued studies, by emphasising the protective role of education in face of the risks existing at the time of labour-market entry. Nonetheless, the findings of this kind of model must be qualified in the light of the restrictive hypotheses on which it is based. First of all, the Hogan and Walker (2002) model, for example, may be seen as a very specific case of the human capital model, namely a *pure schooling* model in which labour supply is presumed to be exogenous. This means that the duration of schooling is the only variable which the individual controls. As a result, the model rules out any possibility of training associated with employment. During the employment period, wages are presumed to increase at an exogenous rate which is identical for all workers. Thus, the average wage differentials observed over the individuals' life cycles are explained solely on the basis of schooling levels. Any possibility of changing the initial situation through post-schooling investment in human capital is excluded. So, the question of investment over the life cycle becomes irrelevant because the individual's future situation is definitively established at the time of the exit from the school system.

Last of all, the model does not distinguish the sources of uncertainty. Since the process of human capital accumulation is not specified, all uncertainty is presumed to come from the labour market through wages. The risky content of school-based learning is totally absent.

Here, we would point out that the diversification of training pathways increasingly encountered in Western economies can also affect uncertainty about the students' productive capacities. This observation suffices to justify taking into account the risk which may exist within the education process. This second path of analysis corresponds to the approach taken by Williams (1979) in a different but more general theoretical context. Williams proposes a portfolio model much closer to the initial human capital model. In fact, he generalises the basic model, where the individual makes a trade-off between non-risky activities, to the case of a trade-off between risky activities. The object of the individual choice is not so much that of the optimal duration of studies as that of the optimal intensity of investment in human capital over the life cycle. This comes from the fact that, for Williams, human capital cannot be reduced solely to initial training. Post-schooling training, especially on-the-job training, is another form of human capital investment taken into account. Williams's frame of analysis appears also more general, to the extent that, on the one hand, the labour supply is endogenous and, on the other, different sources of uncertainty, involving both the educational process and the wage, are taken into account. More precisely, four sources of uncertainty are distinguished. A first source bears on the value of the financial assets in which the individual can invest on the financial market. Two others stem from the process of human capital accumulation: one concerns the efficiency of school-based learning and the other, the depreciation rate of the human capital. The last source of uncertainty has to do with the wage rate by level of skills. The co-variances between the different risky variables play a central role in the analysis, notably in the determination of the overall effect of risk on the investment in human capital. Contrary to Hogan and Walker (2002), Williams concludes that risk has a negative effect: individuals are led to reduce their human capital investment when the risk associated with education increases.

[3] The proof of this is provided in the appendix of Hogan and Walker (2002).

In the section which follows, we show that this result stems from the (ad hoc) hypothesis of the independence of the different markets. By forcing the nullity of certain covariances, particularly those linking the risks relative to the knowledge-acquisition processes to the risk over wages, Williams implicitly brings all the weight of uncertainty to bear on the learning process. This hypothesis is difficult to defend to the extent that the very essence of the economics of education (Willis and Rosen 1979) lies in the study of the mechanisms articulating the schooling process and its recognition on the labour market. The observable and non-observable individual features explaining the differences in education levels also account in part for the differences in wages (Willis 1986, Grilliches 1977).

On the basis of all of these criticisms, we propose a theoretical model which develops that of Williams (1979) in two directions. On the one hand, in order to avoid reducing all uncertainty stemming from the labour market to the risk over wages, we explicitly integrate the risk bearing on the human capital utilisation in future occupations. This allows a better appreciation of imperfect job matching on the labour market and the present phenomenon of overeducation faced by young people in the school-to-work transition. On the other hand, we raise the hypothesis of independence between the random variables describing the process of acquiring knowledge and those relating to the labour market and re-examine the effect of different sources of uncertainty on human capital investment.

3 Specification of the model

In this section, we present a dynamic stochastic model of human capital accumulation over the life cycle. The uncertainty here stems from five sources: the four used in Williams's (1979) model plus an additional one bearing on the utilization rate of human capital in the future occupation. We suppose that a trade-off can exist between the human capital held by the individual and the level of human capital required by the future job. Uncertainty intervenes in the constraints of the accumulation of human capital and financial wealth, which is why the first step in this study consists of correctly deriving the stochastic versions of these accumulation equations.

In this kind of model, the value of the current human capital stock $k(t)$ is defined as the income of the maximum work which the individual can expect on the labour market. The proportion of time which he or she devotes to training during the period t is measured by $e(t)$, while $l(t)$ designates the proportion of time allocated to leisure activities during the period t. Thus, $(1 - e(t) - l(t))$ is the proportion of remaining time devoted to work.

The human capital stock for the period $t + \Delta t$ is equal to the stock of the preceding period $k(t)$ plus the new human capital produced at the beginning of the period $t+\Delta t : \theta(t, t+\Delta t)e(t)k(t)$. This production of human capital is supposed to depend linearly of the proportion of time devoted to education $e(t)$ and a parameter $\theta(t, t+\Delta t)$ which measure the efficiency of training. It is also necessary to subtract the depreciation of the human capital stock during the period t, which is equal to $\delta(t, t + \Delta t)k(t)$:

$$K\left(t+\Delta t\right)=K\left(t\right)+\theta\left(t,t+\Delta t\right)e\left(t\right)K\left(t\right)-\delta\left(t,t+\Delta t\right)K\left(t\right) \tag{1}$$

To take into account the problems faced by young people on the European labour markets, we suppose that, on the labour market, the individual can obtain a job which doesn't correspond completely to its level of skills. More precisely, we suppose that the individual's human capital can be not fully used in a given occupation. Thus, by weighting the stock of human capital (equation 1) by a coefficient h, with $h \in [0, 1[$ one obtains the stock of human capital actually used in a given occupation:

$$h\left(t+\Delta t\right) K\left(t+\Delta t\right) = h\left(t+\Delta t\right)\left[1+\theta\left(t, t+\Delta t\right) e\left(t\right) - \delta\left(t, t+\Delta t\right)\right] K\left(t\right) \quad (2)$$

This equation highlights a phenomenon of overeducation, which is all the more large as h is low [4].

By multiplying each of the two members of the equation above by the price of the human capital "ω", one obtains the money value of the stock of human capital actually used on the labour market :

$$\omega\left(t+\Delta t\right) h\left(t+\Delta t\right) K\left(t+\Delta t\right) = \omega\left(t+\Delta t\right) h\left(t+\Delta t\right)\left[1+\theta\left(t, t+\Delta t\right) e\left(t\right) - \delta\left(t, t+\Delta t\right)\right] K\left(t\right) \quad (3)$$

Assuming that $k\left(t+\Delta t\right) = \omega\left(t+\Delta t\right) h\left(t+\Delta t\right) K\left(t+\Delta t\right)$, the value of the stock of future human capital actually used in the future occupation can be rewritten in the following form:

$$k\left(t+\Delta t\right) = \frac{h\left(t+\Delta t\right)}{h(t)} \frac{\omega\left(t+\Delta t\right)}{\omega\left(t\right)}\left[1-\delta\left(t, t+\Delta t\right)+\theta\left(t, t+\Delta t\right) e\left(t\right)\right] k\left(t\right) \quad (4)$$

Written in this form, this equation indicates that the value of the stock of human capital of the period $t + \Delta t$, depends not only on the variation of the volume of human capital, expressed by the term within the square brackets, but it depends also on the variations between the dates t and $t + \Delta t$, of the value of the stock of human capital on the labour market, expressed at the same time by the variation of the utilization rate of the human capital in future job $\frac{h(t+\Delta t)}{h(t)}$, and by the variation of its market price : $\frac{\omega(t+\Delta t)}{\omega(t)}$.

The uncertainty bears on the parameters $\theta\left(t, t+\Delta t\right)$ and $\delta\left(t, t+\Delta t\right)$, relating to the process of accumulation of human capital in volume, as well as on the parameters $\omega\left(t+\Delta t\right)$ et $h\left(t+\Delta t\right)$, relating to the future value of the human capital on the labour market.

The stochastic parameter $\theta\left(t, t+\Delta t\right)$ represents the uncertainty about the individual's real learning ability. It also takes into account all the non-identifiable inputs entering the education process. The random parameter $\delta\left(t, t+\Delta t\right)$ represents the unknown rate of human capital depreciation in each period.

The random parameter $\omega\left(t+\Delta t\right)$ corresponds to the future wage rate by skill level, unknown at the time t. We suppose that the individual perfectly observes the current wage corresponding to his level of human capital, but doesn't know the distribution of future wages. The randomness of the distribution of wages (for a fixed level of education) is a manner of characterizing future trends in the labour market - in particular, the institutional conditions of wage determination

[4] By definition, the stock of human capital held by an individual corresponds to the maximum labour income to which it can claim on the labour market. Thus, any phenomenon of undereducation, corresponding to an overuse of the human capital in a particular employment, is excluded from this model.

- largely uncontrolled by the individual at the time he makes his decisions. In other words, two individuals having the same characteristics, working in similar jobs today, can perceive different wages in the future, only because they will have obtained a different "draw" in the distribution of wages.

The parameter $h(t+\Delta t)$ represents the uncertainty bearing on the utilization rate of human capital in the future occupation. We suppose that employment is heterogeneous from the point of view of the use of human capital, and that the individual does not perfectly know the distribution of jobs. That means, two individuals having the same level of human capital can obtain different labour incomes in the future, if they reach different jobs or different professional promotions. In other words, even if the individual could perfectly know the wages associated with each job, he cannot know his future labour income a priori, because he does not know if the job which he will reach in the future is fully adapted to its skills. Thus, the parameter $h(t+\Delta t)$ highlights the possible problems of mismatch between the skill requirements of jobs and the skills held by the individuals. In what follows, we will use the term "risk of overeducation" to refer to the risk bearing on $h(t+\Delta t)$[5].

These four random parameters are presumed to be infinitely divisible and to follow lognormal laws of mean μ and variance σ : $\theta(t,t+\Delta t) \sim LN(\mu_\theta,\sigma_\theta)$; $\delta(t,t+\Delta t) \sim LN(\mu_\delta,\sigma_\delta)$; $\omega(t+\Delta t) \sim LN(\mu_\omega,\sigma_\omega)$ et $h(t+\Delta t) \sim LN(\mu_h,\sigma_h)$.

