Voting Models for the Council of Ministers of the European Union

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Abstract

An important question for the European Union is to know whether its institutions will permit it to escape from political deadlocks each time a question is at stake. Two studies [1, 7] suggest, by using the Impartial Culture assumption to model the voting behavior, that the EU could only take a decision in 2% of the cases with its current voting mechanisms. The Impartial Culture model has been criticized from a theoretical point of view [5], a political one [12] and does not fit with experimental data [10]. The generalized Impartial Anonymous Culture assumption we consider in this paper is an improvement of this first model. We here study the probability of approval under this assumption, first for the asymptotic case (reached when the number of countries N goes to infinity), and next with computer enumerations and Monte Carlo simulations for the European Union with 27 members. We consider both the Treaty of Nice and some proposals for the European Constitution.

JEL classification: D7

1 Introduction

In the last five years, a considerable body of research on the choice of the best voting rules for federal union have been inspired by the debates on the Treaty of Nice and the projects for an European Constitution. Without being exhaustive, we can mention the work by Baldwin, Berglof, Giavazzi and Widgren [1], Barberà and Jackson [2], Bobay [4], Feix, Lepelley, Merlin and Rouet [5], Felsenthal and Machover [7, 9], Laruelle and Widgren [11]. All these contributions share a common organization: the authors propose a voting model, and then seek for the voting rule or the constitution that better fits according to some normative criteria.

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Recently a welcome and quite useful discussion between a high level politician (Axel Moberg) and scientists (Dan Felsenthal and Moshe Machover) has developed [12, 8]. At the origin, the scientific analysis of the Treaty of Nice [1, 8] claims that the need of 255 (or 258) votes (on a total of 345) to approve a proposition will result in a serious deadlock at the council of ministers with an *a priori* probability of approval of 2%. A. Moberg disagrees strongly, pointing out that the result ignores the "strong consensual culture of the EU". Who is right ? In fact, the scientific analysis given in [1, 8] is only a part of the full story : it is based on the use of the Impartial Culture (IC hereafter) model, which states that each country chooses to vote 'yes' or 'no' independently with equal probability. In other words, each country flips a fair coin to take a decision! But another model exists, which is more subtle and less easy to compute. This model, called Impartial Anonymous Culture (IAC), asserts that all the distributions of the votes at the Union level are equally likely.[1]

The aim of this study is to show that the use of a model related to the IAC one is able to give answers which are closer to the reality of the European Union with 27 members (EU27 hereafter) and, in some way, takes into account the consensual character of the vote. By departing from the common IC assumption, we obtain a theoretical probability of passing a motion that turns out to be higher. Our result concerns not only the Treaty of Nice with its famous 74.8% majority rule (one key vote), but also one of the decision schemes that have been suggested during the debates for the European Constitution (double key vote: a motion is passed if it is supported by more than 50% of the countries gathering more than 50% of the population).

The paper is organized as follows. In section 2, we present a generalized IAC model (GIAC) and we briefly discuss its adequation to the vote at the council. In section 3, we give the theoretical probability of approval in the asymptotic limit, *i.e.* when the number of countries (denoted by N in what follows) goes to infinity. Section 4 checks the relevance of this asymptotic solution for EU27, by providing numerical simulations, and we present our conclusions in section 5.

2 The different models

We consider binary issue votes 'yes' or 'no' for the N states (elsewhere voters, MPs, etc...) of a federal union. In the simplest IC model, each vote is independent of the others and each voter says 'yes' or 'no' with equal probability $p = 1/2$. IC has serious drawbacks. It describes a vote where either everybody is undecided (no exchange of points of view allowing the emergence of a majority has taken place) or the existence of two blocks of strictly equal importance. In both cases, the vote will be won by a margin going as $N^{1/2}$. This explains the low probability of approval with a quota of 258/345 *i.e.* 74.8% in the Treaty of Nice decision scheme. The idea is consequently to introduce a model where a probability p different from 1/2 has emerged. Moreover, our knowledge of p is itself of a probabilistic nature, it is mathematically described by the function $f(p)$ which is the probability distribution of p. The emergence of a probability p different from 1/2 seems rather natural in an assembly where certainly long discussions, explanations, compromises, package deals, etc... precede each vote (the "consensual culture" of A. Moberg). Notice that, all these discussions are resumed in a $p \neq 1/2$ and that the subsequent

