Zeno's Arrow still Pointed

Paradoxa of Motion and Hund's "Paradox of the Optical Isomers": an unresolved conflict between classical and quantum theory.

For Imre Toth,
friend and mentor

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1. The paradox of the arrow; classical and quantum physics.

   In the Platonic version (cf "Parmenides"), supposedly very close to Zeno's original teaching, the paradox of the arrow is condensed into just one statement: "That which goes towards the goal must first go through the middle". This statement, by iteration, sets up an unending recurrence: on its course from A to Ω the arrow has to be at a sequence M_1, M_2, M_3, ... of "middles"; by definition M_1 lies halfway between A and Ω, M_2 halfway between M_1 and Ω and so on. Thus each of the M_n is clearly different from Ω: the arrow – whatever our senses may tell us – can at no time *be* at Ω.

   This is the perfect, the paradigmatic example of a paradox: faultless logical reasoning ("a theory") has consequences which are at variance with the evidence of the senses ("an experiment").

   Faced with such a contradiction we find only two ways out:

   - either our senses deceive us; this conclusion, it seems, was the intent of the Eleatic school; (in modern terms one would, perhaps, say: the interpretation of the experiments is mistaken)

   - or the reasoning is not as faultless after all; this is evidently what Plato, the mathematician, and his successors through the ages have pursued; (in modern terms: the theory is not adequate).

   Motion cannot *be*: that is the message from Elea. Yet the arrow that Zeno has shot at us through the millenia has indeed moved and arrived at generation after generation of philosophers and mathematicians, causing irritation and pain.

   For its lasting effects on Greek thought and modern mathematics see for example [Toth, 1991], an article with the appropriate subtitle "Zénon et Platon, Eudoxe et Dedekind: une généalogie philosophico–mathematique!" What is elaborated in that paper may be perhaps summarized as follows: The Eleatic challenge has brought forth the two central problems, the two non–ents of Greek philosophy: that of the *infinite*(–simal) and that of the

incommensurate, that mathematicians have struggled with for more than 2000 years.

Their resolution (truly "eine Aufhebung" in the Hegelian double meaning) has only been achieved – on the mathematical side – by the *creation* of "an adequate theory" in the last 200 years: there is a (logically consistent) construction that brings forth a *new realm of the mind*, where the incommensurate, like the diagonal of the unit square, is as *real* as the integer, and where endless sums like the sum of the distances $A \to M_1$, $M_1 \to M_2$, $M_2 \to M_3$, ... *exist* and *do* have a well–defined value ($A \to \Omega$ in the example). This is the wonderfully ordered realm of real numbers, first outlined by very serious adventurers in the late 18th century and firmly erected unter Dedekind and Cantor. (Cf Dedekind's letter [Dedekind, 1932] of 1888, quoted by Toth: "... etwas *Neues*, was der Geist *erschafft*. Wir sind göttlichen Geschlechts ..." What beautiful late 19th century self–confidence!)

For Thomistic minds such usurpation of the divine must, of course, still be blasphemy, as shown by a recent polemic, issuing from the Ponteficia Università Lateranense: "Le radici forti del pensiero debole" [Basti, 1996]: the authors know, through Saint Thomas, that the infinite exists, but is God's prerogative: "Deus cognoscit infinita" and "It is impossible that the divine intellect would not *actually* know the infinite things – infinita etiam actu –", (Summa I^0, 69); the usurpation, however, of the actual infinite by man, in particular by modern mathematics and physics, is not only "weak thinking", but, according to the authors, the root of nihilism and all present day evils. What remarkable late 20th century self–reproach.

And yet this beautiful creation by our divine race still does not quite suffice to resolve the arrow paradox; surely the distance $A \to \Omega$ now exists also as the infinite sum of its ever smaller parts; but how does the arrow *actually* get to Ω? Allow a physicist to sum up 2000 years of debate: this is not a question of mathematics, but of ontology, or, more modestly, of physics as knowledge of the properties of things being; at first motion had to be accepted as *being*, as a state of things being; this pragmatic point of view, from Oresmes to Galileo, gave rise to science. Why was it possible in Paris in 1350 AD to break that ontological link – "being" is "being at rest" – that had remained unbroken from Athens to Aquino? We may speculate whether early science was just another of the countless heresies of the troubled 14th century, or whether it was the advancement of technology since the 12th century – mechanical clockworks, elaborate water mills, textile machines –

that made time and motion, change of place in time an everyday experience. It is precisely Oresmes' concept of velocity – distance travelled per unit of time passed – that takes motion for granted and allows things to be moving and moving things to be. The concept even contains the seed of (Galilean) relativity: once established as a quantity in its own right, this velocity may equally take on any value, hence also the value zero; thus the state of rest, so central to antique thought, has lost all privilege. The giant statues of Aristotle and Thomas, watching over the flock, began to tremble, though it took further revolutions, and more violent ones, to finally overthrow them.