By introducing the temporal dimension, the variations of each of the random parameters are presumed, according to Ito's lemma[6], to follow a Weiner process characterised by the following stochastic differentials :

$$\Delta\theta(t) = \mu_\theta\Delta t + \sigma_\theta\Delta Z(t) \quad (5)$$
$$\Delta\delta(t) = \mu_\delta\Delta t + \sigma_\delta\Delta Z(t) \quad (6)$$
$$\Delta\omega(t) = \omega(t)\left[\mu_\omega\Delta t + \sigma_\omega\Delta Z(t)\right] \quad (7)$$
$$\Delta h(t) = h(t)\left[\mu_h\Delta t + \sigma_h\Delta Z(t)\right] \quad (8)$$

$\mu_\theta, \mu_\delta, \mu_\omega$ et μ_h represent the instantaneous means of the respective stochastic processes. These parameters are constant, following the properties of the Wiener process. $\sigma_\theta, \sigma_\delta, \sigma_\omega$ et σ_h are vectors of constants of $N+4$ dimension. They correspond to the instantaneous standard deviations of each stochastic process. They are indicators of risk in Rothschild-Stiglitz' sense . $Z(t)$ is the standard Wiener process. By definition, it is a random vector of zero mean, zero covariances and variance Δt. In this model, it is also of dimension $N+4$.

On the basis of these four expressions, by making Δt tend towards 0, by replacing these values in (1) and applying Ito's lemma, we obtain the stochastic expression of the human capital accumulation equation in continuous time [7] :

[5] The definition of over-education we adopt here, in term of under-utilisation of human capital on the labour market, is quite restrictive. Now, a large literature covers this issue both in Europe and North America. For a complete treatment of measurement issues, theoretical and empiqiric results and challenges raised by the question of over-education, see the excellent survey of Sloane (2002).

[6] Ito's lemma, also known as the fundamental theorem of stochastic calculus, is the most commonly used result in continuous-time models. It permits the determination of the parameters of any Ito process when the latter depends on a process of the same nature with known parameters. The unidimensional and multidimensional versions of Ito's lemma are given by Rogers (1991), pp. 234-236.

[7] The calculations used to obtain this equation are presented in their totality in appendix A.1.

$$\frac{dk(t)}{k(t)} = (\mu_h + \mu_\omega + (\mu_\theta + \sigma_{h\theta} + \sigma_{\theta\omega}) e(t) - \mu_\delta - \sigma_{h\delta} - \sigma_{\delta\omega} + \sigma_{h\omega}) dt + (\sigma_h + \sigma_\omega + \sigma_\theta e(t) - \sigma_\delta)' dZ(t) \quad (9)$$

This equation establishes that the return on human capital $\frac{dk(t)}{k(t)}$ follows a diffusion process linearly dependent on the time devoted to education $e(t)$. More precisely, an increase in the current allocation to education $e(t)$ simultaneously increases the mean and the variance of the instantaneous human capital growth rate and thus the future earnings over the life cycle.

On the basis of this equation, we may note that the extent of this educational effect crucially depends on the net productivity parameter θ. Only uncertainty about the depreciation of human capital has no weight on the education effect. Unlike Williams (1979), uncertainty about the labour market directly affects the return to the human capital investment through the instantaneous covariances $\sigma_{h\theta}$ and $\sigma_{\theta\omega}$. In Williams's model indeed, jobs are supposed to be homogeneous : human capital is always fully used, so that $h = 1$ and $\sigma_{h\theta} = 0$. In addition, the adjustments occurring on the labour market and in the educational system are presumed to be independent, which implies the nullity of $\sigma_{\theta\omega}$ and other covariances appearing in equation (9).

The originality of our work is precisely to reject this assumption of independence, and suppose, on the contrary, the existence of a nonnull relation between the variables associated with the process of human capital accumulation and those associated with the value of human capital on the labour market. $\sigma_{\theta h}$ identifies the relation between the efficiency of training and the utilization rate of human capital in the future job. $\sigma_{\theta\omega}$ identify the relation between the efficiency of training and the level of the wages (for a fixed level of skills). We suppose that these instantaneous covariances are positive : the most effective individuals, or individuals who have followed the most effective formations, are those who reach higher levels of employment and wages corresponding to their level of human capital. This assumption, rather intuitive, is not contradicted by empirical work. Besides, the aim of a vast econometric literature around the famous "ability bias" is to identify this type of relations (cf Moullet (2000), for a synthesis of this work).

At the same time, individuals are assumed to divide their current financial wealth among three elements: consumption expenditures, an investment in a non-risky asset and an investment in N risky assets on the financial market. The asset without risk is assumed to yield a known interest rate fixed at r. The yields of the risky financial assets are assumed to follow a Wiener process with the stochastic differential written as :

$$dP(t) = P(t)[\mu dt + \Gamma' dZ(t)] \quad (10)$$

where μ is the vector of the average returns per unit of time and $\Sigma \equiv \Gamma\prime\Gamma$ the variances-covariances matrix of the returns of the N risky assets per unit of time, of dimension $(N+4) \times N$. In addition, the returns of the N risky assets show the covariances $\Sigma_\omega \equiv \Gamma'\sigma_\omega$ with wage adjustments by level of education, the covariances $\Sigma_h \equiv \Gamma'\sigma_h$ with over-education adjustments for a given level of skills and the covariances $\Sigma_\theta \equiv \Gamma'\sigma_\theta$ and $\Sigma_\delta \equiv \Gamma'\sigma_\delta$ with the parameters of net productivity and depreciation of human capital. To make drastic simplifications in the calculations, we shall assume these covariances to be equal to 0 in what follows.

In other words, like Williams (1979), we assume that the financial market functions in an independent way, namely that the adjustments of wages and over-education, as well as the values of net productivity and depreciation of productive skills, are determined independently of the forces exerted by the financial market.

If X is the proportion of the wealth invested in risky assets, $c(t)$ the consumption flow at time t, and $y(t)$ the flow of labour income received in t, the variation of the financial wealth between t andΔt, when Δt tends towards 0, may be written :

$$dw\left(t\right)=\frac{wX'dP\left(t\right)}{P\left(t\right)}+r\left(1-X\right)w\left(t\right)dt+y\left(t\right)dt-c\left(t\right)dt$$

By replacing $dP(t)$ by its value given by (7) and rearranging the terms, we obtain the stochastic constraint of financial wealth accumulation:

$$dw\left(t\right)=\left[\left(rw(t)+y\left(t\right)-c\left(t\right)\right)+w\left(t\right)\left(\mu-r1\right)'X\right]dt+w\left(t\right)X'\Gamma'dZ\left(t\right) \quad (11)$$

Equations (6) and (8) constitute the constraints under which individuals are, over the whole of their life cycle, presumed to maximise their time-separable utility function, which depends on consumption, leisure, human capital[8] and terminal wealth[9]. More precisely, the programme to be resolved is posed in the following way :

$$Max\ E_t\int_t^T u\left[c\left(t\right),l\left(t\right),k\left(t\right),t\right]\Delta t+B\left[w\left(T\right),T\right] \quad (12)$$

under the constraints[10]:

$$\begin{aligned}\frac{dk}{k} &= \left(\mu_h+\mu_\omega+\left(\mu_\theta+\sigma_{h\theta}+\sigma_{\theta\omega}\right)e-\mu_\delta-\sigma_{h\delta}-\sigma_{\delta\omega}+\sigma_{h\omega}\right)dt \\ &\quad +\left(\sigma_h+\sigma_\omega+\sigma_\theta e-\sigma_\delta\right)'dZ\left(t\right) && (13)\\ dw &= \left[rw+\left(1-e-l\right)k-c+w\left(\mu-r1\right)'X\right]dt+wX'\Gamma'dZ\left(t\right) && (14)\\ \text{et } c &\geqslant 0,\ 0\leqslant e,l\leqslant 1 && (15)\end{aligned}$$

4 Optimal investment in human capital

On the basis of this initial stochastic programme, Bellman's optimality principle allows us to express the following equivalences :

[8]It is not essential to integrate human capital into the utility function but this allows us to take into account the non-monetary earnings procured by human capital, in the same way that leisure gives satisfaction to the individuals .

[9]When the individual's planning horizon is not infinite, a bequest function $B[w(T),T]$ may be added to the problem.

[10]In what follows, to simplify, we respectively replace $c\left(t\right),l\left(t\right),k\left(t\right),w\left(t\right)$, and $e\left(t\right)$, by c,l,k,w,e.

$$
\begin{aligned}
V[k,w,t,T] &\equiv Max\ E_t \int_t^T u[c,l,k,t]\,\Delta t + B[w(T),T] &(16)\\
&\equiv Max\ E_t \{\int_t^T u[c,l,k,t]\,\Delta t\\
&\quad + Max\ E_{t+\Delta t}\int_{t+\Delta t}^T u[c,l,k,t]\,\Delta t + B[w(T),T]\}\\
&\equiv Max\ E_t \{\int_t^{t+\Delta t} u[c,l,k,t]\,\Delta t\\
&\quad + V[k,w,t+\Delta t,T]\} &(17)
\end{aligned}
$$

$V[k, w, t, T]$ is called the indirect utility function. It corresponds to the maximum utility level which individuals can expect to obtain over the whole of their life cycle if they make an optimal allocation of their time and wealth to the different activities offered to them. It is presumed to be strictly increasing and concave in k and w.

Bellman's optimality principle allows us to show the present and future effects of the variations of the control variables on the optimal trajectory of the state variables. More specifically, the maximum utility obtained in the interval $[t, T]$ results from a series of choices of control variables e, wX, c and l and the evolution of the state variables k, and w. Consequently, the first term $u[c, l, k, t]\Delta t$ gives the value of the direct effects of the decision made at instant t, while the second $V[k, w, t + \Delta t, T]$ measures the indirect effects, namely the cumulative utility which individuals can obtain in the interval $[t + \Delta t, T]$ given the choice made in the interval $[t, t + \Delta t]$.

In this model, human and financial capital are presumed to be imperfect substitutes, which explains why k and w appear separately in the indirect utility function V. This hypothesis stems from the fact that, unlike financial capital, human capital cannot be freely bought or sold on the market. It is non-commercial and partly irreversible.