[1] Notice that the widely used Banzhaf power index relies upon the IC probability assumption. For its part, the IAC model can be associated to the Shapley–Shubik power index. The link between the probability models and power indices was first emphasized by Straffin [13] and Berg [3].

votes are independent. Then the GIAC model is characterized by a given $f(p)$ with $0 \leq p \leq 1$, $f(p) \geq 0$ and $\int_0^1 f(p)\mathrm{d}p = 1$. The function $f(p) = 1$ for all p gives the IAC model. With this model, the average number of votes for which n voters of equal importance, on a total of N, have voted 'yes' reads

$$C_N^n \int_0^1 p^n (1-p)^{N-n}\mathrm{d}p = \frac{1}{N+1}. \tag{1}$$

All values of n (from 0 to N) have the same probability $1/(N+1)$, consequently, for the IAC model, the probability to have n 'yes' on a total of N voters is a flat curve. It is also easily proved that, if both n and N go to infinity while the ratio n/N is kept constant, the probability distribution of n is to first order in $/N$

$$F_N(n) = \frac{1}{N} f(\frac{n}{N}). \tag{2}$$

This result is a direct consequence of the possibility of interpreting a probability as a frequency when the number of random drawings goes to infinity and would be trivial if we were not dealing with a double probability concept as mentioned above.

Now let us suppose that each of the N voters has one vote and that Q votes are needed for an approval. Let $q = Q/N$. Then for the IAC model, the probability of approval is $1-q$, independent of N. For $q = .75$, for example, the IAC model gives a 25% chance for an approval, while the computation using the IC model gives 0.3% for $N = 27$ voters.

3 The probability of approval in the asymptotic limit

In a more generalized case, voter i has a_i votes (or mandates); moreover, two kinds of mandates have been proposed in the EU constitution : each voter has two mandates a_i and b_i, and his (her) vote ('yes' or 'no') is used in two qualified majority games $\mathcal{A}$ and $\mathcal{B}$, the respective quotas being Q_A and Q_B. Notice that for each player i, it is the same vote ('yes' or 'no') which is used to compute the number of mandates obtained respectively with keys $\mathcal{A}$ and $\mathcal{B}$. The two quotas must be reached for final approval, each one being related to a certain type of legitimacy. The EU Constitution project proposes for country i to take $a_i = 1$ and b_i equal to the population of state i.

In order to be able to perform analytical computations, to see the role of $f(p)$ and the influence of the different quotas, we suppose that N is large enough to use asymptotic calculations. At the end, we will compare the obtained asymptotic results to numerical simulations and will see that, as already stated in [6], EU27 can be fairly approximated by this limit at least for the one key vote (the details of the calculations are given in appendix).

For the GIAC model, characterized by $f(p)$, the distribution function for x mandates in favor of approval in the single key case reads

$$F_N(x) = \frac{1}{A} f(x/A), \tag{3}$$

while for the double key case, we obtain

$$F_N(x,y) = \frac{1}{AB}\,\delta(\frac{x}{A}-\frac{y}{B})\,f\left(p=\frac{x}{A}=\frac{y}{B}\right), \tag{4}$$

where δ is the Dirac distribution and x and y are the numbers of mandates in favor of approval for keys $\mathcal{A}$ and $\mathcal{B}$ respectively. Note that x and y are strictly correlated ($\frac{x}{A}=\frac{y}{B}$). In formulae (3) and (4), A and B are defined by

$$A = \sum_{i=1}^{N} a_i, \qquad B = \sum_{i=1}^{N} b_i. \tag{5}$$

For the IAC model ($f(p)=1$) and a single key vote, equation (3) means a flat density from $x=0$ to $x=A$. Still for the IAC model but for the double key vote, the points are located on the segment joining $(x=0, y=0)$ and $(x=A, y=B)$ with a uniform distribution on this segment. In the approximation used to obtain (3) and (4), votes characterized by p give points located at $x=pA$ and $y=pB$.

Note that these results hold for N going to infinity. It can be shown that the first correction (N large but not infinite) provides a diffusion around these points in $N^{-1/2}$. While this scattering slightly modifies the flatness of $F_N(x)$ for the one key vote, it transforms the segment of the two key vote into a long ellipse with a ratio long over small axes in $N^{1/2}$. Coming back to the segment structure, we see that in the double key vote case, with two unequal quotas, it is the one with the highest quota which will set up the frequency of 'yes' votes. Also, we must point out that equation (3) is a generalization of equation (2) obtained for a set of voters with one mandate each (in that case $A=N$). Notice also that integration of (4) respectively on y and x gives

$$F_N(x) = \frac{1}{A}\,f(\frac{x}{A}), \quad \text{and} \quad F_N(y) = \frac{1}{B}\,f(\frac{y}{B}), \tag{6}$$

in agreement with the results of the one key vote.