Long after all these revolutionary deeds, with the convenient self–assuredness of mainstream thinking, it becomes an all too easy task, even for smaller minds, to correct Zeno's "error" (cf. Encycl. Brit. on Zeno): both the sum of the distances travelled and the times required for that are convergent series: Ω is reached at T.

Thus, putting together the intuitive concepts of physics, including Newton's differential calculus, with the exact reasoning of 19th century mathematics, sketched above, theory was finally reconciled with the facts; "rational mechanics", the foundation stone of classical physics, was completed. (Around 1880 the young student Max Planck was disencouraged by a professor of physics from going into this field: "There is not much new there still to be discovered".)

Yet this victory, finally achieved after so many centuries of struggle, could, as we know, only be enjoyed for a very short time; new disturbing observations about the motion of submicroscopic objects, the behaviour of atoms or their constituents, that accumulated around the beginning of our century, did shatter the building of classical physics and, after 25 years of agitation gave rise to quantum theory. Then, with unparallelled intellectual intensity within less than five years nearly all disciplines of physics that have been pursued since in this century were developed by a handful of people, most of them around Arnold Sommerfeld in Munich, Nils Bohr in Copenhagen and Max Born in Göttingen. In all these new fields (atomic and molecular physics, nuclear physics and physics of the solid state, elementary particle– and astrophysics etc.) the predictions of quantum theory have been verified with highest precision in countless experiments.

And yet: "Nobody understands quantum mechanics" (Richard Feynman, Nobel prize winner 1965 for his contributions to quantum physics); the

reason for this non–understanding is that the theory forces us to ascribe a behaviour to its objects, ironically still called particles, that has nothing to do with the familiar objective properties of macroscopic bodies that are the objects of classical physics.

This nonclassical behaviour is best characterized by a number of negations:

- Quantum objects are *not* objective – the properties which can be ascribed to them depend on the context. This behaviour violates the great achievement of classical physics sketched above: that the "being" of an object, more precisely its exact location, and the state of motion of the same object, expressed as its velocity (or its "momentum" in Newtonian physics), are two completely independent, objective properties. This is *not* true for a quantum object: a precise measurement of its location destroys all possible information about its velocity and vice versa; only one or the other of the two properties can be objective. (This is rather like Zenon's arrow, when it is "being" it cannot move and when it is moving, it cannot be. Indeed there is a formulation of quantum theory, where the motion of a particle is described in the Zenonic way: it is continually annihilated in one place and recreated the next moment in a neighbouring location.)

- Quantum events are *not* deterministic: the theory does *not* allow precise predictions about individual events like the disintegration of one nucleus of a radioactive element, but allows the precise calculation of the probability of such events. All attempts to give an additional intrinsic cause, e.g. for the decay of an individual nucleus, have failed.

- The quantum probabilities do not obey the equations of motion of bodies, but rather those of waves. The fundamental equation of quantum dynamics, Schrödinger's wave equation (1926) is a deterministic equation for probability amplitudes, whose motion resembles that of oscillations on a string or waves on the surface of a pond.

On the other hand it is also true that the classical world, the motions of macroscopic bodies from arrows to rockets, from the planetary system to classical technology like railroads and factory machines, are equally well and precisely covered by the laws of classical Galilean–Newtonian physics. Thus we have two equally well confirmed fundamental theories, but, as the qualitative description of quantum properties, given above, suggests, there is no continuous transition from quantum theory to classical theory, there is

no exact limiting process that leads from one to the other, classical physics is *not* contained in quantum physics as a special case. (In more technical terms: if the fundamental quantum constant of nature, Planck's quantum of action h, is allowed to go to zero which should correspond to the classical limit, the quantum equations go to a singularity, across which they cannot be continued.)