In appendix A.2, we show that programme (17) is equivalent to :

$$
\begin{aligned}
0 \equiv\ & Max\ \{u[c,l,k,t] + V_k k\left(\mu_h + \mu_\omega + (\mu_\theta + \sigma_{h\theta} + \sigma_{\theta\omega})\,e - \mu_\delta - \sigma_{\delta h} - \sigma_{\delta\omega} + \sigma_{h\omega}\right)\\
& + V_w\left[rw + (1-e-l)\,k - c + w(\mu - r1)'\,X\right]\\
& + \frac{1}{2}V_{kk}k^2\left(\sigma_h^2 + \sigma_\omega^2 + \sigma_\theta^2 e^2 + \sigma_\delta^2 + 2\sigma_{h\theta}e + 2\sigma_{\theta\omega}e - 2\sigma_{\theta\delta}e + 2\sigma_{h\omega} - 2\sigma_{h\delta} - 2\sigma_{\delta\omega}\right)\\
& + \frac{1}{2}V_{ww}w^2 X'\Sigma X + V_{kw}kw\Sigma_\omega' X + V_t\} \qquad (18)
\end{aligned}
$$

This equation with stochastic partial derivatives is known as a Bellman stochastic optimal control equation or a Hamilton-Jacobi-Bellman equation. Its internal solutions are obtained in the classic way, by positing each of the partial derivatives equal to zero[11]. We thus obtain the optimality conditions which implicitly define the four solutions $c^*(t)$, $l^*(t)$, $wX^*(t)$, and $e^*(t)$.

The first two conditions, for $0 \leq t \leq T$, are immediate :

[11] The subscripts designate the partial derivatives relative to the variables.

$$\frac{u_c}{V_w} = 1 \tag{19}$$

and

$$\frac{u_c}{u_l} = \frac{1}{k} \tag{20}$$

Conditions (19) and (20) are similar to those of models with certainty since the risk-related parameters are not involved in the determination of these two optimality conditions. Furthermore, current consumption and current leisure time are affected in a way which is similar to the certainty case by the increments in financial wealth and human capital. In effect, a marginal increment in financial wealth increases the individual's current consumption and current leisure time[12]. If, like Williams, we assume that the marginal rate of substitution between consumption and leisure is independent of human capital, an increase in current human capital increases the individual's current consumption but reduces current leisure[13]. Indeed, the increment in human capital gives rise to a relative increase in the value of current financial wealth and thus, through (19), increases consumption. However, by increasing the marginal utility of financial wealth, the increment in human capital increases the opportunity cost of leisure and individuals are thus encouraged to reduce their leisure time in favour of work.

The third condition implies that the partial derivative of (18) with respect to wX is equal to 0, which yields :

$$V_w\left(\mu - r1\right) + V_{ww} w X'\Sigma + V_{kw} k \Sigma'_\omega = 0$$

Thus

$$wX(t) = \left(-\frac{V_w}{V_{ww}}\right)\Sigma^{-1}\left(\mu - r1\right) - \frac{V_{kw}}{V_{ww}} k \Sigma^{-1}\Sigma'_\omega \tag{21}$$

Finally, the optimal level of human capital investment is given by :

$$V_k k\left(\mu_\theta + \sigma_{h\theta} + \sigma_{\theta\omega}\right) - V_w k + V_{kk} k^2 \sigma_\theta^2 e - V_{kk} k^2 \sigma_{\theta\delta} + V_{kk} k^2 \sigma_{h\theta} + V_{kk} k^2 \sigma_{\theta\omega} = 0$$

Which, if we rearrange the terms, yields :

$$e(t) = \left(-\frac{V_k}{V_{kk} k}\right)\frac{\left(\mu_\theta + \sigma_{h\theta} + \sigma_{\theta\omega}\right) - \frac{V_w}{V_k}}{\sigma_\theta^2} + \frac{\sigma_{\theta\delta} - \sigma_{h\theta} - \sigma_{\theta\omega}}{\sigma_\theta^2} \tag{22}$$

Condition (21) is similar to the optimality condition of a standard mean-variance portfolio problem. In effect, it establishes that the individual's risky asset portfolio is constituted at the optimum of the two base portfolios. In the general case, the

[12] In fact, if w increases, then, according to (19), $\frac{\partial u_c}{\partial w} = V_{ww}$. Thus $\frac{\partial c}{\partial w} = \frac{V_{ww}}{u_{cc}} > 0$, according to the concavity hypothesis of utlity functions. By applying the same principle, we also shows that leisure increases with financial wealth : $u_l = k u_c = k V_w$. Thus $\frac{\partial u_l}{\partial w} = k V_{ww}$ and $\frac{\partial l}{\partial w} = k \frac{V_{ww}}{u_{ll}} > 0$.

[13] The hypothesis $\frac{\partial}{\partial k}\left(\frac{u_c}{u_l}\right) = 0$ implies that $\frac{\partial c}{\partial k} = -\frac{u_c}{u_l}\frac{u_{cc}}{u_{ll}}\frac{\partial l}{\partial k}$. Utility is a strictly increasing and concave function of c and l, thus : $u_c > 0, u_l > 0, u_{cc} < 0$ and $u_{ll} < 0$. We deduce that $\partial c/\partial k$ and $\partial l/\partial k$ are of opposite sign. However , from (19), $\frac{\partial u_c}{\partial k} = V_{wk}$, thus $\frac{\partial c}{\partial k} = \frac{V_{wk}}{u_{cc}} > 0$. We can conclude that $\frac{\partial l}{\partial k} < 0$.

portfolio $\Sigma^{-1}\Sigma'_\omega$ is composed solely of risky assets and represents the maximum coverage against risk over human capital. If we assume that wage adjustments have no influence on the price of financial assets, this portfolio is then null, since in this case Σ'_ω is null. The other portfolio, $\Sigma^{-1}(\mu - r1)$, is the standard market portfolio.

We may note that the composition of the optimal risky asset portfolio is crucially dependent on the individuals' perception of the risk (Σ), but also on the behaviour adopted in face of the risk. Investment in risky assets is thus proportional to the risk premium $(-\frac{1}{2}\Sigma\frac{V_{ww}}{V_w})$ and inversely proportional to the Arrow-Pratt measure of absolute risk aversion[14] : $(-\frac{V_{ww}}{V_w})$, as well as to the variance of the returns (Σ). It should also be noted that the weight attached to the market portfolio is strictly the inverse of the measure of absolute risk aversion.

Investment in education (22) is directly affected by the parameters associated with the risk inherent in human capital accumulation. In fact, individuals allocate their time to training activities in function of their risk preferences and their judgement about the course of future events. We can observe that investment in human capital is highly dependent on the Arrow-Pratt measure of relative risk aversion $-\frac{V_{kk}k}{V_k}$. In particular, any increase in risk aversion results in a reduction of investment in human capital.

The time allocated to education also depends on the perception of the risk associated with education, as represented by the whole of the instantaneous variances and covariances. It depends directly on the risk associated with learning efficiency, measured by the variance of the marginal product of education σ^2_θ. On the other hand, the risks bearing on the depreciation of existing skills, on wages and on over-education, intervene directly through the intermediary of the covariances $\sigma_{\theta\delta}$,$\sigma_{h\theta}$ and $\sigma_{\theta\omega}$.

The main difference from the Williams (1979) model is that here, the overall effect of the different sources of uncertainty on human capital investment cannot be unambiguously determined, because it depends on the sign and scale of the covariances. In fact, the absence of uncertainty about the distrbution of jobs (over-education risk) and the hypothesis of independence between the risk over wages and risks existing in the training process lead Williams to make the following predictions: the increase of risk over the marginal product of education results in a decrease in human capital investment and earnings over the life cycle. Similarly, assuming a negative relationship between learning efficiency and depreciation ($\sigma_{\theta\delta} < 0$), individuals are encouraged to reduce their investment to protect themselves against an increased risk of skills depreciation.

In our model, these findings remain partly valid but they are insufficient to characterise the overall effect of risk because the risk of over-education in the future job and the risk concerning wages intervene in the determination of $e(t)$. In order to measure the impact of a variation in these two types of risk, it is necessary to specify individual preferences in greater detail, as becomes clear when we observe that the signs of these two effects depend fundamentally on the value of the risk aversion parameter. But this index, constructed from the indirect utility function V, generally absorbs the effect of all the parameters of the model over the time structure of the optimal values. Without additional restrictions on individual preferences, it is not possible to obtain explicit solutions for (18) or to evaluate the impact of the variations in the model's parameters over the life cycle. This is why we specify the individual preferences in greater detail in the following section.

[14]For a detailed presentation of the different measures of risk aversion, *cf.* Laffont (1991).

5 Specification of individual preferences and effects of the different sources of uncertainty

Let us assume that the instantaneous utility function has the following structure[15]:

$$u\left[c\left(t\right),l\left(t\right),k\left(t\right),t\right]=\alpha\left(t\right)\left[c\left(t\right)-\mathbf{c}\left(t\right)\right]^{\gamma}\left[l\left(t\right)k\left(t\right)\right]^{\lambda} \tag{23}$$

with $\alpha\left(t\right),c\left(t\right),\mathbf{c}\left(t\right),l\left(t\right),k\left(t\right)\geqslant 0$, $\gamma,\lambda>0$ and $\gamma+\lambda<1$

$\mathbf{c}\left(t\right)$ represents the individual's minimal indispensable consumption level.

$l\left(t\right)k\left(t\right)$ designates effective leisure (*cf.* Heckman (1976)).