Concerning the double key vote, it is worth noticing that, for any N (not necessarily going to infinity) and for quotas equal to $A/2$ and $B/2$, the voting power of a state X (*i.e.* its probability of being pivotal) is given by

$$P(X) = \frac{P_A(X)+P_B(X)}{2} \tag{7}$$

where $P_A(X)$ and $P_B(X)$ are the voting powers of X with (respectively) keys A and B. This result is valid for IC, IAC and GIAC models when $f(p)=f(1-p)$.

To end this section, a comparison with the results of the IC model is in order. The above treatment with $f(p)=\delta(p-1/2)$ concentrates all the points at the central point ($x=A/2, y=B/2$). This confirms the quickly decreasing probability of approval when the quotas are not very closed to $1/2$. For the double key vote, with the IC model, the authors have carried the computations of the next term to obtain the scattering around the central point. The approval probability obtained analytically for the two relative quotas equal to $1/2$ reads

$$P = \frac{1}{\pi}\arctan(\frac{\sqrt{1+r}}{\sqrt{1-r}}) \tag{8}$$

where $r = \sum_{i=1}^{N} a_i b_i / \left[\sum_{i=1}^{N} a_i^2 \sum_{i=1}^{N} b_i^2\right]^{1/2}$ is the correlation factor between vote $\mathcal{A}$ and vote $\mathcal{B}$. P varies from 1/4 ($r = 0$) to 1/2 ($r = 1$, obtained for $b_i = ka_i$, in fact a single vote).

4 Numerical simulations

In this section, the results of numerical simulations will be shown for the IAC case. Because we want to reach the asymptotic limit which supposes both an important number of elections and a large number of voters, Monte Carlo method should be used. Actually, it is not possible, when the number of voters is large, to enumerate, stock and compute the 2^N configurations because of lack of memories and computation time.

In addition, the Monte Carlo technique will illustrate clearly the double probabilistic character of the IAC model. This method consists to make a random sampling among all the vote configurations, but without taking all of them. Then the contribution of all the samples are gathered.

The method has two steps. First a probability p is chosen at random in the distribution function $f(p)$. Second a vote configuration is chosen accordingly this probability p : for each of the N voters, a random number is taken in a uniform distribution, if this number is lower than p, the voter gives its mandates (it is a 'yes' vote) while he doesn't if the number is higher. This is in fact an acceptation-rejection method and if the number of voters is large, the number of 'yes' voters divided by N will tend toward p. This process is repeated for a large number of elections with, at each election, a choice of a new p into $f(p)$ and so on.

Notice that the results of the IC model could also be obtained by this technique. The probability p of the N voters is then equal to 1/2 which corresponds to $f(p) = \delta(p - 1/2)$.

First we give the obtained results for a large number of voters ($N = 100$) and $M = 50000$ elections, both for a single key and a double key vote using the Monte Carlo technique. For the single key case, figure 1 shows the histogram of the number of configurations, as a function of the related number of mandates. The mandates of the N states have been taken at random in a uniform distribution between 1 and 5, then the sum has been normalized to 100. This normalization does not change the ratio 5 between the highest value of the mandates and the smallest one. The histogram is flat in agreement with equation (3) and the probability of approval is very close to $(1 - q)$.

For the double key case, figure 2 gives the distribution of the M elections performed in the plane (x, y), one point representing one election. For both keys, the mandates have been taken at random in a uniform distribution between 1 and 5, then the sum has been normalized to 100. Because all the points have the same weight, their density gives the value of $F_N(x, y)$ (see equation (4)). As expected, the points are roughly distributed on the segment delimited by the two points $(0, 0)$ and (A, B). In addition to this global behavior, the distribution shows a certain scattering. Consequently the results obtained in the asymptotic limit are recovered (at least for $N = 100$ voters). We have checked that, for this case, the probability of approval is closely given by $1 - sup(Q_A, Q_B)$ as shown table 1. For example, for $Q_A = Q_B = 70\%$, we get 29.3% of approval and for $Q_A = 50\%$ and $Q_B = 80\%$, we get 20.4%.For figure 3 the number of voters has been increased up to $N = 1000$. We observed that the scattering of the points

	Key $\mathcal{A}$				
	14	17	19	22	25
50%	48.66	40.15	30.34	20.53	10.59
60%	40.08	39.11	30.34	20.53	10.59
Key $\mathcal{B}$ 70%	30.42	30.42	29.40	20.53	10.59
80%	20.49	20.49	20.49	19.58	10.59
90%	10.55	10.55	10.55	10.55	9.92

Table 1: Double key vote. Percentage of approval for the simulation presented figure 1 as a function of the two quotas Q_A and Q_B.

decreases as expected.

