Geometrical Working Space, a tool for comparison

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GEOMETRICAL WORKING SPACE, A TOOL FOR COMPARISON

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This theoretical text is nourished by a comparison project (ECOS program 2003-2005) on mathematical curricula between Chile and France. How and what to look at in curricula? What tools could help to produce fruitful comparison? Following the presentation of our theoretical framework, Geometrical Paradigms, the study of an exercise about the determination of inaccessible magnitude, from French and Chilean point de view will lead on to the definition of Geometrical Working Space. With these concepts, we will precise important differences between French and Chilean intended and available curricula, what concerns Geometry between 8th and 10th grade.

INTRODUCTION

Within the context of education research cooperation between Chile and France (aiming at mathematical curriculum comparison) we chose elementary geometry as field of study. We think that Geometry is a good mathematical subject for comparison:

- it is a field studied from infancy to the end of statutory curriculum;
- it is a field in which models are produced with different degrees of complexity: geometric education usually begins by studying and using real material objects (cuboids… graphic lines on a paper sheet or a computer screen), but more stylised than real objects; then it progressively deals with intellectual objects: the mathematician’s square is not the child’s square, it is a construction of the mind which includes an infinite number of points and exists only through its own properties;
- it is a field particularly connected with logical thought, deductive reasoning and proof, a characteristic property of Mathematics.

Our study (Castela & al. 2006) has been carried out on four levels.

- The first level we have studied corresponds to the statutory contents of the syllabus (knowledge, skills and understanding), which international comparison surveys call the *intended curriculum*.

- The second level that is generally described by what we call *accompanying texts* concerns the context, activities and areas of study through which the statutory contents should be taught. According to the countries these texts are mandatory or just pieces of advice.

- The third level is composed of text books that offer an organized list of classroom activities and exercises ready for teaching. We note that the second and third level both concern a part of the *available curriculum*.

- The last level is composed of practise of some teachers from either country and of students' performances confronted to the same geometrical problem.

We (Houdement & Kuzniak 1999, 2002, 2003) have worked on Geometry as it is taught in France and produced a theoretical framework to understand and describe the different meanings determined by the same term of Geometry.

The aim of this text is to show how Geometrical Paradigms and Geometrical Working Space can help to organize a comparative analysis; particularly what concerns *intended curriculum* and *available curriculum* about determination of an inaccessible magnitude.

Let us present Geometrical Paradigms.

GEOMETRICAL PARADIGMS

Our research (Houdement and Kuzniak 1999) following Gonseth (1945-1955) shows how three different paradigms could explain the different forms of geometry. We keep the idea of paradigm from Kuhn (1962; 1970) who used it to explain the development of science. A paradigm is composed of a theory to guide observation, activity and judgement and to permit new knowledge production. A paradigm is shared by a community; the scientific activity of a researcher is guided by the paradigm on which he is working. We made the following hypothesis: Kuhn's analysis of the development of science can be imported into Mathematics, precisely into Elementary Geometry.

We distinguish three paradigms whose names would be easily remembered: Geometry 1, Geometry 2 and Geometry 3. Let us now precise some properties of each paradigm.

Geometry 1

The objects of Geometry 1 are material objects, graphic lines on a paper sheet or virtual lines on a computer screen. Even material, the lines are always consecutive to a first representation of reality. Objects of the sensitive space can be schematised in a micro-space (Berthelot and Salin 1998) by a network of lines. The straight line is a model thus it refuses bumps; the circle is perfect, all its points are at the same distance of the centre. The chosen graphic objects (and their properties) are often in a first time the most convenient to describe reality, hence the name of Natural Geometry for Geometry 1. The objects of Geometry 1 are already the consequences of a first classification that gathers all the objects related by an isometric transformation.

In this paradigm the ordinary techniques are the drawing techniques with ordinary geometrical tools: ruler, set square, compasses but also folding, cutting, superposing…

To produce new knowledge in this paradigm, all methods are allowed: evidence, real or virtual experience and of course reasoning. The backward and forward motion between the model and the real is permanent and enables to prove the assertions: the most important thing is to convince.