Well, in practice the situation is not quite that bad: we do have certain criteria to decide in which cases or to which objects we should apply one or the other theory; there is a beautiful theorem by Ehrenfest (September 1927), which, combined with Heisenberg's uncertainty relation (March 1927), predicts that the center of mass of a "large" quantum object in a "slowly" varying force–field behaves like that of a classical particle, i.e. its motion obeys Newton's equations.

But how large is large? Is the motion of a speck of dust, that of a virus or a protein molecule classical? In these cases the answer is certainly yes, just as it is surely no for the motion of electrons within atoms.

Hence the question: how small must an object, let us say a molecule, be, in order to exhibit typical quantum properties? And: is the transition, as a function of the object's size, a continuous one or is there a jump? With this question we encounter one of the founders of the quantum theory of molecules.

2. Friedrich Hund (1927): The paradox of optical isomers

Friedrich Hund, who celebrated his hundredth birthday, alive and well, on 6 February 1996 in Göttingen, was a collaborator of Max Born there 70 years ago and had just come back from a stay in Copenhagen, when he wrote a series of articles: Zur Deutung der Molekelspektren [Hund, 1927, I, II and III]. In the third of these papers he states "Das Paradoxon der optischen Isomere" and also sets about to resolve it.

Optical isomers occur in a very large class of compounds, in particular of organic origin. Such substances have been known since Pasteur (c. 1850) to exist in two forms of identical chemical composition but with opposing optical properties; as an example the tartrates found as little crystals on the corks of mature wine (first studied by Pasteur), when produced by "good" wine ferments, occur only in one of these forms and, as transparent crystals

or in aqueous solution, rotate polarized light to the right, whereas the other form, produced by "bad" ferments, when separated from the first, rotates the polarization of light in the opposite, left–hand direction.

The atomistic interpretation of molecules, in particular stereochemistry, which proved extremely successful ever since its formulation in 1874, gave a simple explanation for these observations. In this classical picture molecules are portrayed as an assembly of balls of different sizes, representing the atoms, bound together by 'chemical bonds' like apples pinned together with tooth picks. In a molecule of a compound of four or more elements: ABCD ... , the spheres, as it turns out, practically never lie all in a plane; the ABCD–molecule, as a rule, can be conceived as a triangular pyramid, i.e. a tetrahedron, with the atoms A, B, C and D at the corners. Now it is easy to see that this ABCD–pyramid can occur in two 'isomeric' configurations, that relate to each other like a left hand and a right hand; observed from the corner where the A–atom sits, the sequence BCD is either seen in clockwise or in counter clockwise succession. The two isomers are mirror images of each other; no rotation transforms one into the other. As with screws or helixes one may therefore distinguish the left–handed and the right–handed version of such molecules, abbreviated l and r; one may even get an intuitive insight into the "optical activity" described above: the rotation of the polarization of light waves, that is of the direction in which the wave excitation oscillates, is effected by the "handedness" of molecules along the path of such light.

How could one really convert the l–ABCD configuration into the r–form? Only by pushing one of the four spheres, let us say the one that represents A, through the plane of the other three, such that the A–corner appears on the opposite side of this plane. More technically, the two configurations correspond to two minima of the potential energy, separated by an energy barrier; one calls this a double–well–potential; a classical body rests either in one or the other of the two hollows, just like a golf ball would on a ground with two dips. Handedness or, with a word coined by Lord Kelvin, *chirality* appears in this way as a quality that follows from the applicability of the classical concept of bodies to the atomic constituents of a molecule.

The quantum concept, applied to the situation of a double well, gives a completely different result; the quantum states of such a system may be identified with the oscillations of a string that is supported by a bridge in

the middle, where the classical energy barrier is located. There are two types of stationary oscillations of such a string, it turns out (and the Pythagoreans must already have known that), both have equal amplitude in both halves, the first type is symmetric ("even"), the second antisymmetric ("odd") with respect to the bridge; the pitch (in physics terms the frequency) of the two types of oscillation is slightly different. If one excites the string to oscillate only in one half, this excitation after a while is found only in the other half and wanders back and forth periodically with a beat frequency that equals the difference in pitch of the two stationary oscillations.