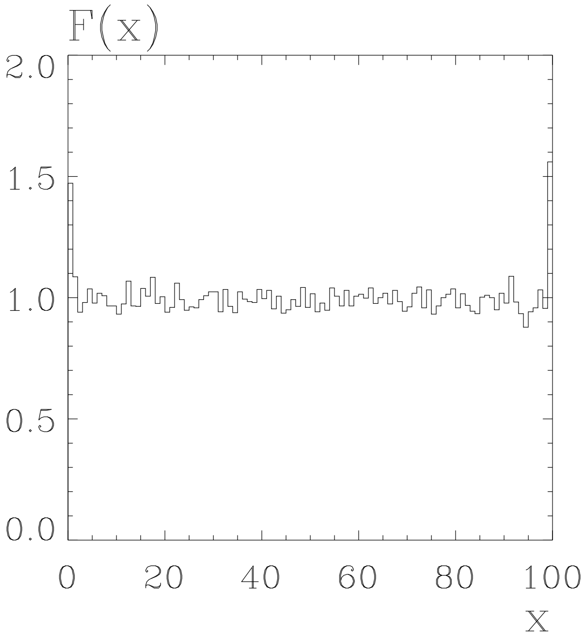

Figure 1: One key vote. Distribution of the results of the votes for $N = 100$ voters and $50,000$ elections using the Monte Carlo technique. The mandates are chosen at random in a ratio 1 to 5 and the sum is normalized to 100.

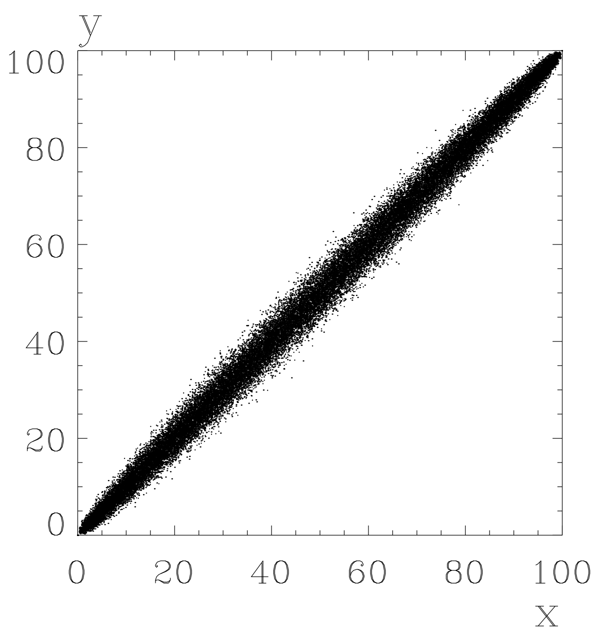

Figure 2: Double key vote. Distribution of the results of the votes for $N = 100$ voters and $50,000$ elections using the Monte Carlo technique. The mandates are chosen at random in a ratio 1 to 5 and the sum is normalized to 100. Each point represents an election.

Now, the question is to know whether or not the asymptotic limit is a good approximation for the EU27[2]. It is here possible to enumerate the 2^{27} vote configurations (taking care of their different weights). For the single key case, figure 4 shows the histogram of the number of configurations as a function of the related number of mandates, which have been taken proportional to the square root of the state populations. This choice is in accordance with the principle used in the EU15 and constitutes a good compromise between the state legitimacy and the citizen legitimacy (see [4]). Again, the curve is rather flat, at least for q between 0.2 and 0.8, indicating

[2]Population data can be found in Moberg [12].