Geometry 2

In Natural Axiomatic Geometry (one model is Euclid's Geometry) the objects are no more material but ideal. Definitions and axioms are necessary to create the objects, but in this paradigm they are as close as possible to the intuition of the sensitive space, therefore the name of Natural Axiomatic Geometry. Geometry 2 stays a model of reality. But, once the axioms fixed, demonstrations inside the system are requested to progress and to reach certainty. In this paradigm the text takes a great importance, all the objects should be defined by texts, drawings are only illustrations, accompaniments of textual propositions. As it is convenient the expert works with drawings, but he knows how to read theses drawing and how all the indications he puts on the drawing are validated by the text.

Geometry 3

Lastly we have Formalist Axiomatic Geometry (Geometry III): in this paradigm the system of axioms itself has no relation with reality, it is complete and independent of its possible applications to the world. This paradigm is not very present in statutory curriculum.

Relationships between the two main paradigms, Geometry 1 and Geometry 2

The true question of geometrical teaching concerns Geometry 1 and Geometry 2. Here a table that resumes the main differences between the two paradigms.

	Geometry 1	**Geometry 2**
Space	Intuitive and physical space	Geometrical Euclidian space
Objects	*Material* objects (or digital ones). *Drawings, models*, products of instrumental activity	*Ideal* objects without dimension Figures (*some areas of space, some relations*). *Definitions, theorems*
Artefacts	Various tools (ruler, set square, template, paper folding….). Dynamic Software.	Physical tools (ruler, compass) with use theoretically justified "Logical-deductive reasoning"
Proof	Evidence, checking by instrument (f.i dragging) OR effective construction	Properties and "pieces of demonstration" (formal proof) Partial of axiomatic
Measuring	Licit: it products knowledge	Non licit for production of

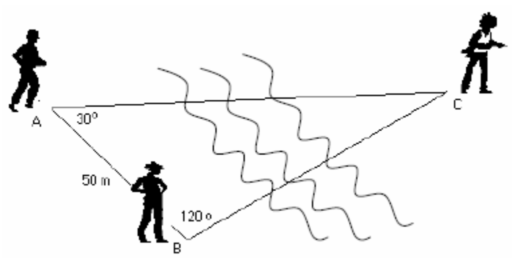
		knowledge, but licit for heuristics
Status of drawing	Object of study and object of validation	Heuristic tool, support of reasoning and “figural concept” (Fischbein 1993)
Privileged aspect	Self-evidence and construction	Properties et demonstration

Table 1: Differences between Geometry 1 and Geometry 2

One paradigm is not superior to the other in their relation to space as shown by the study of the following exercise.

HOW DO GEOMETRICAL PARADIGMS WORK?

A particular study



The drawing shows André and Bernard standing on the same river bank at a distance of 50 meters from each other. Camille stands on the opposite bank.

How far away is André from Camille?

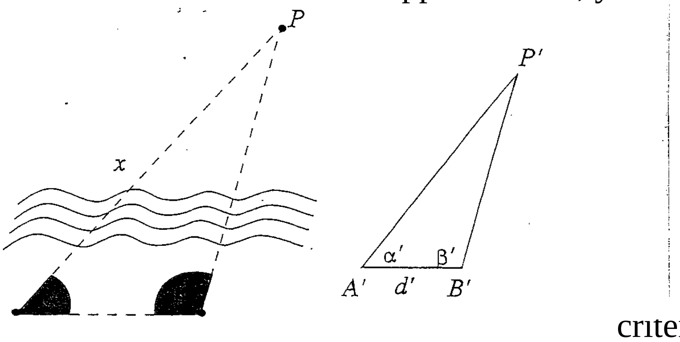
Figure 1: Excerpt coming from *Matemática 2° Medio.* Chile: Arrayan Editores (2001),

Why did we choose this exercise of a Chilean text book for 10th grade -15-16 old students? First it evokes a real problem through a representation of the situation. But the representation is not transparent; it must be read with geometrical knowledge: the given triangle is isosceles, which can not be seen immediately. To be informed of the nature of the triangle it is necessary to deduce it from the information provided by the angles. This first part of geometrical activity is important and related to the “education of sight” in geometrical teaching.