Now, there is a molecule of the ABCD–type that behaves exactly according to this quantum picture, namely the ammonia molecule NH_3; in a stable state of this molecule the probability of finding the A–atom (here, the nitrogen atom N) to the left or to the right of the plane defined by the other three (here the three H–atoms) is exactly the same. In other words NH_3 is neither in the l– nor in the r–configuration, but in a typical quantum state, a (symmetric or antisymmetric) superposition of the two classical configurations. The beat frequency of the odd and the even state of NH_3 is well–known: 31 GHz, the frequency of the ammonia maser, first realized in 1958, the predecessor of all lasers and atomic clocks. The functioning of this ingenious invention is based on exactly the fact that NH_3 behaves as a quantum object and has *no* defined handedness; were it prepared in, let us say, the l–configuration, it would oscillate to r and back with the beat frequency given above; NH_3 shows pure quantum behaviour.

What Hund pointed out as a paradox is the fact that beyond the very simple and light molecule NH_3 there is an immense number of organic molecules, some not much bigger or heavier than ammonia, that occur as stable optical isomers, all sugars and amino–acids are examples, let's say r–$C_6H_{12}O_6$ (dextrose) and l–$C_6H_{12}O_6$ (levose); they show typical classical behaviour.

In fact this is the paradigmatic paradox again: a theory ("micro–objects show quantum properties") and the experiment ("dextrose is chiral and always rotates the polarization of light to the right") contradict each other.

There is one feature that makes this contradiction even more puzzling: life on earth is closely connected with chirality; living cells, from bacteria to homo sapiens, synthesize given isomeric compounds only in one of the two configurations – all sugars in the r–, all amino–acids in the l–form.

A simple proposal to resolve the paradox is already contained in Hund's 1927 paper: quantum theory, he claims, is valid also for compounds with observed chirality, which means that the observed monochiral states are in reality unstable and oscillate back and forth between l and r; the period of this oscillation, however, is so extremely long – thousands of years or more (in contrast to the billionth of a second found for ammonia) that in practice it cannot be observed; it seems to be infinite and hence the chiral states, once synthesized, seem to be stable.

Apparently this plausible hypothesis was generally accepted for half a century, even though the discrepancy between the observed nanosecond beat period of ammonia and the assumed geological time scale for the oscillation of the apparently chiral compounds was somewhat alarming; why are there no substances for which this period is observable and lies in the range of minutes or hours or a year?

Only in the 1970s did a new discussion of these problems set in, motivated by developments in several different areas:

- Chemical industry began to be interested in processes that would allow a large–scale production of monochiral compounds, e.g. only righthanded sugar, from anorganic primary material – maybe with special catalysts that like most enzymes are chiral themselves. (A primary synthesis of, let's say, the compound $C_6H_{12}O_6$ will always produce a fifty–fifty mixture of righthanded dextrose and of lefthanded levose molecules).

- With the advancement of molecular biology the question, already formulated by Pasteur, as to what causes the observed asymmetry in organic nature obtained new relevance.

- The thalidomide disaster – worldwide thousands of children were born in the 1960s with severest deformities after their mothers had taken this supposedly innocuous sleeping drug in early pregnancy – and its later resolution pointed at the drastically different biological effect of the l– and the r–form of many substances. As it turned out, much too late, only l–thalidomid molecules cause malformations of the fetus, because they block the action of a (chiral) enzyme that regulates the synthesis of cartilege in the second month of fetal development.

Around 1980 Hans Primas, a theoretical chemist in Zürich, posed the decisive question: if Hund's hypothesis were correct and the apparent chiral

states were unstable, but of astronomical period, then why do such compounds never occur in their stable quantum mechanical ground state, which is not chiral, but symmetric with respect to the mirror plane of the molecule? Why can one buy a pound of dextrose and a pound of levose in a drugstore, but not a pound of the "neither $r-$ nor $l-$molecules" that should be the stationary quantum ground state?

3. Open quantum systems – a resolution of the paradox?

Several proposals have been made since 1980 that contain Hund's hypothesis of the long lifetimes, but go beyond it in order to answer Primas' question. One obviously needs an additional effect that destabilizes the non–chiral ground state of quantum mechanics and stabilizes the $l-$ and the $r-$configuration once the molecule has been synthesized in one or the other form.