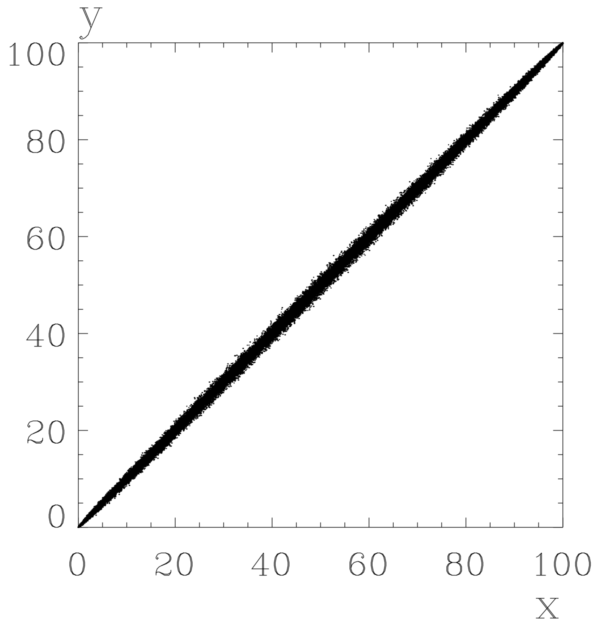

Figure 3: Double key vote. Distribution of the results of the votes for $N = 1000$ voters and 50, 000 elections using the Monte Carlo technique. The mandates are chosen at random in a ratio 1 to 5 and the sum is normalized to 100. Each point represents an election.

		Key $\mathcal{A}$				
		14	17	19	22	25
	50%	45.44	38.27	31.92	21.42	10.71
	60%	39.56	35.60	30.86	21.33	10.71
Key $\mathcal{B}$	70%	31.56	30.20	27.73	20.68	10.71
	80%	22.75	22.54	21.86	18.40	10.55
	90%	13.64	13.64	13.61	12.80	9.03

Table 2: Double key vote. Percentage of approval for the EU27 as a function of the two quotas Q_A and Q_B. The results have been obtained by complete enumeration of all the vote configurations.

that the asymptotic limit could be used for this single key vote.

For the double key case, we turn back to Monte Carlo simulations (although complete enumeration is possible) because each point has the same weight. Then, it is easier to interpret figure 5 which gives the distribution of 2,700 vote configurations in the plane (x, y) (one point represents one election). For key $\mathcal{A}$, all the mandates are equal to 1 (state legitimacy) while for key $\mathcal{B}$, the number of mandates of a state is proportional to its population. The sum of the mandates of key $\mathcal{B}$ has been normalized to 100. Because of the discrete nature of the key $\mathcal{A}$ mandates, the points are aligned on vertical lines distant of 1. The scattering of the points, not negligible, is compatible with the $N^{-1/2}$ law as stated before. Neverthless the rule $1 - sup(Q_A, Q_B)$ for the approval is fairly satisfied as shown by table 2.

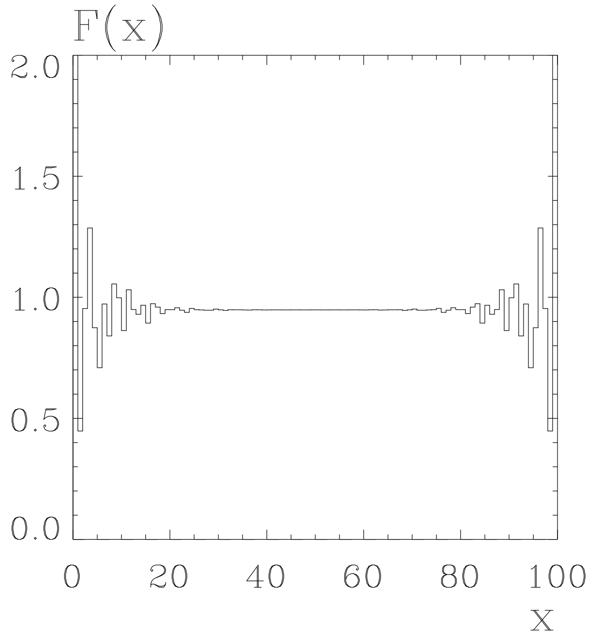

Figure 4: One key vote. Distribution of the results of the votes for the EU27. The mandates are proportional to the square root of the populations of the states and the sum is normalized to 100.