$\alpha\left(t\right)$ is the discount factor, such that the rate of preference for the present $-\frac{\dot{\alpha}(t)}{\alpha(t)}\geqslant 0$

In this case, the bequest function B becomes :

$$B\left[w\left(T\right),T\right]=\alpha\left(T\right)\left[w\left(T\right)-\mathbf{w}\left(T\right)\right]^{\gamma+\lambda} \tag{24}$$

$w\left(t\right)-\mathbf{w}\left(t\right)$ constitutes the individual's effective financial wealth. It corresponds to the difference between current wealth $w(t)$ and the minimum level of wealth $\mathbf{w}\left(t\right)$ needed to satisfy the minimum level of consumption $\mathbf{c}\left(t\right)$. More precisely, $\mathbf{w}\left(t\right)$ corresponds to the actualised sum of minimum consumption and minimum terminal wealth :

$$\mathbf{w}\left(t\right)\equiv\int_{t}^{T}\mathbf{c}\left(\tau\right)e^{-r(\tau-t)}d\tau+\mathbf{w}\left(T\right)e^{-r(T-t)} \tag{25}$$

On the basis of (23) and (24), it is possible to find approximated explicit solutions for the initial problem (12) :[16]:

$$c^{*}\left(t\right)\simeq\mathbf{c}+\left[\frac{\alpha}{\beta}\frac{\lambda^{\lambda}\gamma^{1-\gamma}}{\gamma+\lambda}\right]^{\frac{1}{1-\gamma-\lambda}}\left[\eta k+w-\mathbf{w}\right] \tag{26}$$

$$l^{*}\left(t\right)\simeq\eta\left[\frac{\alpha}{\beta}\frac{\gamma^{\gamma}\lambda^{1-\gamma}}{\gamma+\lambda}\right]^{\frac{1}{1-\gamma-\lambda}}\left[1+\frac{w-\mathbf{w}}{\eta k}\right] \tag{27}$$

$$wX^{*}\left(t\right)\simeq\frac{\Sigma^{-1}\left(\mu-r1\right)}{1-\gamma-\lambda}\left[\eta k+w-\mathbf{w}\right] \tag{28}$$

$$e^{*}\left(t\right)\simeq\frac{\left(\mu_{\theta}+\sigma_{h\theta}+\sigma_{\theta\omega}\right)-1/\eta}{\left(1-\gamma-\lambda\right)\sigma_{\theta}^{2}}\left[1+\frac{w-\mathbf{w}}{\eta k}\right]+\frac{\sigma_{\theta\delta}-\sigma_{h\theta}-\sigma_{\theta\omega}}{\sigma_{\theta}^{2}} \tag{29}$$

With regard to the optimal solutions (26), (27) and (28), we arrive at the results previously set out, identical to those of Williams (1979). A marginal increase in effective financial wealth $w\left(t\right)-\mathbf{w}\left(t\right)$ increases current consumption, current leisure time and current investment in risky assets. An increment in effective human capital

[15] This structure is identicat to that of Williams (1979), itself similar to that of Heckman (1976) in a framework without uncertainty.

[16] The complete resolution is provided in appendix A.3. This problem is subject to constraints (13), (14) and (15), as well as to the following ones.: $\beta\left(T\right)=\alpha\left(T\right)$ and $\eta\left(T\right)=0$. Solutions are computed at time t.

$\eta(t)\,k(t)$[17] increases current consumption and current investment in risky assets but reduces the time devoted to leisure. These results are demonstrated in appendix A.4.

As was previously the case, current investment in risky assets is always reduced when the risk aversion parameter $(1-\gamma-\lambda)$ and the variance of the returns of the risky assets (Σ) increase.

As in the certainty case, an increment in effective human capital reduces the optimal level of human capital investment. On the other hand, any increase in current effective wealth increases the human capital investment and, through equation (9), also increases the mean and variance of earnings over the life cycle. This fundamental property, brought out by Williams (1979) and found here once again, sharply contrasts with the predictions of models without uncertainty. Indeed, in Heckman's model, the structure of which is similar to ours, human capital investment is not affected by variations in financial wealth (*cf.* Heckman 1976, p. 523). As a result, the observed differences in education levels between individuals with different financial resources are commonly attributed to differences in real interest rates. The latter are interpreted as the indirect proof of an imperfection in the capital market. It is likely, as Becker (1993) already pointed out, that this heterogeneity in individual interest rates exists and that it increases the financial constraint of the least wealthy individuals. In our case, however, the heterogeneity of the opportunity cost of education does not stem from the heterogeneity of the interest rate. Other things being equal, individuals with limited financial resources attain levels of education, and thus of wages, which are lower than those of those with greater financial resources because they are confronted with a higher opportunity cost of human capital. This theoretical result, which is fairly realistic, is obtained independently of any imperfection of the capital market.

As emphasised by Williams (1979), variations in effective financial wealth or the ratio of effective financial wealth to effective human capital have an effect not only on the current values of optimal solutions but also on the time structure of these variables. As indicated in appendix A.4, any increase in effective wealth increases consumption and leisure in the current period, but also in future periods if the rate of time preference is sufficiently low. On the other hand, variations in effective financial wealth and effective human capital have an effect on investment in risky assets only in the current period. The proportion of income invested in risky assets is independent of the individual's time horizon. Finally, the increase of the finance capital / human capital ratio increases current human capital investment. This effect is partially compensated over the individual's life cycle because the marginal rate of substitution between human capital and financial wealth decreases over time.

When individual preferences are specified, the effect of certain risk parameters on human capital investment is identical to the case where the utility function is defined implicitly. In particular, increments in the relative risk aversion index $(1-\gamma-\lambda)$, and in the risk over the depreciation of human capital $(\sigma_{\delta\theta})$ reduce current investment in education.

On the other hand, if we assume that individuals with a high marginal product of education have easier access to jobs and wages better corresponding to their education level, in other words, if $\sigma_{h\theta} > 0$ and $\sigma_{\omega\theta} > 0$, then an increase in the risk over the labour market leads to an increase in human capital investment. Indeed, any increase in the risk over future over-education, reflected in a positive increment in $\sigma_{h\theta}$, raises the level of human capital investment. Thus,

[17] Effective human capital corresponds to the current stock of human capital weighted by the marginal rate of substitution between human capital and financial capital (η).

$$\frac{\partial e(t)}{\partial \sigma_{h\theta}} = \frac{1}{(1-\gamma-\lambda)\,\sigma_\theta^2}\left[1+\frac{w-\mathbf{w}}{\eta k}\right]-\frac{1}{\sigma_\theta^2}$$
$$= \frac{1}{(1-\gamma-\lambda)\,\sigma_\theta^2}\left[\frac{w-\mathbf{w}}{\eta k}+\gamma+\lambda\right]>0$$

In the same way, this model predict that individuals will increase their human capital investment in response to an increased risk on wages (the effect is identical to the preceding one).

Finally, contrary to Williams (1979) an increase of the risk bearing on the efficiency of education : σ_θ^2 don't necessarily reduce investment in human capital.. Two contradictory effects are produced : a negative direct effect through σ_θ^2, which is located at the denominator of (29), and a positive indirect effect through $\sigma_{h\theta}$ and $\sigma_{\omega\theta}$.

The economic interpretation of these results is relatively intuitive : The increase in the risks bearing on the process of human capital accumulation (efficiency of training and depreciation of existing skills) increases the possibilitiy that investment in human capital doesn't become profitable, and so, increases the opportunity cost of investment. Conversely, the increase in the risks bearing on the labour market reduces the opportunity cost of the investment, and is thus an incentive for the individual to increase his investment in human capital.

These novel results are in sharp contrast with the clear-cut conclusions of Williams (1979) concerning the effect of risk on human capital investment. The extension which we propose here permits a better understanding of why the uncertainty effect, when it is taken into account in an aggregate form, is difficult to establish without ambiguity, namely because the different sources of uncertainty have contradictory effects on investment. The uncertainty about the process of human capital accumulation exercises a negative direct effect on the human capital investment while the risk over the labour market indirectly encourages that investment. The perception individuals have of these different risks, expressed by the extent of the different covariances, is fundamental in the decision to invest in human capital. When individuals decide to invest in education, if they accord greater weight to the risk of future over-education or the risk over their future wage rate than to the risks associated with their learning process, they will, in order to protect themselves, increase their demand for education. Our model thus offers a theoretical response to the phenomenon of continued studies and more generally to the increase in the demand for education which has been observed in Western economies over the past few decades.

6 Conclusion

This study has as its starting point a critique of the standard human capital model, which is no longer capable of describing a certain number of realities observed in the majority of developed countries. The existence (to varying degrees depending on the country and the immediate economic situation) of the phenomena of over-education and unemployment, which have been particularly marked among young people for several years, no longer allows decisions in the area of educational policy to be based on a model assuming that individuals have a perfect knowledge of their future situation when they make their decisions about education.

In order to go beyond this critique, we have presented a theoretical model which situates the analysis of the demand for education in a context of uncertainty. The hypothesis defended in this study is that the effect of uncertainty cannot be determined ex ante without ambiguity if the different sources of uncertainty, bearing at once on the learning process and the future labour-market situation, are not identified in detail. For this reason, we have proposed an extension of the Williams (1979) model in which five sources of uncertainty are specified.

To reflect the problems observed on the European labour markets, we have integrated the risk of future over-education . In addition, by raising the hypothesis of independence between variables related to the labour market and those defining the process of human capital accumulation, we have been able to show the fundamental role of individuals' perceptions of the future conditions of wage and employment on the labour market. We have shown that the weight accorded to the different sources of uncertainty is determinant in the decision to invest in human capital, because the uncertainty stemming from the labour market and that stemming from the learning process exercise contradictory effects on investment. In particular, education plays a protective role against the risks existing on the labour market and may be encouraged if the risks associated with the learning process are low, or at least if they are perceived as such. This conclusion at the individual level, is also true from a social welfare point of view : education has to be promoted by public autorities. As Anderberg and Anderson (2003) have recently shown, human capital investment has an insurance effect that reduce the complex macroeconomic problem of optimal taxation.

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A Appendix : Resolution of the stochastic optimal control problem using continuous time dynamic programming techniques

A.1 Proof of the stochastic equation (9) of human capital accumulation:

The second order Taylor expansion of equation (4) has the following form :

$$\begin{aligned}
k(t+\Delta t) &= k\left(t\right)+\frac{\partial k(t+\Delta t)}{\partial t}\Delta t+\frac{\partial k(t+\Delta t)}{\partial h(t+\Delta t)}\Delta h+\frac{\partial k(t+\Delta t)}{\partial \omega(t+\Delta t)}\Delta \omega+\frac{\partial k(t+\Delta t)}{\partial \theta(t,t+\Delta t)}\Delta \theta \\
&+\frac{\partial k(t+\Delta t)}{\partial \delta(t,t+\Delta t)}\Delta \delta+\frac{1}{2}\frac{\partial^2 k(t+\Delta t)}{\partial h^2(t+\Delta t)}\left(\Delta h\right)^2+\frac{1}{2}\frac{\partial^2 k(t+\Delta t)}{\partial \omega^2(t+\Delta t)}\left(\Delta \omega\right)^2 \\
&+\frac{1}{2}\frac{\partial^2 k(t+\Delta t)}{\partial \theta^2(t,t+\Delta t)}\left(\Delta \theta\right)^2+\frac{1}{2}\frac{\partial^2 k(t+\Delta t)}{\partial \delta^2(t,t+\Delta t)}\left(\Delta \delta\right)^2+\frac{\partial^2 k(t+\Delta t)}{\partial h(t+\Delta t)\partial \omega(t+\Delta t)}\left(\Delta h\Delta \omega\right) \\
&+\frac{\partial^2 k(t+\Delta t)}{\partial h(t+\Delta t)\partial \theta(t,t+\Delta t)}\left(\Delta h\Delta \theta\right)+\frac{\partial^2 k(t+\Delta t)}{\partial h(t+\Delta t)\partial \delta(t,t+\Delta t)}\left(\Delta h\Delta \delta\right) \\
&+\frac{\partial^2 k(t+\Delta t)}{\partial \omega(t+\Delta t)\partial \theta(t,t+\Delta t)}\left(\Delta \omega\Delta \theta\right)+\frac{\partial^2 k(t+\Delta t)}{\partial \omega(t+\Delta t)\partial \delta(t,t+\Delta t)}\left(\Delta \omega\Delta \delta\right) \\
&+\frac{\partial^2 k(t+\Delta t)}{\partial \theta(t,t+\Delta t)\partial \delta(t,t+\Delta t)}\left(\Delta \theta\Delta \delta\right)+\frac{\partial^2 k(t+\Delta t)}{\partial t^2}\left(\Delta t\right)^2+\frac{\partial^2 k(t+\Delta t)}{\partial t\partial h(t+\Delta t)}\left(\Delta t\Delta h\right) \\
&+\frac{\partial^2 k(t+\Delta t)}{\partial t\partial \omega(t+\Delta t)}\left(\Delta t\Delta \omega\right)+\frac{\partial^2 k(t+\Delta t)}{\partial t\partial \theta(t,t+\Delta t)}\left(\Delta t\Delta \theta\right)+\frac{\partial^2 k(t+\Delta t)}{\partial t\partial \delta(t,t+\Delta t)}\left(\Delta t\Delta \delta\right)+R\left(\Delta t\right)
\end{aligned}$$

By noting that terms in Δt of order 2 and beyond are infinitely "small" compared with those in Δt, they will be systematically neglected in calculations. The last six terms thus disappear from the above development. While also noting that $\frac{\partial^2 k(t+\Delta t)}{\partial h^2}=\frac{\partial^2 k(t+\Delta t)}{\partial \omega^2}=\frac{\partial^2 k(t+\Delta t)}{\partial \theta^2}=\frac{\partial^2 k(t+\Delta t)}{\partial \delta^2}=0$, the Taylor series can be rewritten in the following way :

$$\begin{aligned}
k(t+\Delta t) &= k\left(t\right)+\frac{\partial k(t+\Delta t)}{\partial t}\Delta t+\frac{\partial k(t+\Delta t)}{\partial h\left(t+\Delta t\right)}\Delta h+\frac{\partial k(t+\Delta t)}{\partial \omega\left(t+\Delta t\right)}\Delta \omega \\
&+\frac{\partial k(t+\Delta t)}{\partial \theta(t,t+\Delta t)}\Delta \theta+\frac{\partial k(t+\Delta t)}{\partial \delta(t,t+\Delta t)}\Delta \delta+\frac{\partial^2 k(t+\Delta t)}{\partial h(t+\Delta t)\partial \omega(t+\Delta t)}\left(\Delta h\Delta \omega\right) \\
&+\frac{\partial^2 k(t+\Delta t)}{\partial h(t+\Delta t)\partial \theta(t,t+\Delta t)}\left(\Delta h\Delta \theta\right)+\frac{\partial^2 k(t+\Delta t)}{\partial h(t+\Delta t)\partial \delta(t,t+\Delta t)}\left(\Delta h\Delta \delta\right) \\
&+\frac{\partial^2 k(t+\Delta t)}{\partial \omega(t+\Delta t)\partial \theta(t,t+\Delta t)}\left(\Delta \omega\Delta \theta\right)+\frac{\partial^2 k(t+\Delta t)}{\partial \omega(t+\Delta t)\partial \delta(t,t+\Delta t)}\left(\Delta \omega\Delta \delta\right) \\
&+\frac{\partial^2 k(t+\Delta t)}{\partial \theta(t,t+\Delta t)\partial \delta(t,t+\Delta t)}\left(\Delta \theta\Delta \delta\right)
\end{aligned}$$

The increases in the stochastic processes studied are supposed to be stationary, which implies that $\frac{\partial k(t+\Delta t)}{\partial t}\Delta t=0$. The calculation of partial derivatives and the substitution of $\Delta h, \Delta \omega, \Delta \theta$ and $\Delta \delta$ by their respective value (5), (6), (7), (8) give :

$$
\begin{aligned}
k(t+\Delta t)-k\left(t\right) &= \frac{1}{h\left(t\right)}\frac{\omega\left(t+\Delta t\right)}{\omega\left(t\right)}\left[1-\delta(t,t+\Delta t)+\theta(t,t+\Delta t)e\left(t\right)\right]k\left(t\right)h\left(t\right)\left(\mu_h\Delta t+\sigma_h'\Delta Z\left(t\right)\right) \\
&+\frac{h\left(t+\Delta t\right)}{h\left(t\right)}\frac{1}{\omega\left(t\right)}\left[1-\delta(t,t+\Delta t)+\theta(t,t+\Delta t)e\left(t\right)\right]k\left(t\right)\omega\left(t\right)\left(\mu_\omega\Delta t+\sigma_\omega'\Delta Z\left(t\right)\right) \\
&+\frac{h\left(t+\Delta t\right)}{h\left(t\right)}\frac{\omega(t+\Delta t)}{\omega\left(t\right)}e\left(t\right)k\left(t\right)\left(\mu_\theta\Delta t+\sigma_\theta'\Delta Z\left(t\right)\right) \\
&-\frac{h\left(t+\Delta t\right)}{h\left(t\right)}\frac{\omega(t+\Delta t)}{\omega\left(t\right)}k\left(t\right)\left(\mu_\delta\Delta t+\sigma_\delta'\Delta Z\left(t\right)\right) \\
&+\frac{1}{h\left(t\right)}\frac{1}{\omega\left(t\right)}\left[1-\delta(t,t+\Delta t)+\theta(t,t+\Delta t)e\left(t\right)\right]k\left(t\right)h\left(t\right)\omega\left(t\right) \\
&\times\left(\mu_h\Delta t+\sigma_h'\Delta Z\left(t\right)\right)\left(\mu_\omega\Delta t+\sigma_\omega'\Delta Z\left(t\right)\right) \\
&+\frac{1}{h\left(t\right)}\frac{\omega(t+\Delta t)}{\omega\left(t\right)}e\left(t\right)\left[1-\delta(t,t+\Delta t)+\theta(t,t+\Delta t)e\left(t\right)\right]k\left(t\right)h\left(t\right) \\
&\times\left(\mu_h\Delta t+\sigma_h'\Delta Z\left(t\right)\right)\left(\mu_\theta\Delta t+\sigma_\theta'\Delta Z\left(t\right)\right) \\
&-\frac{1}{h\left(t\right)}\frac{\omega(t+\Delta t)}{\omega\left(t\right)}k\left(t\right)h\left(t\right)\left(\mu_h\Delta t+\sigma_h'\Delta Z\left(t\right)\right)\left(\mu_\delta\Delta t+\sigma_\delta'\Delta Z\left(t\right)\right) \\
&+\frac{h\left(t+\Delta t\right)}{h\left(t\right)}\frac{1}{\omega\left(t\right)}e\left(t\right)k\left(t\right)\omega\left(t\right)\left(\mu_\omega\Delta t+\sigma_\omega'\Delta Z\left(t\right)\right)\left(\mu_\theta\Delta t+\sigma_\theta'\Delta Z\left(t\right)\right) \\
&-\frac{h\left(t+\Delta t\right)}{h\left(t\right)}\frac{1}{\omega\left(t\right)}k\left(t\right)\omega\left(t\right)\left(\mu_\omega\Delta t+\sigma_\omega'\Delta Z\left(t\right)\right)\left(\mu_\delta\Delta t+\sigma_\delta'\Delta Z\left(t\right)\right)
\end{aligned}
$$

By simplifying by $h\left(t\right)$, at the first, fifth, seventh and at the ninth line and by $\omega\left(t\right)$, at the second, fifth, tenth and at the eleventh line; by putting $k\left(t\right)$ in factor and passing it from the left side of the equation; and by developing the last five terms, one obtains :

$$
\begin{aligned}
\frac{\Delta k\left(t\right)}{k\left(t\right)} &= \frac{\omega\left(t+\Delta t\right)}{\omega\left(t\right)}\left[1-\delta(t,t+\Delta t)+\theta(t,t+\Delta t)e\left(t\right)\right]\left(\mu_h\Delta t+\sigma_h'\Delta Z\left(t\right)\right) \\
&+\frac{h\left(t+\Delta t\right)}{h\left(t\right)}\left[1-\delta(t,t+\Delta t)+\theta(t,t+\Delta t)e\left(t\right)\right]\left(\mu_\omega\Delta t+\sigma_\omega'\Delta Z\left(t\right)\right) \\
&+\frac{h\left(t+\Delta t\right)}{h\left(t\right)}\frac{\omega(t+\Delta t)}{\omega\left(t\right)}e\left(t\right)\left(\mu_\theta\Delta t+\sigma_\theta'\Delta Z\left(t\right)\right) \\
&-\frac{h\left(t+\Delta t\right)}{h\left(t\right)}\frac{\omega(t+\Delta t)}{\omega\left(t\right)}\left(\mu_\delta\Delta t+\sigma_\delta'\Delta Z\left(t\right)\right) \\
&+\left[1-\delta(t,t+\Delta t)+\theta(t,t+\Delta t)e\left(t\right)\right]\times \\
&\left[\mu_h\mu_\omega\left(\Delta t\right)^2+\mu_h\sigma_\omega'\Delta Z\left(t\right)\Delta t+\mu_\omega\sigma_h'\Delta Z\left(t\right)\Delta t+\rho_{h\omega}\sigma_h'\sigma_\omega\left(\Delta Z\left(t\right)\right)^2\right] \\
&+\frac{\omega(t+\Delta t)}{\omega\left(t\right)}e\left(t\right)\left[1-\delta(t+\Delta t)+\theta(t+\Delta t)e\left(t\right)\right]\times \\
&\left[\mu_h\mu_\theta\left(\Delta t\right)^2+\mu_h\sigma_\theta'\Delta Z\left(t\right)\Delta t+\mu_\theta\sigma_\omega'\Delta Z\left(t\right)\Delta t+\rho_{h\theta}\sigma_h'\sigma_\theta\left(\Delta Z\left(t\right)\right)^2\right] \\
&-\frac{\omega(t+\Delta t)}{\omega\left(t\right)}\left[\mu_h\mu_\delta\left(\Delta t\right)^2+\mu_h\sigma_\delta'\Delta Z\left(t\right)\Delta t+\mu_\delta\sigma_h'\Delta Z\left(t\right)\Delta t+\rho_{h\delta}\sigma_h'\sigma_\delta\left(\Delta Z\left(t\right)\right)^2\right] \\
&+\frac{h\left(t+\Delta t\right)}{h\left(t\right)}e(t)\left[\mu_\omega\mu_\theta\left(\Delta t\right)^2+\mu_\omega\sigma_\theta'\Delta Z\left(t\right)\Delta t+\mu_\theta\sigma_\omega'\Delta Z\left(t\right)\Delta t+\rho_{\omega\theta}\sigma_\omega'\sigma_\theta\left(\Delta Z\left(t\right)\right)^2\right] \\
&-\frac{h\left(t+\Delta t\right)}{h\left(t\right)}\left[\mu_\omega\mu_\delta\left(\Delta t\right)^2+\mu_\omega\sigma_\delta'\Delta Z\left(t\right)\Delta t+\mu_\delta\sigma_\omega'\Delta Z\left(t\right)\Delta t+\rho_{\delta\omega}\sigma_\delta'\sigma_\omega\left(\Delta Z\left(t\right)\right)^2\right]
\end{aligned}
$$

where ρ denote the instantaneous correlation between stochastic processes.

When Δt tends to 0 : $\frac{h(t+\Delta t)}{h(t)} = \frac{\omega(t+\Delta t)}{\omega(t)} = 1$ and
$[1 - \delta(t, t+\Delta t) + \theta(t, t+\Delta t) e\,(t)] = 1$ from equation (23), thus :

$$\begin{aligned}
\frac{\Delta k\,(t)}{k\,(t)} &= (\mu_h dt + \sigma_h' dZ\,(t)) + (\mu_\omega dt + \sigma_\omega' dZ\,(t)) \\
&+ e\,(t)\,(\mu_\theta dt + \sigma_\theta' dZ\,(t)) - (\mu_\delta dt + \sigma_\delta' dZ\,(t)) \\
&+ \left[\mu_h \mu_\omega\,(dt)^2 + \mu_h \sigma_\omega' dZ\,(t)\,dt + \mu_\omega \sigma_h' dZ\,(t)\,dt + \rho_{h\omega}\sigma_h' \sigma_\omega\,(dZ\,(t))^2\right] \\
&+ e\,(t)\left[\mu_h \mu_\theta\,(dt)^2 + \mu_h \sigma_\theta' dZ\,(t)\,dt + \mu_\theta \sigma_h' dZ\,(t)\,dt + \rho_{h\theta}\sigma_h' \sigma_\theta\,(dZ\,(t))^2\right] \\
&- \left[\mu_h \mu_\delta\,(dt)^2 + \mu_h \gamma_\delta' dZ\,(t)\,dt + \mu_\delta \gamma_h' dZ\,(t)\,dt + \rho_{h\delta}\sigma_h' \sigma_\delta\,(dZ\,(t))^2\right] \\
&+ e(t)\left[\mu_\omega \mu_\theta\,(dt)^2 + \mu_\omega \sigma_\theta' dZ\,(t)\,dt + \mu_\theta \sigma_\omega' dZ\,(t)\,dt + \rho_{\omega\theta}\sigma_\omega' \sigma_\theta\,(dZ\,(t))^2\right] \\
&- \left[\mu_\omega \mu_\delta\,(dt)^2 + \mu_\omega \sigma_\delta' dZ\,(t)\,dt + \mu_\delta \sigma_\omega' dZ\,(t)\,dt + \rho_{\delta\omega}\sigma_\delta' \sigma_\omega\,(dZ\,(t))^2\right]
\end{aligned}$$

Finally, from properties of the standard Wiener process: $(dt)^2 = dZ\,(t)\,dt = o\,(dt)$ and $(dZ\,(t))^2 = dt + o\,(dt)$. When rearranging terms, we find the final expression of the stochastic equation of human capital accumulation (9) :

$$\begin{aligned}
\frac{dk\,(t)}{k\,(t)} &= (\mu_h + \mu_\omega + (\mu_\theta + \sigma_{h\theta} + \sigma_{\theta\omega})\,e\,(t) - \mu_\delta - \sigma_{h\delta} - \sigma_{\delta\omega} + \sigma_{h\omega})\,dt \\
&+ (\sigma_h + \sigma_\omega + \sigma_\theta e\,(t) - \sigma_\delta)'\,dZ\,(t)
\end{aligned}$$

where

$$\begin{aligned}
\sigma_{h\theta} &= \rho_{h\theta}\sigma_h' \sigma_\theta = Cov\,(h, \theta) \\
\sigma_{\theta\omega} &= \rho_{\omega\theta}\sigma_\theta' \sigma_\omega = Cov\,(\omega, \theta) \\
\sigma_{h\delta} &= \rho_{h\delta}\sigma_h' \sigma_\delta = Cov\,(h, \delta) \\
\sigma_{\delta\omega} &= \rho_{\delta\omega}\sigma_\delta' \sigma_\omega = Cov\,(\delta, \omega) \\
\sigma_{h\omega} &= \rho_{h\omega}\sigma_h' \sigma_\omega = Cov\,(h, \omega)
\end{aligned}$$

A.2 Proof of equation (18)

The first term of (17) in the square brackets can be approximated by $u\,[c, l, k, t]\,\Delta t$. Applying to $V\,[k, w, t+\Delta t, T]$ the second order Taylor expansion, and neglecting terms of higher order [18], then we obtain :

$$\begin{aligned}
&V\,[k\,(t)\,, w\,(t)\,, T] \equiv Max\ E_t\ \{u\,[c, l, k, t]\,\Delta t + V\,[k, w, T] \\
&+ V_t \Delta t + V_k \Delta k + V_w \Delta w + \frac{1}{2}\,(\Delta k)'\,V_{kk}\Delta k + \frac{1}{2}\,(\Delta w)'\,V_{ww}\Delta w + V_{kw}\,(\Delta k)'\,\Delta w\}
\end{aligned}$$

[18] That is the sum of terms in $(dt)^\alpha$ with $\alpha > 1$.

By simplifing by $V[k, w, T]$ in each member of the equation and applying Itô's lemma when Δt tends to 0, we find :

$$\begin{aligned}0 &\equiv Max\ E_t\ \{u[c,l,k,t]\,dt \\ &+ \left[V_t + \mu_k V_k + \mu_w V_w + \frac{1}{2}\sigma_k^2 V_{kk} + \frac{1}{2}\sigma_w^2 V_{ww} + \sigma_k \sigma_w V_{kw}\right] dt \\ &+ [\sigma_k V_k + \sigma_w V_w]\, dZ(t)\}\end{aligned} \tag{30}$$

If we define

$$\begin{aligned}dV &= \left[V_t + \mu_k V_k + \mu_w V_w + \frac{1}{2}\sigma_k^2 V_{kk} + \frac{1}{2}\sigma_w^2 V_{ww} + \sigma_k \sigma_w V_{kw}\right] dt \\ &\quad + [\sigma_k V_k + \sigma_w V_w]\, dZ(t)\}\end{aligned} \tag{31}$$

and expand the conditional expectation operator, we obtain the following *stochastic partial derivative equation* (SPDE) :

$$0 \equiv Max\ \{E_t\ u[c,l,k,t]\,dt + E_t\ dV\} \tag{32}$$

This equation can be simplified because Z is a standard Wiener process. Thus $E_t[dZ(t)] = 0$ and $E_t[\sigma_k V_k + \sigma_w V_w]\,dZ(t) = 0$. We can then write :

$$E_t dV = \left[V_t + \mu_k V_k + \mu_w V_w + \frac{1}{2}\sigma_k^2 V_{kk} + \frac{1}{2}\sigma_w^2 V_{ww} + \sigma_k \sigma_w V_{kw}\right] dt \tag{33}$$

By using this equation (33) and the equation

$$E_t u[c,l,k,t] = u[c,l,k,t] \tag{34}$$

one obtains an equation equivalent to (32), which is written :

$$\begin{aligned}0 &\equiv Max\ \{u[c,l,k,t]\,dt \\ &\quad + [V_t + \mu_k V_k + \mu_w V_w + \frac{1}{2}\sigma_k^2 V_{kk} + \frac{1}{2}\sigma_w^2 V_{ww} + \sigma_k \sigma_w V_{kw}]dt\}\end{aligned} \tag{35}$$

Parameters associed with Itô's process are as follows :

$$\begin{aligned}\mu_k &= (\mu_h + \mu_\omega + (\mu_\theta + \sigma_{h\theta} + \sigma_{\theta\omega})\,e - \mu_\delta - \sigma_{h\delta} - \sigma_{\delta\omega} + \sigma_{h\omega}) \\ \mu_w &= rw - c + (1 - e - l)\,k + w\,(\mu - r1)'\,X \\ \sigma_k^2 &= \sigma_h^2 + \sigma_\omega^2 + \sigma_\theta^2 e^2 + \sigma_\delta^2 + 2\sigma_{h\theta} e + 2\sigma_{\theta\omega} e - 2\sigma_{\theta\delta} e + 2\sigma_{h\omega} - 2\sigma_{h\delta} - 2\sigma_{\delta\omega} \\ \text{since } \sigma_k &= (\sigma_h + \sigma_\omega + \sigma_\theta e - \sigma_\delta)' \\ \sigma_w^2 &= w^2 X' \Sigma X \text{ since } \sigma_w = wX'\Gamma' \\ \sigma_k \sigma_w &= kw\Sigma_\omega' X\end{aligned}$$

By dividing the two members of (35) by dt and replacing μ_k, μ_w, σ_k^2, σ_w^2, and $\sigma_k \sigma_w$ by their above expression, one obtains the final version of the initial maximization program (12), which corresponds to equation (18) in the text:

$$
\begin{aligned}
0 \equiv\ & Max\ \{u\left[c,l,k,t\right] + V_k k\left(\mu_h + \mu_\omega + \left(\mu_\theta + \sigma_{h\theta} + \sigma_{\theta\omega}\right)e - \mu_\delta - \sigma_{\delta h} - \sigma_{\delta\omega} + \sigma_{h\omega}\right) \\
& + V_w\left[rw + (1-e-l)k - c + w\left(\mu - r1\right)' X\right] \\
& + \frac{1}{2}V_{kk}k^2\left(\sigma_h^2 + \sigma_\omega^2 + \sigma_\theta^2 e^2 + \sigma_\delta^2 + 2\sigma_{h\theta}e + 2\sigma_{\theta\omega}e - 2\sigma_{\theta\delta}e + 2\sigma_{h\omega} - 2\sigma_{h\delta} - 2\sigma_{\delta\omega}\right) \\
& + \frac{1}{2}V_{ww}w^2 X'\Sigma X + V_{kw}kw\Sigma'_\omega X + V_t\}
\end{aligned}
$$

A.3 Explicit solutions

Knowing the form of the utility function, we recompute the optimality conditions from this function and we replace them in equation (18). We then obtain a new form of the Hamilton-Jacobi-Bellman equation, from which one defines the explicit solutions of the stochastic optimization problem :

$$
\begin{aligned}
0 =\ & u\left(1-\gamma-\lambda\right) - V_w\mathbf{c} + V_k k\left(\mu_h + \mu_\omega - \mu_\delta - \sigma_{h\delta} - \sigma_{\delta\omega} + \sigma_{h\omega}\right) + V_w\left(k + rw\right) \\
& + \frac{1}{2}V_{kk}k^2\left(\sigma_h^2 + \sigma_\omega^2 + \sigma_\delta^2 + 2\sigma_{h\omega} - 2\sigma_{h\delta} - 2\sigma_{\delta\omega}\right) \\
& + \frac{V_{kk}k\left(\sigma_{\theta\delta} - \sigma_{h\theta} - \sigma_{\theta\omega}\right) + V_w - V_k\left(\mu_\theta + \sigma_{h\theta} + \sigma_{\theta\omega}\right)}{V_{kk}\sigma_\theta^2} \times \\
& \{V_k\left(\mu_\theta + \sigma_{h\theta} + \sigma_{\theta\omega}\right) - V_w - V_{kk}\left(\sigma_{\theta\delta} - \sigma_{h\theta} - \sigma_{\theta\omega}\right) \\
& + \frac{1}{2}\left[V_{kk}k\left(\sigma_{\theta\delta} - \sigma_{h\theta} - \sigma_{\theta\omega}\right) + V_w - V_k\left(\mu_\theta + \sigma_{h\theta} + \sigma_{\theta\omega}\right)\right]\} \\
& - \frac{V_w\left(\mu - r1\right) + V_{kw}k\Sigma'_\omega}{V_{ww}\Sigma} \times \\
& \left(V_w\left(\mu - r1\right) - V_{kw}k\Sigma'_\omega + \frac{1}{2}\left(V_w\left(\mu - r1\right) + V_{kw}k\Sigma'_\omega\right)\right) + V_t
\end{aligned} \tag{36}
$$

By observing that "$V_k\left(\mu_\theta + \sigma_{h\theta} + \sigma_{\theta\omega}\right)$" and "$V_w$" disappear inside the second bracket, equation (36) can be simplified in the following way:

$$
\begin{aligned}
0 =\ & u\left(1-\gamma-\lambda\right) - V_w\mathbf{c} + V_t + V_k k\left(\mu_h + \mu_\omega - \mu_\delta - \sigma_{h\delta} - \sigma_{\delta\omega} + \sigma_{h\omega}\right) + V_w\left(k + rw\right) \\
& + \frac{1}{2}V_{kk}k^2\left(\sigma_h^2 + \sigma_\omega^2 + \sigma_\delta^2 + 2\sigma_{h\omega} - 2\sigma_{h\delta} - 2\sigma_{\delta\omega}\right) \\
& - \frac{1}{2}\frac{\left[V_k\left(\mu_\theta + \sigma_{h\theta} + \sigma_{\theta\omega}\right) - V_w - V_{kk}k\left(\sigma_{\theta\delta} - \sigma_{\theta\omega}\right)\right]^2}{V_{kk}\sigma_\theta^2} \\
& - \frac{V_w\left(\mu - r1\right) + V_{kw}k\Sigma'_\omega}{V_{ww}\Sigma}\left(V_w\left(\mu - r1\right) - V_{kw}k\Sigma'_\omega + \frac{1}{2}\left(V_w\left(\mu - r1\right) + V_{kw}k\Sigma'_\omega\right)\right)
\end{aligned} \tag{37}
$$

The development of the last bracket gives :

$$
\begin{aligned}
0 \quad = \quad & u\left(1-\gamma-\lambda\right)-V_w\mathbf{c}+V_t+V_kk\left(\mu_h+\mu_\omega-\mu_\delta-\sigma_{h\delta}-\sigma_{\delta\omega}+\sigma_{h\omega}\right)+V_w\left(k+rw\right)\\
&+\frac{1}{2}V_{kk}k^2\left(\sigma_h^2+\sigma_\omega^2+\sigma_\delta^2+2\sigma_{h\omega}-2\sigma_{h\delta}-2\sigma_{\delta\omega}\right)\\
&-\frac{1}{2}\frac{\left[V_k\left(\mu_\theta+\sigma_{h\theta}+\sigma_{\theta\omega}\right)-V_w-V_{kk}k\left(\sigma_{\theta\delta}-\sigma_{h\theta}-\sigma_{\theta\omega}\right)\right]^2}{V_{kk}\sigma_\theta^2} \qquad (38)\\
&+V_{kw}k\Sigma_\omega'\left(\frac{-V_w\left(\mu-r1\right)-V_{kw}k\Sigma_\omega'}{V_{ww}\Sigma}\right)+\frac{1}{2}V_{ww}\Sigma\left(\frac{-V_w\left(\mu-r1\right)-V_{kw}k\Sigma_\omega'}{V_{ww}\Sigma}\right)^2
\end{aligned}
$$

Factorizing terms into brackets and simplifying, one obtains:

$$
\begin{aligned}
0 \quad = \quad & u\left(1-\gamma-\lambda\right)-V_w\mathbf{c}+V_kk\left(\mu_h+\mu_\omega-\mu_\delta-\sigma_{h\delta}-\sigma_{\delta\omega}+\sigma_{h\omega}\right)+V_w\left(k+rw\right)\\
&+\frac{1}{2}V_{kk}k^2\left(\sigma_h^2+\sigma_\omega^2+\sigma_\delta^2+2\sigma_{h\omega}-2\sigma_{h\delta}-2\sigma_{\delta\omega}\right)\\
&-\frac{1}{2}\frac{\left[V_k\left(\mu_\theta+\sigma_{h\theta}+\sigma_{\theta\omega}\right)-V_w-V_{kk}k\left(\sigma_{\theta\delta}-\sigma_{h\theta}-\sigma_{\theta\omega}\right)\right]^2}{V_{kk}\sigma_\theta^2} \qquad (39)\\
&+\frac{-V_w\left(\mu-r1\right)-V_{kw}k\Sigma_\omega'}{V_{ww}\Sigma}\left(V_{kw}k\Sigma_\omega'-\frac{1}{2}V_w\left(\mu-r1\right)-\frac{1}{2}V_{kw}k\Sigma_\omega'\right)+V_t
\end{aligned}
$$

By simplifying inside the last bracket and arranging the terms, we find the final expression of the Bellman equation:

$$
\begin{aligned}
0 \quad = \quad & u\left(1-\gamma-\lambda\right)-V_w\mathbf{c}-\frac{1}{2}\frac{\left[V_k\left(\mu_\theta+\sigma_{h\theta}+\sigma_{\theta\omega}\right)-V_w-V_{kk}k\left(\sigma_{\theta\delta}-\sigma_{h\theta}-\sigma_{\theta\omega}\right)\right]^2}{V_{kk}\sigma_\theta^2}\\
&-\frac{1}{2}\left\{\left[V_w\left(\mu-r1\right)+V_{kw}k\Sigma_\omega'\right]'\Sigma^{-1}\left[V_w\left(\mu-r1\right)+V_{kw}k\Sigma_\omega'\right]\right\}/V_{ww}\\
&+V_kk\left(\mu_h+\mu_\omega-\mu_\delta-\sigma_{h\delta}-\sigma_{\delta\omega}+\sigma_{h\omega}\right)+V_w\left(k+rw\right)\\
&+\frac{1}{2}V_{kk}k^2\left(\sigma_h^2+\sigma_\omega^2+\sigma_\delta^2+2\sigma_{h\omega}-2\sigma_{h\delta}-2\sigma_{\delta\omega}\right)+V_t \qquad (40)
\end{aligned}
$$

Given the specification of the utility function, the equation above admits an approximate solution:

$$
V\left[k\left(t\right),w\left(t\right),t\right]\simeq\beta\left(t\right)\left[\eta\left(t\right)k\left(t\right)+w\left(t\right)-\mathbf{w}\left(t\right)\right]^{\gamma+\lambda} \qquad (41)
$$

It is then enough to use the following expressions to obtain the explicit solutions of the starting problem (12):

$$
\begin{aligned}
V_k &\simeq \beta\left(t\right)\left(\gamma+\lambda\right)\eta\left(t\right)\left[\eta\left(t\right)k\left(t\right)+w\left(t\right)-\mathbf{w}\left(t\right)\right]^{\gamma+\lambda-1} && (42)\\
V_w &\simeq \beta\left(t\right)\left(\gamma+\lambda\right)\left[\eta\left(t\right)k\left(t\right)+w\left(t\right)-\mathbf{w}\left(t\right)\right]^{\gamma+\lambda-1} && (43)\\
V_{kk} &\simeq \beta\left(t\right)\left(\gamma+\lambda\right)\left(\gamma+\lambda-1\right)\eta^2\left(t\right)\left[\eta\left(t\right)k\left(t\right)+w\left(t\right)-\mathbf{w}\left(t\right)\right]^{\gamma+\lambda-2} && (44)\\
V_{ww} &\simeq \beta\left(t\right)\left(\gamma+\lambda\right)\left(\gamma+\lambda-1\right)\left[\eta\left(t\right)k\left(t\right)+w\left(t\right)-\mathbf{w}\left(t\right)\right]^{\gamma+\lambda-2} && (45)\\
V_{kw} &\simeq \beta\left(t\right)\left(\gamma+\lambda\right)\left(\gamma+\lambda-1\right)\eta\left(t\right)\left[\eta\left(t\right)k\left(t\right)+w\left(t\right)-\mathbf{w}\left(t\right)\right]^{\gamma+\lambda-2} && (46)
\end{aligned}
$$