How could it be solved? A first method consists in constructing a similar triangle A’B’C’ on another scale, measuring A’C’ and deducing AC through calculation. In the French curriculum this method would be accessible in the 7^{th} grade, but rejected in upper grades. Another method, more formal, consists in first deducing from the angle magnitudes that the triangle is isosceles (using the theorem of the sum of three angles in a triangle) and then trying to calculate the unknown length: this calculation requires the drawing of further lines like the right bisector of AC or the perpendicular height from B -to obtain two right angled triangles) and the use of theorems like Pythagoras or cosine. In the French curriculum these methods are expected from 8^{th} to 10^{th} grade.

What does the Chilean text book of the 10th grade suggest? We can deduce it from the study of another activity in the same book, just before the preceding river exercise.

If you want to calculate the distance between a point A that is situated on the river bank and a tree that is situated on the opposite bank, you can act this way:



$$\frac{x}{A'P'} = \frac{d}{d'}$$

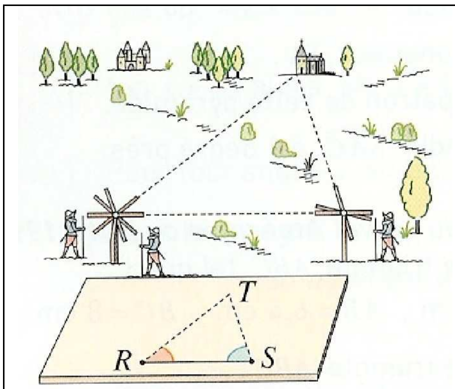
tuate a point B at a determined distance A;
easure off the angles PAB and ABP taking ɔf sight;
easure off the distance AB;
onstruct a scale drawing of a triangle P' similar to the triangle ABP (angular criteria for similarity);
5- measure with a ruler the length of A'P';
6- calculate the length of AP taking into account the similarity ratio of the scale d/d'.

Figure 2: Excerpt coming from *Matemática 2° Medio.* Chile: Arrayan Editores (2001)

The heart of the solution is propositions 4-5-6; the former one helps to transform a space question (to calculate a real distance) into a geometric question.

It is remarkable that the Chilean textbook recommends to draw and to measure on the drawing. The drawing is an object of study and permits to obtain the unknown length by effective measuring.

It would be inconceivable at the same age group in France: the unknown length could only be deduced from given textual information in a way as independent as possible from the drawing in most French text books of 9th grade where no other method is suggested, as it is shown below.



To determine inaccessible magnitude…

A precise point T is taken as sight from situated points R and S whose distance as the crow flies is known.

Then the angles of the triangle RST are measured, which allows to determine the distances with convenient approximation, **because of** :

$$RT = RS\frac{\sin \hat{S}}{\sin \hat{T}} \text{ et } ST = RS\frac{\sin \hat{R}}{\sin \hat{T}}$$

Figure 3: Excerpt coming from *Maths 3°.Cinq sur Cinq*. France : Hachette (2003)

Already in most of the 8th grade (13-14 years old students) French text books there is the assertion « *Seeing or measuring on a drawing is not enough to prove that a* *geometrical phrase is true* » (*Triangle 5ème* Editions Hatier 2001 page 127, *Triangle 4ème* Editions Hatier 2002 page 94…).

Thus the Chilean curriculum accepts and expects a method that is refused at the same grade in the French curriculum. These methods would be accepted in France in lower grades, but such problems whose first work consists in thinking how (and why) to schematize reality (propositions 1-2-3-4) are generally not proposed in lower grades text books.

Consequently in similar questions 10th grade French students prefer not to answer rather than to propose an answer by making a drawing and measuring it.

An analysis with Geometrical Paradigms

In 10th grade even if students are confronted to the same river problem, the answers are not the same: France considers that a treatment in Geometry 1, with the effective use of measures is not convenient. On the contrary in Chile a treatment in Geometry 1 is convenient and recommended by text books as we have seen above.