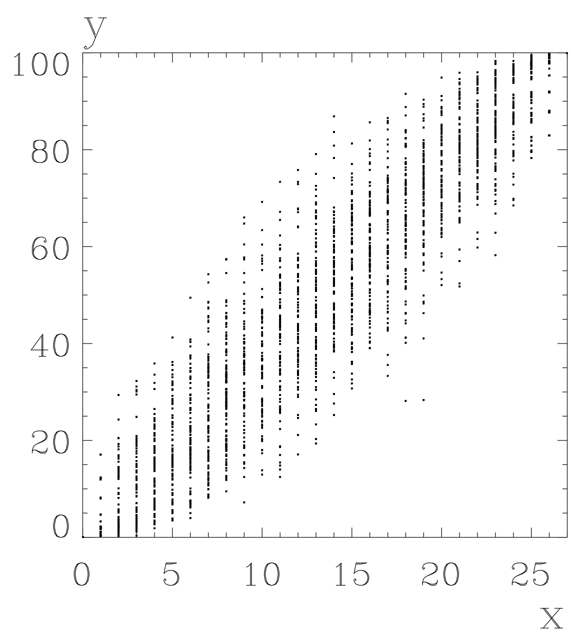

Figure 5: Double key vote. Distribution of the results of the votes for the EU27. For key $\mathcal{A}$ (x variable) all the mandates are equal to 1, for key $\mathcal{B}$ (y variable) the mandates are proportional to the populations of the states and the sum is normalized to 100.

5 Conclusion

Except if we take quotas closed to 1/2, IC and IAC (or GIAC) give results which differ by a large factor. Can we decide which model is the more appropriate ? The question is of great importance if we remember that the two main power indices (Banhzaf and Shapley–Shubik) are respectively based on IC and IAC models. In the first applications of statistical models to voting theory, like the studies computing the Condorcet effect probability, it was often found that the two models were giving very similar results. It is with the study of more sophisticated problems (like the one we have here considered) that the fundamentals differences between the two models become apparent. In [5], the authors have criticized the IC model which describes so tied elections that they can be considered as not having fulfilled their role. In this paper, the GIAC model (with its arbitrary $f(p)$) was introduced and allows us to produce more sensible results. We have shown that the critics of A. Moberg were directed against the IC model but can be easily answered through the use of the IAC model.

Finally, it is of interest to mention a recent study by Gelman *et alii* [10] that gives first insights on the nature of the relevant probability models. The chief merit of this study is that it analyzes data from American and European elections. It is shown that, for elections with a large number of voters N, the $N^{1/2}$ scale for the differences between two issues is not correct and must be replaced by an N^{α} scale [3]. This confirms that the search for the adequate $f(p)$ (which must

[3]Using statistical technique, the authors arrive at $\alpha = .9$, but themselves insist that this value must be taken with caution and that a N scale may be correct.

be reasonably stable from one election to the other) is of crucial importance. Of course, this difference between IAC and IC votes is much less important when the number of voters is small. But, with 27 members, the process of voting may become more frequent and more important, then realistic models must be used. Analysis of the previous votes and “experimental votes” (to test new hypothesis) will be welcome.

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Appendix

We provide in this appendix the details of the calculations leading to relations (3) and (4). We suppose that N is large enough to use asymptotic calculations.

1. One key vote

Let us suppose that after the i first votes, the number x of "yes" mandates has a repartition $F_i(x)$. The next vote brings x at $x + a_{i+1}$ ("yes" vote) with probability p and x stays at its value ("no" vote) with probability $q = 1 - p$. Consequently $F_{i+1}(x)$ reads

$$F_{i+1}(x) = \int F_i(x-a)\, P_{i+1}(a)\mathrm{d}a \tag{9}$$

where $P_{i+1}(a)$ is the probability that x increases of a. Here

$$P_{i+1}(a) = p\,\delta(a - a_{i+1}) + q\,\delta(a) \tag{10}$$

where δ is the Dirac distribution defined by the following properties

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x)\mathrm{d}x = f(x_0) \quad ; \quad \delta(\alpha x) = \frac{1}{|\alpha|}\delta(x)$$

The distribution $\delta(x)$ can be considered as the limit when $\sigma \to 0$ of the Gaussian

$$(2\pi)^{-1/2}\sigma^{-1}\exp(-x^2/2\sigma^2)$$

with $\sigma > 0$.