What these recent proposals have in common is the assumption that the required effect comes from some – even though extremely weak – coupling of our molecule to the rest of the world. This is the "open quantum system" hypothesis: the interaction with everything else that is present in the universe must not be completely neglected even if this interaction can be proven to be very weak.

There is a good quantum mechanical reason for believing in the extreme sensitivity of the molecules in question with respect to arbitrarily weak energy exchanges with the outside world: with the $r-$ $l-$oscillation period very long, its inverse, the beat frequency, becomes very small. Now Planck's original 1899 quantum hypothesis, the foundation of all later quantum theory, says that an oscillation of a given frequency ν is associated with quanta of energy which are integer multiples of the "quantum" $h\nu$ (h: Planck's constant). If the characteristic frequency ν of a quantum object is very small, so is the energy quantum $h\nu$ that may influence its behaviour.

Thus the "open quantum system" hypothesis appears not unreasonable as the basis of a research program; yet considerable difficulties have to be overcome to carry it out

- The "rest of the world" and the kind of interaction it has with our quantum object, the molecule, has to be specified.

- One has to decide, whether one describes the outside world as (possibly infinitely many) quantum systems, which gives rise to enormous mathematical difficulties, or as a "classical" world, in which case there is no consistent theoretical foundation for the type of interaction with the quantum object – one has to invent such a coupling ad hoc.

- In any case the theory to be tackled is by far more difficult than ordinary quantum theory – a closed mathematical description of interacting systems of this kind still does not exist.

To at least reduce these difficulties, our group [Koschany et al, 1994] has recently proposed to apply a very simplified model to the chirality problem: to the wave equation of quantum mechanics one adds — following a suggestion of Birula–Bialynicki — a nonlinear term constructed in such a way that the fundamental principles of quantum mechanics (with the exeption, of course, of the superposition principle which holds only for a linear theory) are not violated. This nonlinearity is meant to represent, as an "effective interaction", the coupling of the molecule to all the rest of the world; it contains only one parameter, a coupling constant b.

The result we obtain is very satisfactory: as a function of the coupling strength b the system undergoes a spontaneous symmetry breaking (or a second–order phase transition); below a critical value b_{crit} the molecule behaves purely quantum–mechanical, i.e. symmetric and non–chiral; at b_{crit} there is a bifurcation, asymmetry sets in (with infinite slope), and for larger b the system's asymmetry soon goes to saturation; the behaviour becomes completely classical with two exactly equivalent positions corresponding to the l– and r–configuration; the figure gives a sketch of this behaviour.

Of the several recent attempts to formulate more realistic, less simplified interaction models, at least two should be mentioned, because they have a solid theoretical foundation, although they yield rather difficult results:

- P. Pfeifer from the Zürich group has, as early as 1980, proposed a model in which the molecule is coupled to the infinitely many degrees of freedom of the quantized radiation field. The result, at least for the ground state, looks very similar to ours: there is an effective coupling constant, determined by molecular data, that is inversely proportional to the beat frequency; the molecule undergoes exactly the same type of phase transition portrayed in the figure, when the coupling is increased, i.e. the beat frequency of the molecule decreased. Thus Pfeifer's model (like ours)
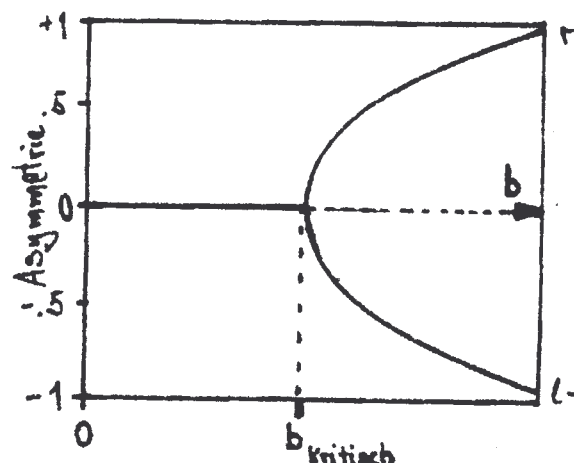
Figure:
Asymmetry of the model
molecule as a function of
the coupling strength b.
(After [Koschany, 1994])

gives a reasonable answer to the question of H. Primas: very small mole-
cules with relatively weak and flexible chemical bonds, like ammonia,
have a high beat frequency and show pure quantum behaviour.