More precisely, from these conditions, one obtains the following solutions[19] :

$$u_c \simeq \alpha\gamma\,(c-\mathbf{c})^{\gamma-1}\,(lk)^{\lambda} \quad (47)$$

$$u_l \simeq \alpha\lambda\,(c-\mathbf{c})^{\gamma}\,(lk)^{\lambda-1}\,k \quad (48)$$

The ratio (47)/(48) gives

$$\frac{u_c}{u_l} \simeq \frac{\gamma}{\lambda}\frac{lk}{c-\mathbf{c}}\frac{1}{k} \quad (49)$$

However

$$\frac{u_c}{u_l} \simeq \frac{1}{k} \qquad \text{d'où} \qquad lk \simeq \frac{\lambda}{\gamma}\,(c-\mathbf{c}) \quad (50)$$

By substituing (50) in (47) one finds :

$$u_c \simeq \alpha\gamma\,(c-\mathbf{c})^{\gamma-1}\left(\frac{\lambda}{\gamma}\,(c-\mathbf{c})\right)^{\lambda} \quad (51)$$

thus by simplifying :

$$u_c \simeq \alpha\lambda^{\lambda}\gamma^{1-\gamma}\,(c-\mathbf{c})^{\gamma+\lambda-1} \quad (52)$$

Notice that $u_c \simeq V_w$ thus

$$\alpha\lambda^{\lambda}\gamma^{1-\gamma}\,(c-\mathbf{c})^{\gamma+\lambda-1} \simeq \beta\,(\gamma+\lambda)\,[\eta k + w - \mathbf{w}]^{\gamma+\lambda-1} \quad (53)$$

Put otherwise

$$(c-\mathbf{c})^{\gamma+\lambda-1} \simeq \frac{\beta}{\alpha}\frac{(\gamma+\lambda)}{\lambda^{\lambda}\gamma^{1-\gamma}}\,[\eta k + w - \mathbf{w}]^{\gamma+\lambda-1} \quad (54)$$

By rearranging (54), one obtains the optimal level of consumption corresponding to the expresson (26) in the text:

$$c^{*} \simeq \mathbf{c} + \left[\frac{\alpha}{\beta}\frac{\lambda^{\lambda}\gamma^{1-\gamma}}{\gamma+\lambda}\right]^{\frac{1}{1-\gamma-\lambda}}\,[\eta k + w - \mathbf{w}]$$

To determine the optimal quantity of leisure, we just replace the optimal value of the consumption defined above in (50). One obtains:

$$l \simeq \frac{\lambda}{\gamma}\frac{1}{k}\left[\frac{\alpha}{\beta}\frac{\lambda^{\lambda}\gamma^{1-\gamma}}{\gamma+\lambda}\right]^{\frac{1}{1-\gamma-\lambda}}\,[\eta k + w - \mathbf{w}] \quad (55)$$

By putting η in factor and inserting $\frac{\lambda}{\gamma}$ into the bracket, one finds (27):

$$l^{*} \simeq \eta\left[\frac{\alpha}{\beta}\frac{\gamma^{\gamma}\lambda^{1-\gamma}}{\gamma+\lambda}\right]^{\frac{1}{1-\gamma-\lambda}}\left[1 + \frac{w-\mathbf{w}}{\eta k}\right] \quad (56)$$

[19] To reduce the writing, we remove the time subscripts "(t)".

To determine the explicit optimal value of investment in financial assets, it is necessary to replace (43), (45) and (46) in

$$wX \simeq \left(-\frac{V_w}{V_{ww}w}\right)\Sigma^{-1}\left(\mu - r1\right) - \left(\frac{V_{kw}k}{V_{ww}}\right)\Sigma^{-1}\Sigma_\omega \tag{57}$$

That give

$$\begin{aligned} wX \quad &\simeq \quad \left(-\frac{\beta\left(\gamma+\lambda\right)\left[\eta k + w - \mathbf{w}\right]^{\gamma+\lambda-1}}{\beta\left(\gamma+\lambda\right)\left(\gamma+\lambda-1\right)\left[\eta k + w - \mathbf{w}\right]^{\gamma+\lambda-2} w}\right)\Sigma^{-1}\left(\mu - r1\right) \\ &\quad - \left(\frac{\beta\left(\gamma+\lambda\right)\left(\gamma+\lambda-1\right)\eta\left[\eta k + w - \mathbf{w}\right]^{\gamma+\lambda-2} k}{\beta\left(\gamma+\lambda\right)\left(\gamma+\lambda-1\right)\left[\eta k + w - \mathbf{w}\right]^{\gamma+\lambda-2}}\right)\Sigma^{-1}\Sigma_\omega \end{aligned} \tag{58}$$

By simplifying, one directly obtains

$$wX \simeq \frac{\Sigma^{-1}\left(\mu - r1\right)}{1-\gamma-\lambda}\left[\eta k + w - \mathbf{w}\right] + \eta k \Sigma^{-1}\Sigma_\omega \tag{59}$$

Assuming that $\Sigma'_\omega \simeq 0$, we finally obtain expression (28):

$$wX^* \simeq \frac{\Sigma^{-1}\left(\mu - r1\right)}{1-\gamma-\lambda}\left[\eta k + w - \mathbf{w}\right] \tag{60}$$

Finally, the optimal level of human capital investment is given by equation (22) in text :

$$e \simeq \left(-\frac{V_k}{V_{kk}k}\right)\frac{\left(\mu_\theta + \sigma_{h\theta} + \sigma_{\theta\omega}\right) - V_w/V_k}{\sigma_\theta^2} + \frac{\sigma_{\theta\delta} - \sigma_{h\theta} - \sigma_{\theta\omega}}{\sigma_\theta^2} \tag{61}$$

By substituting (42), (43) and (44) into (61) one obtains

$$\begin{aligned} e \quad &\simeq \quad \left(-\frac{\beta\left(\gamma+\lambda\right)\eta\left[\eta k + w - \mathbf{w}\right]^{\gamma+\lambda-1}}{\beta\left(\gamma+\lambda\right)\left(\gamma+\lambda-1\right)\eta^2\left[\eta k + w - \mathbf{w}\right]^{\gamma+\lambda-2} k}\right) \\ &\quad \times \frac{\left(\mu_\theta + \sigma_{h\theta} + \sigma_{\theta\omega}\right) - \frac{\beta(\gamma+\lambda)[\eta k + w - \mathbf{w}]^{\gamma+\lambda-1}}{\beta(\gamma+\lambda)\eta[\eta k + w - \mathbf{w}]^{\gamma+\lambda-1}}}{\sigma_\theta^2} + \frac{\sigma_{\theta\delta} - \sigma_{h\theta} - \sigma_{\theta\omega}}{\sigma_\theta^2} \end{aligned} \tag{62}$$

That is, by simplifying

$$e \simeq \left(-\frac{\eta k + w - \mathbf{w}}{\left(\gamma+\lambda-1\right)\eta k}\right)\frac{\left(\mu_\theta + \sigma_{h\theta} + \sigma_{\theta\omega}\right) - 1/\eta}{\sigma_\theta^2} + \frac{\sigma_{\theta\delta} - \sigma_{h\theta} - \sigma_{\theta\omega}}{\sigma_\theta^2} \tag{63}$$

Put otherwise, we find the explicit solution (29) :

$$e^*\left(t\right) \simeq \frac{\left(\mu_\theta + \sigma_{h\theta} + \sigma_{\theta\omega}\right) - 1/\eta}{\left(1-\gamma-\lambda\right)\sigma_\theta^2}\left[1 + \frac{w - \mathbf{w}}{\eta k}\right] + \frac{\sigma_{\theta\delta} - \sigma_{h\theta} - \sigma_{\theta\omega}}{\sigma_\theta^2} \tag{64}$$

A.4 The marginal effect of effective human capital and effective financial wealth on the pattern of optimal variables over the life cycle

The effect of effective financial wealth on consumption is positive :

$$\frac{\partial c}{\partial (w-\mathbf{w})} = \left[\frac{\alpha}{\beta}\frac{\lambda^{\lambda}\gamma^{1-\gamma}}{\gamma+\lambda}\right]^{\frac{1}{1-\gamma-\lambda}} > 0$$

Over the life cycle, this effect is still positive if $\dot{\alpha}$ is low. Otherwise, the marginal propensity to consume decrease over the life cycle.

$$\frac{d}{dt}\left[\frac{\partial c}{\partial (w-\mathbf{w})}\right] = \frac{1}{\beta}\frac{1}{1-\gamma-\lambda}\frac{\lambda^{\lambda}\gamma^{1-\gamma}}{\gamma+\lambda}\left[\frac{\alpha}{\beta}\frac{\lambda^{\lambda}\gamma^{1-\gamma}}{\gamma+\lambda}\right]^{\frac{\gamma+\lambda}{1-\gamma-\lambda}}\left(\dot{\alpha}-\alpha\frac{\dot{\beta}}{\beta}\right)$$

The marginal effect of financial wealth (expressed in proportion of effective human capital) on leisure, is the same :

$$\frac{\partial l}{\partial \frac{(w-\mathbf{w})}{\eta k}} = \left[\frac{\alpha}{\beta}\frac{\gamma^{\gamma}\lambda^{1-\gamma}}{\gamma+\lambda}\right]^{\frac{1}{1-\gamma-\lambda}} > 0$$

The instantaneous effect of an increase in the ratio of effective human capital over effective financial capital on human capital investment is positive:

$$\frac{\partial e}{\partial \frac{(w-\mathbf{w})}{\eta k}} = \frac{(\mu_{\theta}+\sigma_{h\theta}+\sigma_{\theta\omega})-1/\eta}{(1-\gamma-\lambda)\,\sigma_{\theta}^{2}} > 0$$

The derivative of this effect with respect to time depends on the current marginal rate of substitution between human and financial capital. If it drops during time $\left(\dot{\eta}<0\right)$, as one may expect, the investment in human capital (e) decreases over the life cycle:

$$\frac{d}{dt}\left[\frac{\partial e}{\partial \frac{(w-\mathbf{w})}{\eta k}}\right] = \frac{\dot{\eta}}{\eta^{2}}\frac{1}{(1-\gamma-\lambda)\,\sigma_{\theta}^{2}} < 0$$

Changes in effective financial and human capital have no effect on the variations over time of investment in risky assets. Only current investment in risky assets is positively affected by an increase in $(w-\mathbf{w})$ or ηk :

$$\frac{d}{dt}\left[\frac{\partial wX}{\partial (w-\mathbf{w})}\right] = 0 \qquad \frac{d}{dt}\left[\frac{\partial wX}{\partial \eta k}\right] = 0$$