We recognize in equation (9) a convolution product which is changed into an ordinary product using a Fourier transform. Calling $\hat{F}_{i+1}(k)$, $\hat{F}_i(k)$ and $\hat{P}(k)$ the Fourier transforms of $F_{i+1}(x)$, $F_i(x)$ and $P(x)$ respectively, equation (10) reads

$$\hat{P}_{i+1}(k) = p\,\exp(ika_{i+1}) + q, \tag{11}$$

while the Fourier transform of the final $F_N(x,p)$ reads

$$\hat{F}_N(k,p) = \prod_{i=1}^{N} (p\,\exp(ika_i) + q)\,. \tag{12}$$

The inverse Fourier transform gives the expression of $F_N(x,p)$:

$$F_N(x,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \prod_{i=1}^{N} (p\,\exp(ika_i) + q)\,\exp(-ikx)\mathrm{d}k. \tag{13}$$

Notice that equation (13) is an exact formula valid for any N and describes exactly the 2^N possible configurations associated to the N votes. In the asymptotic limit, the term given by equation (12) only contributes to the integral (13) around $k = 0$. In this operation, we break the true nature of $F_N(x,p)$ which is a sum of 2^N Dirac and turn $F_N(x,p)$ into a continuum. Consequently to obtain the asymptotic solution, we expand this term to first order in k :

$$p\,\exp(ika_i) + q = p(1 + i\,ka_i) + q = 1 + i\,p\,ka_i. \tag{14}$$

Taking the logarithm to compute the product we get

$$\hat{F}_N(k,p) = \exp(ikp\sum_{i=1}^{N} a_i). \tag{15}$$

Calling $A = \sum_{i=1}^{N} a_i$, $F_N(x,p)$ reads

$$F_N(x,p) = \delta(x - pA). \tag{16}$$

But equation (16) is again a quite understandable result. It states that after N votes with probability p of "yes", the number of mandates obtained is pA.

Taking into account in equation (14) the k^2 terms would bring a diffusion around this value. The average displacement varies as A (*i.e.* as N) when the ignored diffusion has a scale of $\sqrt{A}$ (*i.e.* varies as $N^{-1/2}$). The last step is the p integration with

$$F_N(x) = \int_0^1 f(p)\,\delta(x - pA)\mathrm{d}p = \frac{1}{A} f(x/A). \tag{17}$$

Equation (17) is strictly identical to (2) where all voters have the same number of mandates (one, for example). We simply have to replace N by A. In all cases, keeping only the leading term of the expansion, the distribution of x/A reproduces the distribution $f(p)$ which characterizes the GIAC.

2. Double key vote

For the double key vote, the computation is quite similar to the preceding one. The probability $P_{i+1}(a,b)$ that voter $i+1$ moves the score (x,y) to $(x+a, y+b)$ is

$$P_{i+1}(a,b) = p\,\delta(a - a_{i+1})\,\delta(b - b_{i+1}) + q\,\delta(a)\,\delta(b) \tag{18}$$

with, as usual, $q = 1 - p$. The double Fourier transform of equation (18) reads

$$\hat{P}_{i+1}(k,l) = p\,\exp(i\,k\,a_{i+1} + i\,l\,b_{i+1}) + q \tag{19}$$

and the double Fourier transform $\hat{F}_N(k,l,p)$ of $F_N(x,y,p)$ is

$$\hat{F}_N(k,l,p) = \prod_{i=1}^{N} \left[p\,\exp\left(i(k\,a_i + l\,b_i)\right) + q\right]. \tag{20}$$

Again equation (20) is an exact formula. Now to compute its Fourier transform, we expand up to first order in k and l and obtain in this approximation

$$\hat{F}_N(k,l,p) = \exp\left(ip(k\sum_{i=1}^{N} a_i + l\sum_{i=1}^{N} b_i)\right), \tag{21}$$

which is a straightforward generalization of equation (15). Introducing $A = \sum_{i=1}^{N} a_i$ and $B = \sum_{i=1}^{N} b_i$, taking the inverse Fourier transform, we get for these votes characterized by p

$$F_N(x,y,p) = \delta(x - pA)\,\delta(y - pB) \tag{22}$$

which is a generalization of equation (16). For the last step, the p integration, we use the relation

$$\int_{-\infty}^{\infty} \delta(p-\alpha)\,\delta(p-\beta)\,f(p)\mathrm{d}p = \delta(\alpha-\beta)\,f(\xi)$$

where $\alpha = \beta = \xi$. Finally

$$F_N(x,y) = \frac{1}{AB}\,\delta(\frac{x}{A}-\frac{y}{B})\,f\left(p=\frac{x}{A}=\frac{y}{B}\right). \tag{23}$$