To solve practically the problem, the first method, drawing at scale that takes place in Geometry 1 is sufficient and effective. The other methods, in Geometry 2 because they don't depend on the drawing, consider ideal situations and use conceptual results: they bring more precision and allow generalisation without new drawings. But precision and generalisation are not required in the river problem. The other methods enable to solve other questions than the determination of that distance only.

It looks as though in France, Geometry 2 takes the place of Geometry 1 and makes it disappear, whereas it is easy to see how complimentary both paradigms are.

Knowledge and practise of Geometry 1 is always necessary first to realise a convenient drawing (see the first exercise), more generally to treat space professional problems with drawing as schematisation; secondly to visualize specific configurations in this drawing (add right further lines to divide the first triangle into two right angled triangles): Duval (1998) already studied the importance of visualization.

Geometry 2 often permits generalisation and logical justification of action in Geometry 1. Geometry 1 is necessary to Geometry 2 as an experience field (Boero 1994), but could not be reduced to an application of Geometry 2.

We now need a new concept to conciliate Geometry 1 and Geometry 2, *Geometrical Working Space* (Kuzniak 2004).

GEOMETRICAL WORKING SPACE: GWS

The Geometrical Working Space (GWS) is the place organized to ensure the geometrical work and to integrate the play between both paradigms. It puts the three following components in a network:

- the objects whose nature depends on the geometrical paradigm,

- the artifacts like drawings tools, computers but also rules of deduction used by the geometrician,

- a theoretical system of reference possibly organized in a theoretical model depending on the geometrical paradigm.

The Geometrical Working Space becomes manageable only when its user can link and master the three components above mentioned. An expert solving a problem of geometry creates a *suitable GWS* to work. This GWS must comply with two conditions: its components should be sufficiently powerful to handle the problem in the right geometrical paradigm and its various components should be mastered and used in a valid way. When the expert has decided what geometrical paradigm is convenient for the problem, s/he can organize the use of artifacts and the type of reasoning thanks to the GWS which suits this paradigm.

When a person (student or professor) is confronted to a problem, this person handles the problem with his/her *personal GWS*. This *personal GWS* generally depends on the knowledge of the person but also on the institution where the person works: what kind of geometrical productions are accepted or valorised by the institution at any time?

Through the organization of the geometrical different contents by grade, the teaching recommendations to the teachers and the notes about how a student can learn geometry, the curricula define specific geometrical environments that can also be seen as GWS: we will call them *institutional GWS*.

THE INSTITUTIONAL GWS OF A PARTICULAR THEME

Taking an example "figures of same shapes", it is easy to make clear the difference between Chile and France, only through a syllabus reading.

In France the different notions: enlargement-reduction (4^{th} and 5^{th}), scale representation and lengths (7^{th}), Thales theorem (8^{th} and 9^{th}), similar triangles (10^{th}), enlargement transformation (11^{th} in speciality) are successively taught in different grades with a perspective strongly focused on Geometry 2 from 8^{th} (following syllabus and textbooks). Thus scale representation (and plan reading) could not be functional either in mathematical activities (it becomes fast forbidden to measure on drawing) or in practical problems (not practised in classrooms).

In Chile students meet enlargement-reduction activities first in 6^{th} grade, similar triangle and scale representation in 8^{th} with a Geometry 1 perspective on lengths and angles and in relation to proportionality. But in 10^{th} grade all these notions are taught again in a network with mathematics' complements (Thales theorem, enlargement transformations) and also history and arts complements about the theory of proportions. The main perspective is always Geometry 1 to create relationships between different notions of a same theme and construct the students' practical culture, even nourished by some theoretical results of Geometry 2 (like Thales theorem).

We think that relating mathematical teaching to reality including in the succession of the different notions of a same theme is also a way to define *institutional GWS.* The Chilean curriculum permits a play between both paradigms from 10th grade; the French curriculum does not officially permit that different ways to solve a problem meet, for it officially rejects Geometry 1 already from 8th grade.

The study of the *institutional GWS* has become our first work to precise the difference between both curricula.

BACK TO GENERAL COMPARISON THROUGH INSTITUTIONAL GWS

We will try to precise particularly the crucial differences between Chile and France for the period between 8th and 10th grade.

The system of reference

Both curricula don't act with the same *institutional GWS*. The **French** reference is Geometry 2: the unique authorized public reasoning concerns ideal objects and even conceptual objects and logic deduction. Geometry 1 is not a suitable paradigm in French 10th grade curriculum; it is not officially integrated in the institutional GWS; it must stay private. **In Chile** Geometry 1 is an assumed reference and plays a public role in the institutional GWS. Geometry 2 can exist too, but it is entirely under the teacher's responsibility.

The place of drawing

In Chile the drawing is taken as a field of experience (Boero 1994) and also a validation object: a field of experience because students are taught to experiment on drawings, to look for reasons of regularities on drawings, to extend validity of observed regularities on drawing; a validation object because constructing a drawing allows to check regularities and to convince of the plausibility of an assertion.

The drawing with usual geometrical tools is considered as a prime model of reality: for example the triangle is introduced as the simplest non deformable structure to show its interest for construction.

A special teaching time is dedicated to techniques of drawing and construction drills (not directly but through various activities).

In France geometrical drawing has no official place; it must stay private and only serve as a support for a conjecture. But how it can serve for geometrical thinking is not taught, thus it can not constitute an experience field. Out of the private mind, drawing is simply and purely forbidden.

Construction activity (for example with ruler and compasses) is not emphasized (it disappears in France from 6^{th} grade) and in the textbooks each spatial problem is immediately illustrated by a drawing, so that students are always in front of a schematised situation. The construction act appears as not very important for geometrical thinking in French curricula.

Validation

In France the only recognized validation is that which verifies the non contradiction inside mathematics; a new proposition is accepted as valid only if it can be logically deduced from other accepted propositions.

In Chile two levels of validation are accepted and distinguished: first conformity to reality, reality of the sensible world, the graphic line on paper; this conformity can be a pretext for a declaration that is recognized and accepted as ‘plausible’; this declaration must be demonstrated to become true in mathematics.

The geometrical objects

From French 8^{th} grade, licit geometrical objects are definitions and theorems, hence only textual declarations that can be accompanied by drawing (as ‘figural concept’ Fischbein 1993). Thus all objects are conceptual, that means ideal but coherent with and inside a theory (Bunge 1983). There is no recognized place for other objects (material or virtual), even if they are used inside the classroom.

In Chile all the objects are accepted, material (like drawings), ideal, but the quality of the declaration made about the drawing does not have the same conceptual quality as that made by the teacher quoting mathematics.

CONCLUSION

For our comparison we have studied syllabus, accompanying texts and text books through a particular filter: *institutional GWS*. GWS organizes different components of geometrical activity: what objects, what licit tools and what licit validation, what play between both paradigms? Let us resume the main differences.

The study of the nature of objects and the validation precise what paradigm is referent and what type of reasoning is valid inside the *institutional GWS*. **Chile** accepts explicitly two levels of reasoning, thus implicitly two paradigms (Geometry 1 and Geometry 2). **France** only considers a deductive organisation of discourse (reference Geometry 2) as licit to produce valid declarations.

The study of drawing is related to licit tools (and the use of these tools and the teaching of the use of these tools); the given status of drawing contributes to define the institutional GWS. In **Chile** Geometry 1 and all the work on drawing is considered as the heart of geometry, the experience field on what the students could constitute their prime experience and confront their declarations. In **France**

Geometry 1 is considered as a perturbation of geometrical teaching that must be forgotten to access to “true geometry”.

Our very few effective class practices seem to confirm these differences but a larger survey would be necessary to take a sight of *implemented curriculum* and *attained curriculum.*

We hope our readers will be convinced that an entry through the *institutional GWS* in different grades of curricula could produce rich comparison at least in *intended curriculum* and *available curriculum* and open new perspectives for geometrical teaching in his/her own country.

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