As one considers heavier and heavier molecules, the mass, strength and
rigidities of the chemical bonds go up and, as Hund has already shown
in 1927, the beat period (corresponding, as one should remember, to the
transition of one atom through the plane formed by the others) increases
exponentially. Hence the the molecular world is clearly divided into a
small group of light, nonchiral compounds, and the enormous number
of "classical" molecules that exhibit handedness and optical activity,
providing an excellent agreement with reality.

Nevertheless some mathematical approximations contained in this result
have been severely criticized; it is also not clear wether the result is stable
with respect to noise, like a coupling to a world of finite temperature;
these questions are presently under investigation.

• A completely different hypothesis tries to establish a connection between
asymmetry/chirality on the level of molecules and the cosmic asymmetry
of matter and antimatter. The connecting agent here is supposed to be
the so–called electroweak interaction, which contains a coupling through

the "heavy intermediary bosons" like the Z_0 particle whose existence had long been predicted by theory before they were found about 10 years ago; a coupling involving these particles is intrinsically chiral or "parity–violating". Indeed in heavy atoms a (very weak) optical activity, caused by a slight handedness of their electronic shell, has been found and explained successfully by the electroweak theory, in good agreement with the experiments. (Pasteur had already speculated that the asymmetry in the chemistry of living cells had something to do with cosmic asymmetry.)

Unfortunately this neat explanation, when applied to molecules, is much less satisfactory; it does not yield the discontinuous phase transition from symmetric and completely nonchiral to asymmetric and completely chiral compounds that was discussed above, but rather exhibits — as one goes from small to large molecules — a quite gradual increase from small to large asymmetry. Accordingly even the lightest four–atom compounds of the ammonia type should show some asymmerty which seems not to be the case; moreover a given chiral compound, e.g. our standard example, $C_6H_{12}O_6$–sugar, when produced anorganically and without a chiral catalyst, should occur with a slight preponderance of one handednes over the other due to the extrinsic asymmetry of the forces, a result that is also, so it seems, not confirmed in reality.

It turns out, then, that the paradox formulated by Friedrich Hund 70 years ago has not yet lost its irritating and creative power. The connection and transition between the realm of quantum objects and the macroscopic, the classical world remains to the present day a dark chapter in the world of physics. In spite of decades of efforts, of hundreds of textbook–authors who portray quantum mechanics as a complete theory, of the pragmatic negligence that most physicists show with respect to problems that seem to be marginal, there remain some gaps – problems that have a striking similarity with Hund's paradox:

- The theory of measurements in quantum physics must, by definition, contain an interaction between the quantum objects and classical objects like the pointer of some instrument. For want of a fundamental theory of this coupling the "reduction postulate" has been introduced early on as a kind of axiom: the classical apparatus "reduces" the probability before measurement to certainty after the measurement. Even though this prescription seems to work there is growing dissatisfaction with it

and a renewed search for a "quantum theory without reduction" (the title of a recent collection of essays [Cini, 1990]).

- Classical objects may and do exhibit (in the generic case of more than one degree of freedom) a behaviour that had already been studied in the Celestial Mechanics of Poincaré (1890) and — and after its rediscovery in the 1960s — has been named chaotic, they show sensitive dependence on initial conditions which makes long term predictions for such systems impossible. It has been understood in the last years that this behaviour is fundamental for the Second Law of Thermodynamics and gives a solid basis for the application of statistical mechanics.

  Systems obeying pure quantum dynamics, however, do not show any of this irregular behaviour. Again it appears that only a coupling to the rest of the world, i.e. a theory of open quantum systems may reconcile the two "worlds".

- Finally a problem should be mentioned that has been presented in two books of R. Penrose [Penrose 1989, 1994]: quantum theory and the nature of "intelligence". To be sure activity of the human brain (and in fact of all nervous systems) is based on quantum processes that somehow interact with the macroscopic world, with the body and with all the other "bodies". As long as we do not understand this interaction, that is Penrose's argument, all efforts to create something like "artificial intelligence" must fail. (Well, even the artificial stupidity of the computers has created enough problems. Do we really need more of that?)

Hund's paradox is thus embedded in a large class of problems that may lead to a more fundamental theory, underlining once again the challenge created by paradoxa – a late hommage to Zeno whose challenge has – after a rather long time – led to modern science and mathematics.

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Acknowledgement: