Stochastic dominance, risk and disappointment: a synthesis

Thierry Chauveau

UNIVERSITÉ PARIS 1 – PANTHÉON - SORBONNE –

Documents de Travail du Centre d'Economie de la Sorbonne

CES Working Papers

Stochastic dominance, risk and disappointment: a synthesis

Thierry CHAUVEAU

2014.54

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://centredeconomiesorbonne.univ-paris1.fr/
ISSN : 1955-611X

Stochastic dominance, risk and disappointment: a synthesis.

*June 2014**

Thierry Chauveau**

ABSTRACT: Although it is endowed with many interesting properties, the theory of decision-making under risk by Loomes and Sugden [1986] has never been given an axiomatics. In this paper, we make up for this omission because their lottery-dependent functional is endowed with many interesting properties to which little attention has been paid up to now. In particular, investors whose preferences are represented by the functional are rational in that (a) they actually behave differently if they are risk averse or risk prone, (b) risk is defined in a consistent way with risk aversion, (c) the functional is but the opposite to a convex measure of risk (Föllmer and Schied [2002]) when constant marginal utility is assumed and (d) violations of the second-order stochastic dominance property are allowed for when "utils" are substituted for monetary values. Moreover, the partial weak order induced by stochastic dominance over utils is as "close" to the weak order of preferences as possible and utility functions may be elicited.through experimental testing.

JEL classification: D81. Key-words: disappointment, risk-aversion, expected utility, risk premium, stochastic dominance, subjective risk.

RESUME: La théorie de la décision en univers risqué de Loomes and Sugden [1986] possède des propriétés fort intéressantes –qui n'ont guère été, jusqu'à aujourd'hui, mises en évidence–: les investisseurs dont des préférences sont représentées par une fonctionnelle "lottery-dependent" font preuve de rationalité parce que (a) ils se comportent vraiment différemment, selon qu'ils ont de l'aversion ou du goût pour le risque (b) leur aversion pour le risque est définie de façon cohérente avec la notion de risque, (c) si l'utilité marginale de leur richesse est constante, la fonctionnelle de leurs préférences n'est que l'opposé d'une mesure de risque convexe à la Föllmer et Schied [2002] et *(d)* ils ne sont pas astreints à respecter la propriété de dominance stochastique de second ordre. En dépit de ses qualités, la théorie de Loomes et Sugden est demeurée sans fondement axiomatique. C'est cette lacune que nous nous efforçons de combler. A noter que si l'on substitue aux revenus monétaires les utilités qu'ils procurent, la dominance stochastique de second ordre, dite subjective, sera bien respectée, que le préordre partiel que cette relation binaire engendre est aussi proche que possible du préordre des préférences et que la fonction d'utilité peut être déterminée à partir de tests expérimentaux.

Classification JEL: D81. Mots-clés: aversion pour le risque, déception, prime de risque, risque subjectif, utilité espérée.

* This paper will be presented at the 2014 FUR conference in Rotterdam. It was first presented at the Paris1 Finance Seminar on the 20th of June 2012. It was entitled: Subjective risk and disappointment. An intermediate version was published as a *CES working paper* in January 2013.

** Université Paris-I-Panthéon-Sorbonne. e-mail: chauveau@univ-paris1.fr

1 Introduction

The theory of decision-making under risk which has been developed by Loomes and Sugden [1986] (henceforth L&S) is endowed with four important properties: (*i*) it allows for an attitude towards risk which does not require any assumption about marginal utility, (*ii*) it yields coherent definitions of risk and risk aversion, (*iii*) it is close to Artzner's *et alii* [1997] approach of a measure of risk and (*iv*) it allows for many behavioural anomalies.

To understand what property (*i*) actually means, recall that an investor is generally assumed to be sensitive to the utility of his wealth. For instance, when expected utility theory (henceforth EU theory) is valid, the investor's welfare is a probability weighted average of the utilities of the possible outcomes. As a consequence, he is risk-averse (prone) if his elementary utility function is concave (convex). Anyway, in any case, the investor takes into account but an average of the results of a gamble to the results of which he is sensitive. Hence, whatever his attitude towards risk (risk-aversion, risk-proneness or neutrality), he actually behaves in the same way. This is a well-known paradox. By contrast, according to L&S', an investor who averages utilities is risk-neutral. He is risk-averse (prone) if and only if (henceforth *iff*) his welfare includes expected disappointment (elation)[1] in addition to the expected utility of his wealth, and, consequently, the paradox vanishes.

Contrarily to property (*i*), properties (*ii*) and (*iii*) have not, up to now, been pointed out in the litterature. Property (*ii*) means that risk may be defined in a consistent way with risk aversion. This occurs in L&S's approach since, as first pointed out by Allais [1979], any risk premium may be split into into elementary risk premia, each of which may be identified to the product of a quantity of risk by a specific risk-aversion. Property (*iii*) is met when constant marginal utility is assumed: the certainty equivalent of a prospect is then the opposite to a convex measure of risk *à la* Föllmer and Schied [2002]. Property (*iv*) consists in allowing for violations of the independence and/or second-order stochastic dominance properties since both of them are commonly observed in experimental tests. Finally these four properties constitute a strong incentive for favouring the approach of L&S.

An additional reason is that this approach makes possible the elicitation of the utility function of an investor. To see this, recall that an investor is assumed to be rational. As a consequence, since he takes into into account the utilities of the outcomes rather than their monetary values, there is no reason why the second order stochastic dominance (henceforth *SD2*) property should be met. By contrast, this may no longer occur when *second-order subjective stochastic dominance* (henceforth *SSD2*) is under review, *i.e.* when statistical tests are undertaken with units of welfare (hencefortfh "utils") instead of monetary units. note that it is clearly met when EU theory is valid, when investors are risk-averse. In the theory of L&S, a similar result may be obtained.[2] Actually,

[1] Elation/disappointment depends on the gap between the utility of the actual outcome and the expected utility of the prospect.

[2] Whatever the sign of the second derivative of the elementary utility function

among the convex or concave utility functions,[3] there exists one of them which is most likely to represent the investor's preferences since (a) it allows for no violation of *SSD2* and (b) it makes the preorder induced by *SSD2* be as "close" as possible to that of the investor's preferences.[4] Last but not least, it happens that this utility function may be elicited from a sequence of binary choices.

Unfortunately, the theory of L&S lacks an axiomatic basis. As a consequence, it seems of interest to make up for this omission, *i.e.* to develop a fully choice-based theory of decison making under risk which will encompass, as particular cases, EU theory and that of L&S. Moreover, we may prove that the lottery-dependent functional[5] of L&S is well endowed with the four properties mentioned above and that the corresponding model is empirically testable.

The rest of this article is organized as follows: first, stochastic dominance is revisited and the definition of a rational investor is clarified. Next, the axiomatization of a general theory of disappointment is developed. Section 4 deals with the elicitation property and Section 5 concludes.

2 Revisiting stochastic dominance

2.1 *Preliminary definitions*

From now on, let $\mathbb{W}$ be a set of random prospects whose outcomes are monetary and belong to a bounded interval $[a, b] \subset \mathbb{R}$. An element of $\mathbb{W}$ will be labelled $\widetilde{w}$ and its cumulative distribution function $F_{\widetilde{w}}(.)$. If a random prospect $\widetilde{w}$ has a discrete support $\{w_1, w_2, ..., w_K\}$, it will also be denominated $[w_1, w_2, ..., w_K \ ; \ p_1, p_2, ..., p_K]$ where $p_k = \Pr(\widetilde{w} = w_k)$. A probability mixture of $\widetilde{w}_1$ and $\widetilde{w}_2$, will be denoted $\alpha\widetilde{w}_1 \oplus (1-\alpha)\widetilde{w}_2$, where α belongs to $[0, 1]$. The degenerate lottery whose outcome is w with certainty is $\delta(w)$. Preferences over prospects will be denoted $\precsim$, with $\prec$ (strict preference) and $\sim$ (indifference). The certainty equivalent of the prospect $\widetilde{w} \in \mathbb{W}$ is labelled $\mathbf{c}(\widetilde{w})$, *i.e.* $\widetilde{w} \sim \delta(\mathbf{c}(\widetilde{w}))$. A normalized elementary utility function (henceforth *n. e. u.* function) is a continuously derivable and strictly increasing function mapping $[a, b]$ on to $[0, 1]$. The set of *n. e. u.* functions will be denoted $\mathbb{U}$.

Partial weak orders may be defined independently of preferences: they include first and second-order stochastic dominance (henceforth *SD1* and *SD2*). *SD1* (*SD2*) will be denoted $\precsim_{\mathbf{1}}$,($\precsim_{\mathbf{2}}$) with $\prec_{\mathbf{1}}$ ($\prec_{\mathbf{2}}$) for strict dominance. A partial weak order induced by *SD1* (*SD2*) is consistent with the total weak order of preferences if we have the following implication: $\widetilde{w}_1 \precsim_{\mathbf{1}} \widetilde{w}_2 \Rightarrow \widetilde{w}_1 \preceq \widetilde{w}_2$ ($\widetilde{w}_1 \precsim_{\mathbf{2}} \widetilde{w}_2 \Rightarrow \widetilde{w}_1 \preceq \widetilde{w}_2$). As already said, an investor is assumed to be sensitive but to the utility of an outcome, and, consequently, the consistency of his behaviour must be checked for with a test of stochastic dominance over utils instead of monetary outcomes. Hence it is of interest to substitute the former for the latter *i.e.* to define subjective stochastic dominances as indicated below:

[3] Strictly speaking we should speak of normalized elementary utility functions. See below.

[4] The exact definition of the closeness of the two preorders is postponed untill the next section.

[5] Lottery-dependent functionals were first presented by Becker and Sarin [1967].

Definition 1 (*First-order and second-order subjective stochastic dominance (henceforth SSD1 and SSD2)*). *Let $\widetilde{w}_1$ and $\widetilde{w}_2$ be two arbitrary random prospects. Let $u(.)$ be a n. e. u. function and let $\widetilde{\boldsymbol{\omega}}_i = u(\widetilde{w}_i)$ for $i = 1, 2$. It is equivalent to state that $\widetilde{w}_1$ dominates $\widetilde{w}_2$ by SSD1 (SSD2) or that $\widetilde{\boldsymbol{\omega}}_1$ dominates $\widetilde{\boldsymbol{\omega}}_2$ by SD1 (SD2), i.e.*

$$\widetilde{w}_1 \precsim_{\mathbf{1}}^{u} \widetilde{w}_2 \Leftrightarrow \widetilde{\boldsymbol{\omega}}_1 \precsim_{\mathbf{1}} \widetilde{\boldsymbol{\omega}}_2 \qquad (\widetilde{w}_2 \precsim_{\mathbf{2}}^{u} \widetilde{w}_1 \Leftrightarrow \widetilde{\boldsymbol{\omega}}_2 \precsim_{\mathbf{2}} \widetilde{\boldsymbol{\omega}}_1),$$

where SSD1 (SSD2) is denoted $\precsim_{\mathbf{1}}^{u}$ ($\precsim_{\mathbf{2}}^{u}$).

Of course, looking at levels of outcomes may be equivalent to looking at utilities. This happens to be the case when first-order dominance is looked at. Indeed, first-order stochastic dominance is a property which is conservative through the change of random variable: $\widetilde{\boldsymbol{\omega}} = u(\widetilde{w})$.[6] By contrast, this result is no longer valid, when second-order stochastic dominance is considered. Actually, the following characterization of *SSD2* holds:

Proposition 1 (characterization of *SSD2*). *Let $\widetilde{w}_1$ and $\widetilde{w}_2$ be two arbitrary random prospects, let $u(.)$ be a n. e. u. function and let $\widetilde{\boldsymbol{\omega}}_i = u(\widetilde{w}_i)$ for $i = 1, 2$. It is equivalent to state:*

(a) $\widetilde{w}_1$ dominates $\widetilde{w}_2$ by SSD2 (i.e. $\widetilde{w}_2 \precsim_{\mathbf{2}}^{u} \widetilde{w}_1$) or

(b) $\int_a^z u'(x)\left(F_{\widetilde{w}_1}(x) - F_{\widetilde{w}_2}(x)\right)dx \leq 0$ for any $z \in [a, b]$

Proof. It is given in Appendix 2.□.

As a consequence, it is convenient to set the following definition

Definition 2 (consistency/inconsistency).

A n. e. u. function $u(.)$ is consistent with the weak order of preferences $\preceq$ if the partial weak order induced by SSD2 is consistent with the total weak order induced by preferences, i.e. $\widetilde{w}_2 \precsim_{\mathbf{2}}^{u} \widetilde{w}_1 \Longrightarrow \widetilde{w}_2 \precsim \widetilde{w}_1$ for any pair of prospects $(\widetilde{w}_1, \widetilde{w}_2)$.

A n. e. u. function $u(.)$ is inconsistent, if the partial weak order induced by SSD2 contradicts the total weak order induced by preferences, i.e. if there exists at least one pair of prospects $(\widetilde{w}_1, \widetilde{w}_2)$ such that simultaneously $\widetilde{w}_2 \precsim_{\mathbf{2}}^{u} \widetilde{w}_1$ and $\widetilde{w}_2 \succ \widetilde{w}_1$, or such that $\widetilde{w}_2 \prec_{\mathbf{2}}^{u} \widetilde{w}_1$ and, simultaneously, $\widetilde{w}_2 \succsim \widetilde{w}_1$.

The above definition implies that a *n. e. u.* function is either consistent or inconsistent and the weak order induced by *SSD2* is partial. Hence the next definition will make sense:

Definition 3 (comparable prospects). *Two prospects are comparable with respect to the n.e.u. utility function $u(.)$ –or, in short, comparable– iff either $\widetilde{w}_1$ dominates $\widetilde{w}_2$ by SSD2 or if $\widetilde{w}_2$ dominates $\widetilde{w}_1$ by SSD2. The subset of prospects which are comparable with respect to $u(.)$ will be denominated $\mathbf{W}_{\mathbf{2}}^{u}$.*

If $u(.)$ is the affine function $u(x) = x$, it is equivalent to state that two prospects are comparable or that one of them is riskier –according to Rothschild and Stiglitz $[1970]'$s definition of risk– than the other. Hence a generalization of their point of view may be the following: if $\widetilde{w}_2 \precsim_{\mathbf{2}}^{u} \widetilde{w}_1$ and that $\int_a^b u'(x)\left(F_{\widetilde{w}_1}(x) - F_{\widetilde{w}_2}(x)\right)dx = 0$ we shall say that $\widetilde{w}_1$ *are RS-comparable* $\widetilde{w}_2$, and we shall write $\widetilde{w}_1$ **RS** $\widetilde{w}_2$.

[6] The proof of this statement is trivial.

A first example of consistent $n.$ $e.$ $u.$ function is that of a risk-averse investor obeying EU theory since we have $\mathbf{E}\left[u\left(\widetilde{w}\right)\right]=1-\int_a^b u'\left(x\right)F_{\widetilde{w}}\left(x\right)dx$. A second example is that of an investor whose preferences are represented by the following functional

$$\mathbf{U}\left(\widetilde{w}\right)\overset{def}{=}\int_a^b\left(u\left(x\right)+\mathcal{E}\left(u\left(x\right)-\mathbf{E}\left[u\left(\widetilde{w}\right)\right]\right)\right)dF_{\widetilde{w}}\left(x\right)\quad \textbf{(Eq. 1).}$$

where function $\mathcal{E}\left(.\right)$ meets the following additional conditions: *(a)* $\mathcal{E}\left(0\right)=0$, *(b)* $0<\mathcal{E}'\left(x\right)<1$, *(c)* $\mathcal{E}''\left(x\right)<0$ and *(d)* $\sup\mathcal{E}'(z)\leq 1$,[7]:The functional $\mathbf{U}\left(.\right)$ will be called, from now on, a *LS-functional* (after Loomes and Sugden [1986]) It is endowed with the following property:

Proposition 2. *The $n.$ $e.$ $u.$ utility function of a LS functional is consistent.*

Proof. It is given in Appendix 2.□.

Now let $\mathbf{W_2^{u+}}$ ($\mathbf{W_2^{u-}}$) consist in the subset of pairs of prospects $(\widetilde{w}_1,\widetilde{w}_2)$ over which the two weak orders, $\precsim_\mathbf{2}^u$ and $\preceq$, coincide (disagree). A $n.$ $e.$ $u.$ function $u\left(.\right)$ is all the more a good candidate for representing the preferences of an investor that $\mathbf{W_2^{u+}}$ is larger and $\mathbf{W_2^{u-}}$ more tiny. Now, since we want to describe the behaviour of a rational investor – *i.e.* since we want to rule out violations of *SSD2–,* we focus on consistent $n.$ $e.$ $u.$ functions. If two $n.$ $e.$ $u.$ functions, $u\left(.\right)$ and $v\left(.\right)$, are consistent, then $\mathbf{W_2^{u-}}=\mathbf{W_2^{v-}}=\varnothing$ and $u\left(.\right)$ will be "better" than $v\left(.\right)$ *iff* $\mathbf{W_2^{v+}}\subseteq\mathbf{W_2^{u+}}$. However, note that there may exist two consistent $n.$ $e.$ $u.$ functions $u\left(.\right)$ and $v\left(.\right)$ such that neither $\mathbf{W_2^{u+}}\subset\mathbf{W_2^{v+}}$ nor $\mathbf{W_2^{v+}}\subset\mathbf{W_2^{u+}}$.

Finally, one may define a binary relation over the weak orders induced by *SSD2* as follows: the weak order $\precsim_\mathbf{2}^u$ is closer to the weak order $\preceq$ than the weak order $\precsim_\mathbf{2}^v$ *iff* either $\mathbf{W_2^{u-}}\subset\mathbf{W_2^{v-}}$ or $\mathbf{W_2^{u-}}=\mathbf{W_2^{v-}}$ and $\mathbf{W_2^{v+}}\subseteq\mathbf{W}$ and we are looking for a consistent $n.$ $e.$ $u.$ function $u\left(.\right)$, such that, among the weak orders induced by *SSD2*, $\precsim_\mathbf{2}^u$ will be the closest to $\preceq$. In other words, for any other consistent function $v\left(.\right)$, we shall have $\mathbf{W_2^{v+}}\subseteq\mathbf{W_2^{u+}}$. Unfortunately, no function will exhibit such a property unless some additional restrictions are put to the $n.$ $e.$ $u.$ functions, as illustrated by the following proposition:

Proposition 3. *Let $u\left(.\right)$ and $v\left(.\right)$ be two $n.$ $e.$ $u.$ functions such that $u\left(.\right)$ is more concave (i.e. less convex) than $v\left(.\right)$. Then, the following implication will hold:*

$$\widetilde{w}_1\precsim_\mathbf{2}^v\widetilde{w}_2\Rightarrow\widetilde{w}_1\precsim_\mathbf{2}^u\widetilde{w}_2$$

and so will the following inclusions:

$$\mathbf{W_2^{v+}}\subseteq\mathbf{W_2^{u+}}\ ;\ \mathbf{W_2^{v-}}\subseteq\mathbf{W_2^{u-}}$$

Proof. It is given in Appendix 2.□

Clearly the above proposition means that concavifying utility functions increases the size of the subset of comparable pairs of prospects *i.e.* both the size of the subset of the pairs of prospects which do not violate *SSD2* and that of the subset of the pairs of prospects which do violate *SSD2.* The more concave will be the utility function $u\left(.\right)$, (a) the more numerous will be the pairs of comparable random prospects and (b) the more numerous will be the violations of *SSD2.*

[7] The meaning of the four conditions are as follows: *(a)* no elation/disappointment is experimented if the actual outcome coincides with its expected value; *(b and c)* elation/disappointment is an increasing and concave function of the difference between the utility of the actual outcome and that of its expected value; *(d)* is a technical condition.

Hence, the following question rises: does there exist a concave or convex *n. e. u.* function which dominates the others in that it makes the two preorders $\precsim_{\mathbf{2}}^{u}$ and $\preceq$, never disagree and coincide on a maximum number of pairs of prospects? The answer to the question is given by the following proposition.

Proposition 4. (canonical utiliy function). *There exists a unique n. e. u. function* $\mathbf{u}(.)$ *such that any concave or convex n. e. u. function* $u(.)$ *which is more concave than* $\mathbf{u}(.)$ *is inconsistent. Function* $\mathbf{u}(.)$ *is concave or convex and it will be called, from now on, a canonical utility function.*

Proof. See Appendix 2.□

As a consequence,an investor's welfare is valued through his (canonical) utility function and he is well rational in that no *SSD2.*violations may occur. However appealing is this result, it is not sufficient to provide a representation of the investor's preferences. Hence we now develop a simple theory of decision making under risk from which a functional *à la* L&S will be derived. The investor will compare, on the one hand, the (canonical) utility of his actual outcome and, on the other, a reference utility which will be an average of the (canonical) utilities of the outcomes of the prospect, namely the expected canonical utility of the prospect $\mathbf{E}[\mathbf{u}(\widetilde{w})]$..This level of reference will be called *zero-disappoinment utlity* and the *zero-disappointment equivalent* of $\widetilde{w}$ will be defined by the following equality: $\mathbf{z}(\widetilde{w}) = \mathbf{u}^{-1}(\mathbf{E}[\mathbf{u}(\widetilde{w})])$.

3 A simple theory of decision making under risk

A fully choice-based theory of decision making under risk is now presented.

3.1 *The axiomatics*

The first step consists in assuming that preferences obey the two first axioms of EU theory.

Axiom 1 (total ordering of $\preceq$). *The binary relation $\preceq$ is a complete weak order.*

Axiom 2 (continuity of $\preceq$). *For any prospect $\widetilde{w} \in \mathbb{W}$ the sets $\{\widetilde{v} \in \mathbb{W} \mid \widetilde{v} \preceq \widetilde{w}\}$ and $\{\widetilde{v} \in \mathbb{W} \mid \widetilde{w} \preceq \widetilde{v}\}$ are closed in the topology of weak convergence.*

Axioms 1 and 2 imply that, if $\widetilde{w}'' \preceq \widetilde{w} \preceq \widetilde{w}'$, then there exists $\alpha \in [0,1]$ such that $\alpha\widetilde{w}' \oplus (1-\alpha)\,\widetilde{w}'' \sim \widetilde{w}$. They also imply that there exists a continuous utility functional, $\mathcal{U}(.)$, mapping $\mathbb{W}$ on to an interval of $\mathbb{R}$ which represents the investor's preferences. It is defined up to a strictly continuous and increasing transformation. From now on, the set which includes the functionals such as $\mathcal{U}(.)$ will be denominated $\mathbb{V}$. To get stronger results one (or more) additional axiom(s) must be set. In EU theory, a third axiom, namely the independence axiom is set. It is valid over the whole set $\mathbb{W}$. However, to account for anomalies, one must weaken the axiom. This can be done through assuming that the independence property is met only on subsets of prospects exhibiting the same zero-disappoinment equivalent The reason for this is that the zero-disappointment equivalent of a mix of two prospects exhibiting the same zero-disappoinment equivalents will be the common value of these two equivalents. Hence, we set the following axiom:

Axiom 3 (weak independence axiom). *The independence property is met over each subset of prospects exhibiting the same* zero-disappoinment equivalent.

whose well-known consequence is the following proposition:

Proposition 5. *Under Axioms 1 to 3, the weak order of preferences $\preceq$ may be represented over $\mathbb{W}_\pi$ by the following functional:*

$$\mathbf{U}_\pi\left(\widetilde{w}\right) \overset{def}{=} \int_a^b \upsilon_\pi(x) dF_{\widetilde{w}}(x),$$

where $\mathbb{W}_\pi \overset{def}{=} \{\widetilde{w} \in \mathbb{W} \mid \mathbf{E}\left[\mathbf{u}\left(\widetilde{w}\right)\right] = \pi\}$. and where $\boldsymbol{\upsilon}_\pi\left(.\right)$ is a continuous and increasing function mapping $[a, b]$ on to $[\boldsymbol{\upsilon}_\pi(a), \boldsymbol{\upsilon}_\pi(b)]$ which is defined up to an affine and positive transformation.

Proof. See, for instance, Fishburn [1970].□

From now on, we set the following normalization conditions:

$$\boldsymbol{\upsilon}_\pi(\mathbf{u}^{-1}\left(\pi\right)) = \pi \quad \text{and} \quad \pi\boldsymbol{\upsilon}_\pi\left(b\right) + \left(1-\pi\right)\boldsymbol{\upsilon}_\pi\left(a\right) = \mathbf{u}_\pi \ (\textbf{Eq. 2}),$$

where $\mathbf{u}_\pi \overset{def}{=} \mathbf{u}(\mathbf{c}(\widetilde{w}_\pi^{a,b}))$. As a consequence, $\boldsymbol{\upsilon}_\pi(.)$ is, from now on, unambiguously defined. Note that $\mathbf{U}_\pi(\widetilde{w}_\pi^{a,b}) = \mathbf{u}_\pi$ and that $\mathbf{U}_\pi\left(\delta\left(\mathbf{u}^{-1}(\pi)\right)\right) = \pi$. Clearly, any random prospect $\widetilde{w} \in \mathbb{W}_\pi$ is such that $0 \leq \mathbf{U}_\pi(\widetilde{w}) \leq 1$ or, equivalently such that $\widetilde{w}_\pi^{a,b} \preceq \widetilde{w} \preceq \delta\left(\mathbf{u}^{-1}(\pi)\right)$ (**Ineq. 1**)

We now turn to some other important consequences of the above set of axioms. First, for any $\widetilde{w} \in \mathbb{W}_\pi$, there exists a unique real number $\alpha \in [0, 1]$ such that $\widetilde{w} \sim \mathcal{L}_\pi\left(\alpha\right)$.where:

$$\mathcal{L}_\pi\left(\alpha\right) \overset{def}{=} \alpha\delta\left(\pi\right) \oplus \left(1-\alpha\right)\widetilde{w}_\pi^{a,b}.$$

Its existence is a consequence of Axiom 2. Moreover, α is well unique since $\mathcal{L}_\pi\left(\alpha_1\right)$ will strictly dominate $\mathcal{L}_\pi\left(\alpha_2\right)$ by *SSD2 iff* $\alpha_1 > \alpha_2$. Next, since both $\widetilde{w}$ and $\mathcal{L}_\pi\left(\alpha\right)$ belong to $\mathbb{W}_\pi$, then, from Proposition 5, we get that:

$$\mathbf{U}_\pi\left(\widetilde{w}\right) = \mathbf{U}_\pi\left(\mathcal{L}_\pi\left(\alpha\right)\right) = \alpha\mathbf{U}_\pi(\delta\left(\pi\right)) + (1-\alpha)\mathbf{U}_\pi(\widetilde{w}_\pi^{a,b}) = \alpha\pi + (1-\alpha)\mathbf{u}_\pi.$$

Now recall that from (**Ineq. 1**) we get that the utility of the certainty equivalent $\mathbf{c}(\widetilde{w})$ of $\widetilde{w}$ (and of $\mathcal{L}_\pi\left(\alpha\right)$) is a convex combination of π and $\mathbf{u}_\pi$ whose weights $(\beta, 1-\beta)$ are yet unknown, *i.e.*, we have:

$$\mathbf{u}(\mathbf{c}(\widetilde{w})) = \beta\pi + (1-\beta)\mathbf{u}_\pi = \beta\mathbf{U}_\pi(\delta\left(\pi\right)) + (1-\beta)\mathbf{U}_\pi(\widetilde{w}_\pi^{a,b})$$

Clearly, consistency will be reached *iff* $\alpha = \beta$ *i.e. iff* $\delta\left(\mathbf{c}(\widetilde{w})\right)$ is equivalent to $\mathcal{L}_\pi\left(\alpha\right)$ (otherwise there would be a contradiction). Finally, we are led to set the following axiom, which is self-explanatory:

Axiom 4 (consistency). *The compound lottery $\alpha\delta\left(\pi\right) \oplus \left(1-\alpha\right)\widetilde{w}_\pi^{a,b}$ is equivalent to the degenerate lottery $\delta\left(\mathbf{u}^{-1}\left(\alpha\pi + \left(1-\alpha\right)\mathbf{u}(\mathbf{c}(\widetilde{w}_\pi^{a,b}))\right)\right)$.*

As a consequence, we get the following proposition:

Proposition 6. *Under Axioms 1 to 4, the weak order of preferences $\preceq$ may be represented over $\mathbb{W}$ by the following lottery-dependent functional*:

$$\mathbf{U}\left(\widetilde{w}\right) \overset{def}{=} \int_a^b \boldsymbol{\upsilon}_{\mathbf{E}[\mathbf{u}(\widetilde{w})]}(x) dF_{\widetilde{w}}(x), \qquad \textbf{(6)}$$

where $\boldsymbol{\upsilon}_{\mathbf{E}[\mathbf{u}(\widetilde{w})]}$ is a continuous and increasing function mapping $[a, b]$ on to $\left[\boldsymbol{\upsilon}_{\mathbf{E}[\mathbf{u}(\widetilde{w})]}(a), \boldsymbol{\upsilon}_{\mathbf{E}[\mathbf{u}(\widetilde{w})]}(b)\right]$ which satisfies the normalization conditions (**Eq. 2**).

Proof. It is trivial since we have $\mathbf{u}(\mathbf{c}(\widetilde{w})) = \mathbf{U}(\widetilde{w})$.□

3.2 *Properties of LS models*

Many examples of models where preferences satisfy Axioms 1 to 4 can be given. Before we review their properties, we now consider $\boldsymbol{v}_z(x)$ as a function of the two variables, namely z and x, and, consequently, we set $f(z,x) \overset{def}{=} \boldsymbol{v}_z(x)$. As a preliminary, consider the case when *marginal utility is constant* ($\mathbf{u}(x) = x$), *i.e.* when the functional reads:

$\mathbf{U}(\widetilde{w}) = \mathbf{c}(\widetilde{w}) = \int_a^b f(\mathbf{E}[\tilde{w}], x) dF_{\tilde{w}}(x)$ **(Eq. 3).**

From the properties of $\boldsymbol{v}_z(x)$ we get that $f(z,x)$ is is strictly increasing with respect to x and meets the following condition: $f(0,0) = 0$. It is of interest to particularize $f(z,x)$ to get a more operational specification. This can be done through assessing additional conditions to preferences.

Consider the risk premium $\mathbf{\Pi}(\tilde{w}) \overset{def}{=} \mathbf{E}[\tilde{w}] - \mathbf{c}(\widetilde{w})$ of an arbitrary prospect $\widetilde{w} \in \mathbb{W}$. One may assume that *risk premia are translation-invariant*, *i.e.*, $\mathbf{\Pi}(\tilde{w} + x) = \mathbf{\Pi}(\tilde{w})$ or, equivalently, $\mathbf{c}(\widetilde{w} + y) = \mathbf{c}(\widetilde{w}) + y$. Under reasonable mathematical assumptions, one may show that a necessary and sufficient condition for $\mathbf{\Pi}(.)$ to exhibit the invariance property is that $f(z,x) = x + \mathcal{E}(x - z)$ where $\mathcal{E}(.)$ is strictly increasing and meets the requirement: $\mathcal{E}(0) = 0$.

Now, we focus on *risk-averse investors*, which means that any prospect $\widetilde{w} \in \mathbb{W}$, will exhibit a negative risk premia, *i.e.* we must have $\mathbf{E}[\mathcal{E}(\widetilde{w} - \mathbf{E}[\widetilde{w}])] \leq 0$. A sufficient condition[8] for this to hold is to assume that $\mathcal{E}(.)$ is concave, which will be, from now on, done.

Finally, the functional is that of a *disappointment model* where elation/disappointment is an increasing and concave function of the excess of the actual outcome over its expected value. It is a particular case of the model developed by L&S. It may also be viewed as the opposite to a *convex measure of risk* (in the sense of Föllmer and Schied [2002]), since one may set:[9]

$\mathbf{r}(\tilde{w}) \overset{def}{=} -\mathbf{c}(\widetilde{w}) = -(\mathbf{E}[\tilde{w}] + \int_a^b \mathcal{E}(x - \mathbf{E}[\tilde{w}]) dF_{\tilde{w}}(x)) = -\mathbf{U}(\widetilde{w})$ **(Eq. 5).**

The interest of the above result is that it allows for *grounding a convex measure of risk on a theory of the behaviour of economic agents towards risk*. The risk controller is then assumed to behave according to Axioms 1 to 4 and to have preferences endowed with the translation invariance property.

Moreover, from **(Eq. 5)** we also get a decomposition of the risk premium $\mathbf{\Pi}(\tilde{w}) \overset{def}{=} \mathbf{E}[\tilde{w}] - \mathbf{c}(\widetilde{w})$ into elementary premia, which can be viewed as the contributions of the variance, the skewness, the kurtosis ... of a random prospect to the total risk premium which is demanded by an investor. If $\mathcal{E}(.)$ is smooth enough, one may write:

$\mathbf{\Pi}(\widetilde{w}) = -\sum_{n=2}^{+\infty} \mathbf{E}[(\widetilde{w} - \mathbf{E}[\widetilde{w}])^n] \mathcal{E}^{(n)}(\mathbf{E}[\widetilde{w}])/n!$ **(Eq. 6)**.

The total risk premium is then an infinite sum of elementary premia, each of which is proportional to the product of two terms: the nth order centered moment of the random variable $\widetilde{w}$, *i.e.* $\mathbf{E}[(\widetilde{w} - \mathbf{E}[\widetilde{w}])^n]$, and the nth order

[8] This is a direct consequence of Jensen's inequality.

[9] The proof of this statement may be found in Chauveau and Thomas [2014]: Valuing non-quoted CDS with consistent default probabilities, forthcoming *CES working paper*

derivative of $\mathcal{E}(.)$ taken at point $z = \mathbf{E}[\widetilde{w}]$. Any even moment is but a quantity of a "symmetric" risk and its coefficient must be negative if the investor is risk averse, whatever the considered definition of risk. An odd moment may be viewed as a quantity of an "asymmetric" risk and its coefficient must be positive if the investor is risk averse. Finally, Equation **(Eq. 6)** may be viewed as a *theoretical grounding of the multimoment approach of the Capital Asset Pricing Model.* Now recall that EU theory is often violated by experiments and that no general agreement has yet been found about the explaining power of its challengers, *i.e.* Non-EU theories. Hence it is interesting to point out that, because of its flexibility, the functional **(Eq. 6)** is compatible with many of the anomalies of financial theory.

We now turn to the general case of *variable marginal utility*. The functional $\mathbf{U}(\widetilde{w})$ will now read: $\mathbf{U}(\widetilde{w}) = \mathbf{u}(\mathbf{c}(\widetilde{w})) = \int_a^b f(\mathbf{E}[\mathbf{u}(\tilde{w})], x) dF_{\widetilde{w}}(x)$. Here again $f(z, x)$ is strictly increasing and concave with respect to x and meets the following condition: $f(0, 0) = 0$. Since investors care but about "utils", the risk premium of an arbitrary prospect $\widetilde{w} \in \mathbb{W}$ is now defined as $\mathbf{\Pi}(\tilde{w}) \overset{def}{=} \mathbf{E}[\mathbf{u}(\tilde{w})] - \mathbf{u}(\mathbf{c}(\widetilde{w}))$ and one may again assume that risk premia are translation-invariant when they are expressed in utils, *i.e.*: $\mathbf{u}(\mathbf{c}(\widetilde{w}) + y) = \mathbf{u}(\mathbf{c}(\widetilde{w})) + \mathbf{u}(y)$. Under reasonable mathematical assumptions, the functional may be identified to a LS-functional which expresses as: $\mathbf{U}(\widetilde{w}) \overset{def}{=} \int_a^b \boldsymbol{v}_{\mathbf{E}[\mathbf{u}(\tilde{w})]}(x) dF_{\widetilde{w}}(x)$ $= \mathbf{E}[\mathbf{u}(\tilde{w})] + \int_a^b \mathcal{E}(u(x) - \mathbf{E}[\mathbf{u}(\tilde{w})])) dF_{\widetilde{w}}(x)$. Elation/disappointment is an increasing and concave function of the excess of the actual outcome over its expected utility. We here also get a decomposition of the risk premium $\mathbf{\Pi}(\tilde{w})$ $\overset{def}{=} \mathbf{E}[\mathbf{u}(\tilde{w})] - \mathbf{u}(\mathbf{c}(\widetilde{w}))$ into elementary premia, which can be viewed as the contributions of the variance, the skewness, the kurtosis ... of the utility of a random prospect to the total risk premium which is demanded by an investor. This was Allais' [1979] original intuition. If $\mathcal{E}(.)$ is smooth enough, one may now write:

$$\mathbf{\Pi}(\widetilde{w}) = -\sum_{n=2}^{+\infty} \mathbf{E}\left[(\mathbf{u}(\widetilde{w}) - \mathbf{E}[\mathbf{u}(\widetilde{w})])^n\right] \mathcal{E}^{(n)}(\mathbf{E}[\mathbf{u}(\widetilde{w})]) / n! \qquad \textbf{(Eq. 6 bis)}.$$

Anyway, the above results are of interest *iff* $\mathbf{u}(.)$ can be elicited. This question is now going to be addressed.

4 The elicitation property.

Actually, as shown in Chauveau and Nalpas [2010], an important property of LS-models is the *elicitation property.* We briefly recall, in this Section, their results. As a preliminary, we consider a new binary relation over $\mathbb{W}$.

Definition 4 (strong indifference). *Two prospects $\widetilde{w}_1$ and $\widetilde{w}_2$ are strongly equivalent iff (a) they are equivalent and (b) they meet the betweenness property.*[10] *The binary relation "$\widetilde{w}_1$ and $\widetilde{w}_2$ are strongly equivalent" will be labelled "$\widetilde{w}_1 \approx \widetilde{w}_2$".*

[10] Recall that two prospects share the betweenness property iff for any $\alpha \in [0, 1]$, $\widetilde{w}_1 \preceq \widetilde{w}_2$ $\Rightarrow \widetilde{w}_1 \preceq \alpha \widetilde{w}_1 \oplus (1 - \alpha) \widetilde{w}_2 \preceq \widetilde{w}_2$.

The binary relation $\approx$ is obviously an equivalence relation over $\mathbb{W}$ and strong indifference implies indifference in the usual sense. The properties of LS-functionals may then be summed up in the following propositions where $\widetilde{w}_p^{a,x}$ ($\widetilde{w}_{1-q}^{y,b}$) denotes the binary lottery $[a, x; 1-p, p]$ ($[y, b; q, 1-q]$).

Proposition 7 (strong indifference). *If preferences are represented by a LS-functional, two prospects $\widetilde{w}_1$ and $\widetilde{w}_2$ are strongly equivalent iff they exhibit the same certainty equivalent and the same zero-disappointment equivalent, what formally reads:*

$\widetilde{w}_1 \approx \widetilde{w}_2 \Leftrightarrow \mathbf{c}(\widetilde{w}_1) = \mathbf{c}(\widetilde{w}_2)$ *and* $\mathbf{E}[\mathbf{u}(\widetilde{w}_1)] = \mathbf{E}[\mathbf{u}(\widetilde{w}_2)]$

Proposition 8 (strong equivalents). *If preferences are represented by a LS-functional, there exists exactly one binary lottery of the $\widetilde{w}_p^{a,x}$ type (of the $\widetilde{w}_{1-q}^{y,b}$ type) which is. strongly equivalent to $\widetilde{w}$. Lottery $\widetilde{w}_p^{a,x}$ ($\widetilde{w}_{1-q}^{y,b}$) will be called the left (right) strong equivalent of $\widetilde{w}$. The degenerate lottery $\delta(z)$ (the binary lottery $\widetilde{w}_{\mathbf{u}(z)}^{a,b}$) is a maximal (minimal) element in $\mathbb{W}_{\mathbf{u}(z)}$, i.e. $\widetilde{w}_{\mathbf{u}(z)}^{a,b} \preceq z \preceq \delta(z)$.*

Proofs. The proofs are given in Appendix 2.□

Let $w \in [a, b]$ ($\pi \in [0, 1]$) be an arbitrary level of wealth (probability). Consider the sequence of binary lotteries labelled $\{\widetilde{w}_{p_n}^{a,x_n}\}_{n\in\mathbb{N}}$ which meets the below requirements:

$x_0 = w,\ p_0 = \pi$ and $\widetilde{w}_{1-p_{n+1}}^{x_{n+1},b} \approx \widetilde{w}_{p_n}^{a,x_n}$

where $\widetilde{w}_{1-p_{n+1}}^{x_{n+1},b}$ is the right strong equivalent of $\widetilde{w}_{p_n}^{a,x_n}$. Clearly, $\{x_n\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence. The difference between the expected utilities of two consecutive binary lotteries, $\widetilde{w}_{p_n}^{a,x_n}$ and $\widetilde{w}_{p_{n+1}}^{a,x_{n+1}}$, is equal to the second weight $(1-p_{n+1})$ of the right strong equivalent $\widetilde{w}_{1-p_{n+1}}^{x_{n+1},b}$ of $\widetilde{w}_{p_n}^{a,x_n}$, what formally reads:

$\mathbf{E}\left[\mathbf{u}(\widetilde{w}_{p_n}^{a,x_n})\right] - \mathbf{E}\left[\mathbf{u}(\widetilde{w}_{p_{n+1}}^{a,x_{n+1}})\right] = 1 - p_{n+1}.$

Consequently, the expected utility of the initial lottery –*i.e.* $\pi\mathbf{u}(w)$– satisfies the following equality:

$\pi\mathbf{u}(w) = \mathbf{E}\left[\mathbf{u}(\widetilde{w}_{p_n}^{a,x_n})\right] + \sum_{i=1}^{n}(1-p_i)$ **(Eq. 10).**

As shown below, the sequence $\{\widetilde{w}_{p_n}^{a,x_n}\}_{n\in\mathbb{N}}$ it converges, in LS-models, towards $\delta(a)$. The result is valid, whatever the value of π.

Proposition 9 (elicitation property). *Assume that preferences are represented by a LS-functional. Then, $\{x_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of real numbers converging towards a. The sequence $\{1-p_n\}_{n\in\mathbb{N}}$ is increasing and converges towards $\pi\ell$ where ℓ does not depend on π and is a strictly increasing function of w, mapping $[a, b]$ on to $[0, 1]$.*

Proof. It is given in Appendix 2.□

Finally, in LS-models, we have the following equality:

$\ell = \lim_{n\to\infty}(\sum_{i=1}^{n}(1-p_i))/\pi$ **(Eq. 11)**

and we may set $\mathbf{u}(w) = l$. It is left to the reader to transpose the above results when the sequence $\{\widetilde{w}_{q_n}^{y_n,b}\}_{n\in\mathbb{N}}$ –where $y_0 = w; q_0 = \pi$ and $\widetilde{w}_{q_{n+1}}^{a,y_{n+1}} \approx \widetilde{w}_{1-q_n}^{y_n,b}$– is substituted for $\{\widetilde{w}_{p_n}^{a,x_n}\}_{n\in\mathbb{N}}$. Finally, note that Axioms 1 to 4 are, at least in principle, experimentally testable since their checking comes down to making choices between binary loteries.

5 Concluding remarks

In this paper, a fully choice-based theory of disappointment has been developed that can be viewed as an axiomatic foundation of models *à la* L&S (1986). LS models are endowed with many interesting properties which have been presented and/or recalled above. Note that the results do not depend on the assumption that the set of possible outcomes is bounded. Indeed, extensions to $\mathbb{R}$ itself are straightforward. Moreover, under the assumption of constant relative risk aversion(s), one can easily implement the above approach to value any financial asset.[11]

References

[1] Artzner P., F. Delbaen, J.M. Eber , D. Heath, 1999. Coherent Measures of Risk, *Mathematical Finance*, **9**, *3*, 203-228.

[2] Allais, M. 1979. The Foundations of a Positive Theory of Choice Involving Risk and a Criticism of the Postulates and Axioms of the American School. In M. Allais and O. Hagen (Eds). *Expected Utility Hypothesis and the Allais Paradox.* D. Reidel Publishing Co, Dordrecht, Holland.

[3] Becker, J.L., R. Sarin. 1987. Lottery Dependent Utility. *Management Science* **33** 1367-1382.

[4] Chauveau Th., N. Nalpas 2010. Disappointment models: an axiomatic approach, *CES working paper*, 2010.102

[5] Föllmer H., A. Schied 2002. Convex measures of risk and trading constraints. *Finance Stoch.* ***6***, 429-447.

[6] Fishburn, P.C. 1970. *Utility Theory for Decision Making.* Wiley, New-York.

[7] Kahneman, D., A. Tversky. 1979. Prospect Theory: an analysis of Decision under Risk. *Econometrica* **47** 263-291.

[8] Loomes, G., R. Sugden. 1986. Disappointment and Dynamic Consistency in Choice under Uncertainty. *Review of Economic Studies* **53** 271-282.

[9] Rothschild M., J. E. Stiglitz 1970. Increasing Risk; I A Definition, *Journal of Economic Theory* **2** 225-243.

6 Appendix 1 (Further developments)

Up to now, it has been implicitly assumed that *n. e. u.* functions of interest were concave or convex over $[a, b]$. However, such an assumption may seem too

[11] An example of this valuation for CDS's may be found in Chauveau and Thomas [2014]. See footnote 7.

restrictive and an straightforward generalization of the above results consists in taking into account smooth *n. e. u.* functions whose graph includes **N** successive concave and convex sections. In this paper, we focus on the simple case when the investor's behaviour may differ, as suggested by Kahneman and Tversky [1979], when he faces gains or losses, *i.e.* when his *ex-post* wealth $\widetilde{w}$ is higher or lower than a reference level w_0 which may be either his initial (certain) wealth or his expected terminal wealth. For instance,the investor may then be more severely disappointed when he looses money (*i.e.* when $\widetilde{w} \in [a, w_0]$) than when he gets more money than he initially had (*i.e.* when $\widetilde{w} \in [w_0, b]$).

As a preliminary to thge study of this new case, note that *n.e.u.* utility functions could have been normalized differently. Indeed the normalization conditions $\mathbf{u}(a) = 0$ and $\mathbf{u}(b) = 1$, play no particular role in the above analysis. Hence it is equivalent to state that $\mathbf{u}(x)$ is a rationalizing function with the standard normalization conditions or that $\boldsymbol{v}(.)$ is also a rationalizing function checking the following conditions $\boldsymbol{v}(a) = \underline{\boldsymbol{u}} \neq 0$ and $\boldsymbol{v}(b) = \overline{\boldsymbol{u}} \neq 1$, *iff.*$\boldsymbol{v}(x) = \underline{\boldsymbol{u}} + (\overline{\boldsymbol{u}} - \underline{\boldsymbol{u}})\,\mathbf{u}(x)$.

Now, from Proposition 4, we get that there exist a unique *n.e.u.* function $\mathbf{u}_1(.)$ ($\mathbf{u}_2(.)$) which rationalizes the investor's preferences over $[a, w_0]$ ($[w_0, b]$). Let $\mathbf{u}_1(x) = \mathbf{u}_1(x)$ ($\mathbf{u}_2^{\boldsymbol{\lambda}}(x) = \mathbf{u}_2(x)$) if $x \in [a, w_0]$ ($x \in [w_0, b]$) and 0 elsewhere. Then $\mathbf{u}_1^{\boldsymbol{\lambda}}(x) = \boldsymbol{\lambda}\mathbf{u}_1(x)$ and $\mathbf{u}_2^{\boldsymbol{\lambda}}(x) = \boldsymbol{\lambda} + (1 - \boldsymbol{\lambda})\,\mathbf{u}_2(x)$.($\mathbf{u}_2^{\boldsymbol{\lambda}}(x)$) is the rationalizing function over $[a, w_0]$ ($[w_0, b]$) checking: $\mathbf{u}_1^{\boldsymbol{\lambda}}(w_0) = 0$ and $\mathbf{u}_1^{\boldsymbol{\lambda}}(w_0) = \boldsymbol{\lambda}$ ($\mathbf{u}_2^{\boldsymbol{\lambda}}(w_0) = \boldsymbol{\lambda}$ and $\mathbf{u}_2^{\boldsymbol{\lambda}}(b) = 1$). Let $\mathbf{u}(x) = \mathbf{u}_1^{\boldsymbol{\lambda}}(x) + \mathbf{u}_2^{\boldsymbol{\lambda}}(x)$. Then $\mathbf{u}(.)$ is a rationalizing function over $[a, w_0]$ and over $[w_0, b]$, To be smooth, it must check the additional condition that its left-hand size derivative, calculated at point w_0 , is equal to the correesponding right-hand size one, *i.e.* $\mathbf{u}_2^{\boldsymbol{\lambda}\prime}(w_0) = (1 - \boldsymbol{\lambda})\,\mathbf{u}_2'(w_0)$ or, equivalently: $\boldsymbol{\lambda} = \mathbf{u_2'}(\mathbf{w_0}) / (\mathbf{u}_1'(w_0) + \mathbf{u}_2'(w_0))$. The condition is clearly sufficient. As a consequence, the real number $\boldsymbol{\lambda}$ corresponding to the smooth function $\mathbf{u}(.)$ is well defined.and the following result holds:[12]

Proposition 10. *Let w_0 be an arbirtrary level of wealth belonging to $]a, b[$. There exists a unique smooth n.e.u. function $\mathbf{u}(.)$ mapping $[a, b]$ on to $[0, 1]$ which is such that its restriction over $[a, w_0]$ ($[w_0, b]$) rationalizes the investor's preferences over $[a, w_0]$ ($[w_0, b]$) It is concave or convex over each subinterval. It will be denominated, from now on, the w_0-rationalizing function of the investor's preferences.*

Proof. See the above discussion.□

Note that if $\mathbf{u}_1(.)$ and $\mathbf{u}_2(.)$ are both concave (both convex), then $\mathbf{u}(x)$ is concave (convex) and we go back to the case initially studied..

7 Appendix 2 (Proofs)

Proof of Proposition 1

Let $\widetilde{\boldsymbol{\omega}}_i = u(\widetilde{w}_i)$ for $i = 1, 2$. By definition of *SSD2*, it is equivalent to state (a) $\widetilde{w}_2 \precsim_{\mathbf{2}}^{u} \widetilde{w}_1$, ($b$) $\widetilde{\boldsymbol{\omega}}_2 \precsim_{\mathbf{2}} \widetilde{\boldsymbol{\omega}}_1$, or ($c$) $\int_0^v [F_{\widetilde{\boldsymbol{\omega}}_1}(t) - F_{\widetilde{\boldsymbol{\omega}}_2}(t)]\,dt \leq 0$ for $v \in [0, 1]$,. This last condition is, in its turn equivalent to the following one:

[12] A geometrical illustration of Proposition 5 is given on Figures 2A to C.

$\int_a^z u'(x)\,(F_{\widetilde{w}_1}(x) - F_{\widetilde{w}_2}(x))dx \leq 0$ for any $z \in [a,b]$, because of the following equality:

$\int_0^v [F_{\widetilde{\boldsymbol{\omega}}_1}(t) - F_{\widetilde{\boldsymbol{\omega}}_2}(t)]\,dt = \int_a^{u^{-1}(v)} (F_{\widetilde{w}_1}(x) - F_{\widetilde{w}_2}(x))u'(x)\,dx.$ $\square$

Proof of Proposition 2

Proof. We want to prove that if $\widetilde{w}_1$ dominates $\widetilde{w}_2$ by *SSD2*, then $\widetilde{w}_1$ is preferred to $\widetilde{w}_2$. To do so, consider two prospects $\widetilde{w}_1$ and $\widetilde{w}_2$. Let

$$\int_a^b u'(t)\,F_{\widetilde{w}_i}(t)\,dt = -\int_a^b u(x)\,dF_{\widetilde{w}_i} = -\mathbf{E}[\widetilde{w}_i] = -\lambda_i$$

for $i = 1,2$. Assume that $\widetilde{w}_1$ dominates $\widetilde{w}_2$ by *SSD2*, we get:

$$\int_a^b u'(t)\,(F_{\widetilde{w}_1}(t) - F_{\widetilde{w}_2}(t))dt = -(\lambda_1 - \lambda_2) \leq 0 \Rightarrow \lambda_1 - \lambda_2 \geq 0$$

We must now show that $\widetilde{w}_1$ is preferred to $\widetilde{w}_2$, or, equivalently, that $\mathbf{U}(\widetilde{w}_1) - \mathbf{U}(\widetilde{w}_2) \geq 0$. Since $\mathcal{E}(.)$ is strictly increasing we also get:

$$\mathcal{E}(u(x) - \lambda_1) \leq \mathcal{E}(u(x) - \lambda_2)$$

and the difference between the two functionals, $\mathbf{U}(\widetilde{w}_1) - \mathbf{U}(\widetilde{w}_2) = \Delta\mathbf{U}$ expresses as:

$$\begin{aligned}\Delta\mathbf{U} &= \int_a^b (u(x) + \mathcal{E}(u(x) - \lambda_1))dF_{\widetilde{w}_1}(x) - \int_a^b (u(x) + \mathcal{E}(u(x) - \lambda_2))dF_{\widetilde{w}_2}(x)\\ &= (\lambda_1 - \lambda_2) + \int_a^b \mathcal{E}(u(x) - \lambda_1)\,dF_{\widetilde{w}_1}(x) - \int_a^b \mathcal{E}(u(x) - \lambda_2)\,dF_{\widetilde{w}_2}(x)\end{aligned}$$

and we get: $\Delta\mathbf{U} = T1 + T2$, where:

$$T1 = (\lambda_1 - \lambda_2) + \int_a^b (\mathcal{E}(u(x) - \lambda_1) - \mathcal{E}(u(x) - \lambda_2))\,dF_{\widetilde{w}_2}(x)$$

and:

$$T2 = \int_a^b \mathcal{E}(u(x) - \lambda_1)\,(dF_{\widetilde{w}_1}(x) - dF_{\widetilde{w}_2}(x))$$

Straightworward calculations give:

$$T1 = (\lambda_1 - \lambda_2) - \int_a^b (\lambda_1 - \lambda_2)\,\mathcal{E}'(u(x) - \lambda_1 + \theta_1(\lambda_1 - \lambda_2))\,dF_{\widetilde{w}_2}(x)$$

with $\theta_1 \in [0,1]$ and:

$$
\begin{aligned}
T2 &= \left[\mathcal{E}\left(u\left(x\right)-\lambda_1\right)\left(F_{\widetilde{w}_1}\left(x\right)-F_{\widetilde{w}_2}\left(x\right)\right)\right]_a^b \\
&- \int_a^b \mathcal{E}'\left(u\left(x\right)-\lambda_1\right)u'\left(x\right)\left(F_{\widetilde{w}_1}\left(x\right)-F_{\widetilde{w}_2}\left(x\right)\right)dx \\
&= \mathcal{E}\left(-\lambda_1\right)\left(F_{\widetilde{w}_1}\left(a\right)-F_{\widetilde{w}_2}\left(a\right)\right) - \int_a^b \mathcal{E}'\left(u\left(x\right)-\lambda_1\right)u'\left(x\right)\left(F_{\widetilde{w}_1}\left(x\right)-F_{\widetilde{w}_2}\left(x\right)\right)dx \\
&= \mathcal{E}\left(-\lambda_1\right)\left(F_{\widetilde{w}_1}\left(a\right)-F_{\widetilde{w}_2}\left(a\right)\right) - \mathcal{E}'\left(1-\lambda_1\right)\int_a^b u'\left(t\right)\left(F_{\widetilde{w}_1}\left(t\right)-F_{\widetilde{w}_2}\left(t\right)\right)dt \\
&+ \int_a^b \mathcal{E}''\left(u\left(x\right)-\lambda_1\right)u'\left(x\right)\left[\int_a^x u'\left(t\right)\left(F_{\widetilde{w}_1}\left(t\right)-F_{\widetilde{w}_2}\left(t\right)\right)dt\right]dx
\end{aligned}
$$

Clearly, if $[1-\sup\mathcal{E}'(z)] \geq 0$ we get: $sign\left(T1\right) = sign\left(\lambda_1-\lambda_2\right)$ and $T1$ is positive since $\lambda_1-\lambda_2 \geq 0$.

The term $T2$ is also positive, since it is the sum of three positive terms: indeed

[redacted]he first term, which reads $\mathcal{E}\left(-\lambda_1\right)\left(F_{\widetilde{w}_1}\left(a\right)-F_{\widetilde{w}_2}\left(a\right)\right)$, is positive because $\mathcal{E}\left(-\lambda_1\right)$ is negative and so is $\left(F_{\widetilde{w}_1}\left(a\right)-F_{\widetilde{w}_2}\left(a\right)\right)$ (from *SSD2*)

The second term is positive because $\mathcal{E}'\left(1-\lambda_1\right)$ is positive, and the integral $\int_a^b u'\left(t\right)\left(F_{\widetilde{w}_1}\left(t\right)-F_{\widetilde{w}_2}\left(t\right)\right)dt$ is negative (from *SSD2*).

The last term is positive because $\mathcal{E}''\left(u\left(x\right)-\lambda_1\right)$ is negative, $u'\left(x\right)$ is positive and $\int_a^x u'\left(t\right)\left(F_{\widetilde{w}_1}\left(t\right)-F_{\widetilde{w}_2}\left(t\right)\right)dt$ is negative (from *SSD2*). Finally, $\mathbf{U}\left(\widetilde{w}_1\right) - \mathbf{U}\left(\widetilde{w}_2\right) \geq 0$ $\square$

Proof of Proposition 3.

As a preliminary, recall that $u\left(.\right)$ is more concave than $v\left(.\right)$ if and only if $u \circ v^{-1}\left(.\right)$ is concave *i.e.* if there exists $g\left(.\right)$ mapping $[0,1]$ on to itself and such that:$u\left(x\right) = g \circ v\left(x\right)$ with $g'\left(.\right) > 0$ and $g''\left(.\right) < 0$

The proof is grounded on the following calculations:

Let $\Delta \overset{def}{=} \int_a^z u'\left(x\right)\left(F_{\widetilde{w}_1}\left(x\right)-F_{\widetilde{w}_2}\left(x\right)\right)dt$, we get:

$$
\begin{aligned}
\Delta &= \int_a^z g'\left(v\left(x\right)\right)v'\left(x\right)\left(F_{\widetilde{w}_1}\left(x\right)-F_{\widetilde{w}_2}\left(x\right)\right)dx \\
&= \left[g'\left(v\left(x\right)\right)\int_a^x v'\left(t\right)\left(F_{\widetilde{w}_1}\left(t\right)-F_{\widetilde{w}_2}\left(t\right)\right)dt\right]_a^z \\
&- \int_a^z g''\left(v\left(x\right)\right)v'\left(x\right)\left[\int_a^x v'\left(t\right)\left(F_{\widetilde{w}_1}\left(t\right)-F_{\widetilde{w}_2}\left(t\right)\right)dt\right]dx \\
&= g'\left(v\left(z\right)\right)\int_a^z v'\left(t\right)\left(F_{\widetilde{w}_1}\left(t\right)-F_{\widetilde{w}_2}\left(t\right)\right)dt \\
&- \int_a^z g''\left(v\left(x\right)\right)v'\left(x\right)\left[\int_a^x v'\left(t\right)\left(F_{\widetilde{w}_1}\left(t\right)-F_{\widetilde{w}_2}\left(t\right)\right)dt\right]dx
\end{aligned}
$$

Finally, we get the following equivalences and/or implications which hold *for any* z:

$$\int_a^z v'(t)\left(F_{\widetilde{w}_1}(t) - F_{\widetilde{w}_2}(t)\right)dt < 0 \Rightarrow \int_a^z u'(x)\left(F_{\widetilde{w}_1}(x) - F_{\widetilde{w}_2}(x)\right)dx < 0$$

or:

$$\widetilde{w}_1 \precsim_{\mathbf{2}}^{v} \widetilde{w}_2 \Rightarrow \widetilde{w}_1 \precsim_{\mathbf{2}}^{u} \widetilde{w}_2 \quad \Leftrightarrow \quad \mathbf{W}_{\mathbf{2}}^{v} \subset \mathbf{W}_{\mathbf{2}}^{u}$$

and, as a consequence:

$$\mathbf{W}_{\mathbf{2}}^{v-} \subseteq \mathbf{W}_{\mathbf{2}}^{u-} \text{ and } \mathbf{W}_{\mathbf{2}}^{v+} \subseteq \mathbf{W}_{\mathbf{2}}^{u+}$$

□

Proof of Proposition 4.

Let $\mathbb{U}^* \subset \mathbb{U}$ denote the subset of concave or convex *n.e.u.* functions and let $\mathbb{U}_{\mathrm{I}}^*$ ($\mathbb{U}_{\mathrm{C}}^*$) be the subset of inconsistent (consistent) concave or convex *n.e.u.* functions. We have $\mathbb{U}^* = \mathbb{U}_{\mathrm{I}}^* \cup \mathbb{U}_{\mathrm{C}}^*$ and .$\mathbb{U}_{\mathrm{I}}^* \cap \mathbb{U}_{\mathrm{C}}^* = \{\mathbf{f}(.)\}$ where $\mathbf{f}(.)$ is the *n. e. u.* affine function defined by $\mathbf{f}(x) = (x-a)/(b-a)$. Two cases may occur, according to the fact that (standard) *SD2* is violated or not.

A .$\mathbb{U}_{\mathrm{C}}^* \neq \varnothing$ *i.e.* we first assume that *SD2* is not violated. As a consequence, there exists at least one concave function which is consistent. It is the *n. e. u.* affine function $\mathbf{f}(.)$.

1. A first subcase is when $\mathbb{U}_{\mathrm{C}}^* = \{\mathbf{f}(.)\}$; Proposition 4 is then clearly valid.

2. We now leave aside this trivial subcase and assume that $\mathbb{U}_{\mathrm{C}}^*$ includes at least one strictly concave *n.e.u.* function.

Let $\mathbb{H} = \bigcap_{u \in \mathbb{U}_{\mathrm{I}}^*} hypo(u)$. where $hypo(u)$ is the strict hypograph of $u \in \mathbb{U}_{\mathrm{I}}^*$.Since the hypographs are convex, so is $\mathbb{H}$ and so is its "northern" frontier which may be defined from the following equality: $hypo(\mathbf{u}) \overset{def}{=} \bigcap_{u \in \mathbb{U}_{\mathrm{I}}^*} hypo(u)$. Clearly function $\mathbf{u}(.)$ is concave. Since the hypographs $hypo(u)$ are open, we do not know yet whether $\mathbb{H}$ is closed –*i.e* whether $\mathbf{u}(.)$ belongs to $\mathbb{H}$ and, consequently is consistent– or not. Finally, we are going to prove directly that $\mathbf{W}_{\mathbf{2}}^{\mathbf{u}-} = \varnothing$. The proof is three-step.

(*a*) *The first step consists in defining a consistent concave n. e. u. function* $u(.)$ *which is close to* $\mathbf{u}(.)$. Now let $u(.)$ be defined by the following equality:

$$u(x) \overset{def}{=} \mathbf{u}(x) - y(x)$$

where

$$y(x) = \eta\left(\frac{x-a}{b-a}\right) - \eta\left(\frac{x-a}{b-a}\right)^2$$

Clearly, $y(x) \geq 0$ for $x \in [a,b]$, $y'(x) \geq 0$ for $x \in [a, a+(b-a)/2]$, $y'(x) \leq 0$ for $x \in [a+(b-a)/2,]\,b$, $y(a) = y(b) = 0$, $a + (b-a)/2 = Arg\max[y(x)]$ and

$\max\left[y\left(x\right)\right] = \eta/4$. A sufficient condition for $u\left(.\right)$ to be concave is that:

$$\eta < \frac{1}{2}\left(b-a\right)^{2}\inf_{x\in[a,b]}\left(-\mathbf{u}''\left(x\right)\right)$$

Moreover, $u\left(.\right)$ will be strictly increasing if $u'\left(x\right)$ is strictly positive. A sufficient condition for this is that:

$$\eta < \left(b-a\right)\inf_{x\in[a,b]}\left(\mathbf{u}'\left(x\right)\right)$$

and, finally, $u\left(.\right)$ is concave and strictly increasing if the real number η satisfies the below inequality:

$$\eta < \min\left[\frac{1}{2}\left(b-a\right)^{2}\inf_{x\in[a,b]}\left(-\mathbf{u}''\left(x\right)\right),\left(b-a\right)\inf_{x\in[a,b]}\left(\mathbf{u}'\left(x\right)\right)\right] \tag{1}$$

Since $u\left(a\right)=\mathbf{u}\left(a\right)=0$ and $u\left(b\right)=\mathbf{u}\left(b\right)=1$, $u\left(.\right)$ is normalized and since $\eta>0$, the hypograph of $\mathbf{u}\left(.\right)$ strictly includes that of $u\left(.\right)$. As a consequence, may not be inconsistent otherwise we would have $hypo(u)\subset hypo\left(\mathbf{u}\right)$ and simultaneously $u\left(.\right)\in\mathbb{U}_{\mathrm{I}}^{*}\mathrm{N}$. This would contradict the fact that $\mathbb{H}=\bigcap_{u\in\mathbb{U}_{\mathrm{I}}^{*}}hypo(u)$. Finally$u\left(.\right)$ is well consistent. Finally, the function $u\left(.\right)$ is a concave n. e. u. function if *(1)* is met.

(b) *The second step consists in looking for an upper bound for the following difference*:

$$\Delta=\left|\int_a^z\mathbf{u}'\left(x\right)\left(F_{\widetilde{w}_1}\left(x\right)-F_{\widetilde{w}_2}\left(x\right)\right)dx-\int_a^z u'\left(x\right)\left(F_{\widetilde{w}_1}\left(x\right)-F_{\widetilde{w}_2}\left(x\right)\right)dx\right|$$

Integrating by parts yields:

$$\begin{aligned}\Delta &= \left|\int_a^z(\mathbf{u}'\left(x\right)-u'\left(x\right))(F_{\widetilde{w}_1}\left(x\right)-F_{\widetilde{w}_2}\left(x\right))dx\right|\\ &= \left|\left(\mathbf{u}\left(z\right)-u\left(z\right)\right)\left(F_{\widetilde{w}_1}\left(z\right)-F_{\widetilde{w}_2}\left(z\right)\right)+\int_a^z\left(\mathbf{u}'\left(x\right)-u'\left(x\right)\right)\left(dF_{\widetilde{w}_1}\left(x\right)-dF_{\widetilde{w}_2}\left(x\right)\right)\right|\end{aligned}$$

and, consequently:

$$\Delta\leq\left|\left(\mathbf{u}\left(z\right)-u\left(z\right)\right)\left(F_{\widetilde{w}_1}\left(z\right)-F_{\widetilde{w}_2}\left(z\right)\right)\right|+\left|\int_a^z\left(\mathbf{u}'\left(x\right)-u'\left(x\right)\right)\left(dF_{\widetilde{w}_1}\left(x\right)-dF_{\widetilde{w}_2}\left(x\right)\right)\right| \tag{2}$$

The first term is bounded indicated as below:

$$\left|\left(\mathbf{u}\left(z\right)-u\left(z\right)\right)\left(F_{\widetilde{w}_1}\left(z\right)-F_{\widetilde{w}_2}\left(z\right)\right)\right|\leq\left|\left(\mathbf{u}\left(z\right)-u\left(z\right)\right)\right|\leq\sup_{z\in[a,b]}\left|\mathbf{u}'\left(z\right)-u'\left(z\right)\right|$$

We now show that he second term may be bounded as indicated below

$$\left|\int_a^z\left(\mathbf{u}'\left(x\right)-u'\left(x\right)\right)\left(dF_{\widetilde{w}_1}\left(x\right)-dF_{\widetilde{w}_2}\left(x\right)\right)\right|\leq 2\sup_{z\in[a,b]}\left|\mathbf{u}'\left(z\right)-u'\left(z\right)\right|$$

Indeed, we have

$$\left|\int_a^z\left(\mathbf{u}'\left(x\right)-u'\left(x\right)\right)\left(dF_{\widetilde{w}_1}\left(x\right)-dF_{\widetilde{w}_2}\left(x\right)\right)\right|\leq\begin{array}{c}\left|\int_a^z\left(\mathbf{u}'\left(x\right)-u'\left(x\right)\right)dF_{\widetilde{w}_1}\left(x\right)\right|\\ +\left|\int_a^z\left(\mathbf{u}'\left(x\right)-u'\left(x\right)\right)dF_{\widetilde{w}_2}\left(x\right)\right|\end{array}$$

and, for $i = 1, 2.$:

$$\begin{aligned}\left|\int_a^z \left(\mathbf{u}'(x) - u'(x)\right) dF_{\widetilde{w}_i}(x)\right| &\leq \sup_{z\in[a,b]} \left|\mathbf{u}'(z) - u'(z)\right| \int_a^z dF_{\widetilde{w}_i}(x) \\ &\leq \sup_{z\in[a,b]} \left|\mathbf{u}'(z) - u'(z)\right|\end{aligned}$$

Finally, an upper bound of is given by the following inequality:

$$\Delta \leq 3 \sup_{z\in[a,b]} \left|\mathbf{u}'(z) - u'(z)\right|$$

Now, recall that $\sup_{z\in[a,b]} \left|\mathbf{u}'(z) - u'(z)\right| = \sup_{z\in[a,b]} \left|\eta\left(\frac{z-a}{b-a}\right) - \eta\left(\frac{z-a}{b-a}\right)^2\right| = \eta/4$. As a consequence, we get

$$\Delta \leq 3\eta/4$$

(c) *The last step consists in showing that if* $\mathbf{u}(.)$ *were not consistent, then we would get a contradiction.* Indeed if $\mathbf{u}(.)$ were not consistent there would exist two prospects $\widetilde{w}_1$ and $\widetilde{w}_2$ such that $\widetilde{w}_1 \preceq \widetilde{w}_2$ and, simultaneously, there would exist $z \in [a, b]$, such that $\int_a^z \mathbf{u}'(x)\left(F_{\widetilde{w}_1}(x) - F_{\widetilde{w}_2}(x)\right)dx > 0$. In other words, there would exist a strictly positive real number ϵ such that

$$\int_a^z \mathbf{u}'(x)\left(F_{\widetilde{w}_1}(x) - F_{\widetilde{w}_2}(x)\right)dx \geq \epsilon > 0$$

Since $u(.)$ is consistent, we must have $\int_a^z u'(x)\left(F_{\widetilde{w}_1}(x) - F_{\widetilde{w}_2}(x)\right)dx < 0$ and, consequently: we get:

$$\Delta = \int_a^z \mathbf{u}'(x)\left(F_{\widetilde{w}_1}(x) - F_{\widetilde{w}_2}(x)\right)dx + \left|\int_a^z u'(x)\left(F_{\widetilde{w}_1}(x) - F_{\widetilde{w}_2}(x)\right)dx\right| \geq \epsilon$$

and, finally:

$$\epsilon \leq 3\eta/4$$

Hence, if η is small enough, *i.e.* if $\eta < 4\epsilon/3$, we get a contradiction and, finally, $\mathbf{u}(.)$ is well consistent.

B. $\mathbb{U}_{\mathrm{C}}^* = \varnothing$. *i.e.* we now assume that *SD2* is not violated. No concave *n. e. u.* functions may be consistent. By contrast, the subset of convex *n. e. u.* functions is never empty since it always includes the following function: $\underline{u}(x) = 0$ for $x \in [a, b[$ and $\underline{u}(b) = 1$. The rest of the proof is analogous to the above one.□

Proof of Proposition 7.

The first part of the proof consists in proving that, in LS-models, two equivalent prospects $\widetilde{w}_1$ and $\widetilde{w}_2$ which have the same expected utility $\overline{\mathbf{u}}$ and the same certainty equivalent $\mathbf{c}$, are strongly equivalent. Let $\widetilde{w}_1$ and $\widetilde{w}_2$ exhibit the same expected utility $\overline{\mathbf{u}}$ and the same certainty equivalent $\mathbf{c}$. From **(Eq. 1)** we get, for $i = 1, 2$:

$$\mathbf{u}(\mathbf{c}) = \overline{\mathbf{u}} + \sum_{n=1}^{N} p_n^i \left(\mathcal{E}\left(\mathbf{u}(w_n) - \overline{\mathbf{u}}\right)\right)$$

where $\widetilde{w}_i = \left[w_1, ..., w_N \ ; \ p_1^i, ..., p_N^i\right] \ (i = 1, 2)$ and where $\overline{\mathbf{u}} = \sum_{n=1}^N p_n^i \mathbf{u}(w_n)$. As a consequence, we have:

$$\sum_{n=1}^N p_n^1 \mathcal{E}\left(\mathbf{u}(w_n) - \overline{\mathbf{u}}\right) - \sum_{n=1}^N p_n^2 \mathcal{E}\left(\mathbf{u}(w_n) - \overline{\mathbf{u}}\right) = 0 \tag{3}$$

Now, consider the compound lottery

$$\widetilde{w}_\alpha \overset{def}{=} \alpha \widetilde{w}_1 \oplus (1-\alpha)\widetilde{w}_2 = \left[w_1, ..., w_N \ ; \ \alpha p_1^1 + (1-\alpha) p_1^2, ..., \alpha p_N^1 + (1-\alpha) p_N^2\right]$$

Its expected utility is:

$$\mathbf{E}\left[\mathbf{u}(\widetilde{w}_\alpha)\right] = \sum_{n=1}^N (\alpha p_n^1 + (1-\alpha) p_n^2) \mathbf{u}(w_n) = \overline{\mathbf{u}}$$

From **(Eq. 1)** we also get:

$$\mathbf{u}(\mathbf{c}(\widetilde{w}_\alpha)) = \overline{\mathbf{u}} + \sum_{n=1}^N \left(\alpha p_n^1 + (1-\alpha) p_n^2\right) \mathcal{E}\left(\mathbf{u}(w_n) - \overline{\mathbf{u}}\right)$$

where $\mathbf{c}(\widetilde{w}_\alpha)$ is the certainty equivalent of $\widetilde{w}_\alpha$ and, finally:

$$\mathbf{u}(\mathbf{c}(\widetilde{w}_\alpha)) - \mathbf{u}(\mathbf{c}) = \alpha\left(\sum_{n=1}^N p_n^1 \mathcal{E}\left(\mathbf{u}(w_n) - \overline{\mathbf{u}}\right) - \sum_{n=1}^N p_n^2 \mathcal{E}\left(\mathbf{u}(w_n) - \overline{\mathbf{u}}\right)\right) = 0$$

The proof of the converse is as follows.We must show that if $\widetilde{w}_1$ and $\widetilde{w}_2$ are strongly equivalent –*i.e.* if they have the same certainty equivalent and if they exhibit the betweenness property–, then they exhibit the same expected utility. To do so, we consider two discrete prospects:

$$\widetilde{w}_i = \left[w_1, ..., w_N \ ; \ p_1^i, ..., p_N^i\right] \quad i = 1, 2$$

and their probability mixture:

$$\alpha \widetilde{w}_1 \oplus (1-\alpha)\widetilde{w}_2 = \left[w_1, ..., w_N \ ; \ \alpha p_1^1 + (1-\alpha) p_1^2, ..., \alpha p_N^1 + (1-\alpha) p_N^2\right]$$

where $\alpha \in [0, 1]$.

We assume that they have the same certainty equivalent. Hence, we have, for $i = 1, 2$:

$$\mathbf{u}(c) = \mathbf{u}(\mathbf{c}(\widetilde{w}_i)) = \overline{\mathbf{u}}_i + \sum_{n=1}^N p_n^i \mathcal{E}\left(\mathbf{u}_n^i\right) \tag{4}$$

where:

$$\overline{\mathbf{u}}_i = \sum_{n=1}^N p_n^i \mathbf{u}(w_n) \text{ and } \mathbf{u}_n^i = \mathbf{u}(w_n) - \overline{\mathbf{u}}_i \tag{5}$$

Now, recall that, by definition, we have:

$$\begin{aligned}\mathbf{u}(\mathbf{c}(\alpha \widetilde{w}_1 \oplus (1-\alpha)\widetilde{w}_2)) &= \alpha \overline{\mathbf{u}}_1 + (1-\alpha)\overline{\mathbf{u}}_2 \\ &\quad + \sum_{n=1}^N \left[\alpha p_n^1 + (1-\alpha) p_n^2\right] \mathcal{E}\left(\alpha \mathbf{u}_n^1 + (1-\alpha)\mathbf{u}_n^2\right)\end{aligned}$$

and, from (4) and (5), we get :

$$\begin{aligned}\alpha \mathbf{u}(\mathbf{c}(\widetilde{w}_1)) + (1-\alpha)\mathbf{u}(\mathbf{c}(\widetilde{w}_2)) &= \alpha \overline{\mathbf{u}}_1 + (1-\alpha)\overline{\mathbf{u}}_2 \\ &\quad + \sum_{n=1}^N \alpha p_n^1 \mathcal{E}\left(\mathbf{u}_n^1\right) + \sum_{n=1}^N (1-\alpha) p_n^2 \mathcal{E}\left(\mathbf{u}_n^2\right)\end{aligned}$$

Substracting the two above equations from one another and using the betweenness property yields:

$$\sum_{n=1}^{N} p_n^1 \alpha \mathcal{E}\left(\mathbf{u}_n^1\right) + \sum_{n=1}^{N} p_n^2 \left(1-\alpha\right) \mathcal{E}\left(\mathbf{u}_n^2\right) = \alpha \left(\sum_{n=1}^{N} p_n^1 \mathcal{E} \left(\begin{array}{c} \alpha \mathbf{u}_n^1 \\ +\left(1-\alpha\right) \mathbf{u}_n^2 \end{array} \right) \right) + \left(1-\alpha\right) \left(\sum_{n=1}^{N} p_n^2 \mathcal{E} \left(\begin{array}{c} \alpha \mathbf{u}_n^1 + \\ \left(1-\alpha\right) \mathbf{u}_n^2 \end{array} \right) \right)$$

or, equivalently:

$$\begin{aligned} & \sum_{n=1}^{N} p_n^1 \alpha \mathcal{E}\left(\mathbf{u}_n^1\right) + \sum_{n=1}^{N} p_n^2 \left(1-\alpha\right) \mathcal{E}\left(\mathbf{u}_n^1\right) + \sum_{n=1}^{N} p_n^2 \left(1-\alpha\right) \left(\mathcal{E}\left(\mathbf{u}_n^2\right) - \mathcal{E}\left(\mathbf{u}_n^1\right)\right) \\ = & \sum_{n=1}^{N} \left[\alpha p_n^1 + \left(1-\alpha\right) p_n^2\right] \mathcal{E}\left(\alpha \mathbf{u}_n^1 + \left(1-\alpha\right) \mathbf{u}_n^2\right) \end{aligned}$$

and, finally:

$$\sum_{n=1}^{N} \left[\alpha p_n^1 + \left(1-\alpha\right) p_n^2\right] \left[\begin{array}{c} \mathcal{E}\left(\mathbf{u}_n^1\right) - \\ \mathcal{E} \left(\begin{array}{c} \alpha \mathbf{u}_n^1 + \\ \left(1-\alpha\right) \mathbf{u}_n^2 \end{array} \right) \end{array} \right] = \sum_{n=1}^{N} p_n^2 \left(1-\alpha\right) \left(\mathcal{E}\left(\mathbf{u}_n^1\right) - \mathcal{E}\left(\mathbf{u}_n^2\right)\right)$$

$$\sum_{n=1}^{N} \left[\begin{array}{c} \alpha p_n^1 + \\ \left(1-\alpha\right) p_n^2 \end{array} \right] \left[\begin{array}{c} \mathcal{E}\left(\mathbf{u}\left(w_n\right) - \overline{\mathbf{u}}_1\right) \\ -\mathcal{E}\left(\mathbf{u}\left(w_n\right) - \overline{\mathbf{u}}_\alpha\right) \end{array} \right] \left(1-\alpha\right)^{-1} = \sum_{n=1}^{N} p_n^2 \left(\begin{array}{c} \mathcal{E}\left(\mathbf{u}\left(w_n\right) - \overline{\mathbf{u}}_1\right) \\ -\mathcal{E}\left(\mathbf{u}\left(w_n\right) - \overline{\mathbf{u}}_2\right) \end{array} \right)$$

$$\sum_{n=1}^{N} \varpi_n\left(\alpha\right) \left(\mathbf{u}_n^1 - \mathbf{u}_n^2\right) \mathcal{E}' \left(\begin{array}{c} \mathbf{u}\left(w_n\right) - \overline{\mathbf{u}}_1 \\ +\theta_n\left(\alpha\right) \left(\mathbf{u}_n^1 - \mathbf{u}_n^2\right) \end{array} \right) = \sum_{n=1}^{N} p_n^2 \left(\mathbf{u}_n^1 - \mathbf{u}_n^2\right) \times \mathcal{E}' \left(\begin{array}{c} \mathbf{u}\left(w_n\right) - \overline{\mathbf{u}}_1 \\ +\zeta_n \left(\mathbf{u}_n^1 - \mathbf{u}_n^2\right) \end{array} \right)$$

$$\left(\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2\right) \left\{ \sum_{n=1}^{N} \varpi_n\left(\alpha\right) \mathcal{E}' \left(\begin{array}{c} \mathbf{u}\left(w_n\right) - \overline{\mathbf{u}}_1 \\ +\theta_n\left(\alpha\right) \left(\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2\right) \end{array} \right) \right\} = \left(\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2\right) \times \left\{ \sum_{n=1}^{N} p_n^2 \mathcal{E}' \left(\begin{array}{c} \mathbf{u}\left(w_n\right) - \overline{\mathbf{u}}_1 \\ +\zeta_n \left(\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2\right) \end{array} \right) \right\}$$

$$\left(\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2\right) F\left(\alpha\right) = \left(\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2\right) \Lambda$$

Since $F\left(\alpha\right)$ cannot be equal to Λ for any value of α, we must have $\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2 = 0$.□

Proof of Proposition 8.

Let:

$$\widetilde{w}_\pi^{a,b} \overset{def}{=} \left[a, b; 1-\pi, \pi\right], \quad \widetilde{w}_{z,\pi}^{\alpha} \overset{def}{=} \alpha \delta\left(z\right) \oplus \left(1-\alpha\right) \widetilde{w}_\pi^{a,b}$$

and:

$$\overline{\mathbf{u}} \overset{def}{=} \alpha \mathbf{u}\left(z\right) + \left(1-\alpha\right) \pi = \mathbf{E}\left[\mathbf{u}(\widetilde{w}_\pi^{\alpha})\right]$$

In LS-models we get:

$$\mathbf{U}(\widetilde{w}_{z,\pi}^{\alpha}) \overset{def}{=} \overline{\mathbf{u}} + \alpha \mathcal{E}(\mathbf{u}\left(z\right) - \overline{\mathbf{u}}) + \left(1-\alpha\right) \pi \mathcal{E}(1 - \overline{\mathbf{u}}) + \left(1-\alpha\right) \left(1-\pi\right) \mathcal{E}(-\overline{\mathbf{u}})$$

and, consequently:

$$\mathbf{U}(\widetilde{w}_{z,\pi}^{\alpha}) = \alpha \mathbf{U}(\delta\left(z\right)) + \left(1-\alpha\right) \mathbf{U}(\widetilde{w}_\pi^{a,b}) + EXP$$

where:

$$EXP \overset{def}{=} \alpha\mathcal{E}(\mathbf{u}(z)-\overline{\mathbf{u}}) + (1-\alpha)\left\{\begin{array}{c}\pi\left(\mathcal{E}(1-\overline{\mathbf{u}})-\mathcal{E}(1-\pi)\right)\\+(1-\pi)(\mathcal{E}(-\overline{\mathbf{u}})-\mathcal{E}(-\pi))\end{array}\right\}$$

We get:

$$\begin{aligned}EXP &= \alpha(\mathbf{u}(z)-\overline{\mathbf{u}})\mathcal{E}'(\theta(\mathbf{u}(z)-\overline{\mathbf{u}})) + (1-\alpha)\pi(\pi-\overline{\mathbf{u}})\mathcal{E}'(1-\pi+\zeta(\pi-\overline{\mathbf{u}}))\\&\quad+(1-\alpha)(1-\pi)(\pi-\overline{\mathbf{u}})\mathcal{E}'(-\pi+\xi(\pi-\overline{\mathbf{u}}))\end{aligned}$$

or:

$$\begin{aligned}EXP &= \alpha(1-\alpha)(\mathbf{u}(z)-\pi)\mathcal{E}'(\theta(1-\alpha)(\mathbf{u}(z)-\pi))\\&\quad+(1-\alpha)\alpha(\pi-\mathbf{u}(z))\left\{\begin{array}{c}\pi\mathcal{E}'(1-\pi+\zeta\alpha(\pi-\mathbf{u}(z)))\\+(1-\pi)\mathcal{E}'(-\pi+\xi\alpha(\pi-\mathbf{u}(z)))\end{array}\right\}\end{aligned}$$

and, finally:

$$EXP = \alpha(1-\alpha)(\mathbf{u}(z)-\pi)\left[\begin{array}{c}\mathcal{E}'(\theta(1-\alpha)(\mathbf{u}(z)-\pi))\\-\left\{\begin{array}{c}\pi\mathcal{E}'(1-\pi+\zeta\alpha(\pi-\mathbf{u}(z)))\\+(1-\pi)\mathcal{E}'(-\pi+\xi\alpha(\pi-\mathbf{u}(z)))\end{array}\right\}\end{array}\right]$$

The above condition can be rewritten as:

$$EXP = \alpha(1-\alpha)(\mathbf{u}(z)-\pi)\left\{\mathcal{E}'(\theta(1-\alpha)(\mathbf{u}(z)-\pi)) - exp\right\}$$

where:

$$\begin{aligned}exp &= \pi\mathcal{E}'(1-\pi+\zeta\alpha(\pi-\mathbf{u}(z))) + (1-\pi)\mathcal{E}'(-\pi+\xi\alpha(\pi-\mathbf{u}(z)))\\&= \pi\mathcal{E}'(0)+\pi\zeta\alpha(\pi-\mathbf{u}(z))\mathcal{E}''(\zeta s\alpha(\pi-\mathbf{u}(z)) + (1-\pi)\mathcal{E}'(0)\\&\quad+(1-\pi)\zeta\alpha(\pi-\mathbf{u}(z))\mathcal{E}''(\xi k\alpha(\pi-\mathbf{u}(z))\end{aligned}$$

Note that we have:

$$\begin{aligned}\mathcal{E}'(\theta(1-\alpha)(\mathbf{u}(z)-\pi)) - exp &= \mathcal{E}'(0) - \{\pi\mathcal{E}'(0)+(1-\pi)\mathcal{E}'(0)\}\\&\quad+\theta(1-\alpha)(\mathbf{u}(z)-\pi)\,\mathcal{E}''(\theta t(1-\alpha)(\mathbf{u}(z)-\pi))\\&\quad-\pi\zeta\alpha(\pi-\mathbf{u}(z))\mathcal{E}''(\zeta s\alpha(\pi-\mathbf{u}(z)))\\&\quad-(1-\pi)\zeta\alpha(\pi-\mathbf{u}(z))\mathcal{E}''(\xi k\alpha(\pi-\mathbf{u}(z))\end{aligned}$$

and, finally:

$$\begin{aligned}EXP &= \alpha(1-\alpha)(\mathbf{u}(z)-\pi)^2\left\{\begin{array}{c}\theta(1-\alpha)\,\mathcal{E}''(\theta t(1-\alpha)(\mathbf{u}(z)-\pi))\\+\pi\zeta\alpha\mathcal{E}''(\zeta s\alpha(\pi-\mathbf{u}(z)))\\+(1-\pi)\zeta\alpha\mathcal{E}''(\xi k\alpha(\pi-\mathbf{u}(z))\end{array}\right\}<0\\&\simeq \alpha(1-\alpha)(\mathbf{u}(z)-\pi)^2\mathcal{E}''(0)\left\{\theta(1-\alpha)+\zeta\alpha\right\}\end{aligned}$$

and the condition $EXP = 0$ implies $\mathbf{u}(z)-\pi = 0$.□

Proof of Proposition 9.

First, note that *n.e.u.* utility functions could be normalized differently. Indeed the normalization conditions $\mathbf{u}(a) = 0$ and $\mathbf{u}(b) = 1$, play no particular role in the above analysis. Hence it is equivalent to state that $\mathbf{u}(x)$ is a rationalizing function with the standard normalization conditions or that $\boldsymbol{v}(.)$ is also a rationalizing function checking the following conditions $\boldsymbol{v}(a) = \underline{\boldsymbol{u}} \neq 0$ and $\boldsymbol{v}(b) = \overline{\boldsymbol{u}} \neq 1$, *iff.*$\boldsymbol{v}(x) = \underline{\boldsymbol{u}} + (\overline{\boldsymbol{u}} - \underline{\boldsymbol{u}})\,\mathbf{u}(x)$.

Now let: $\mathbf{u}_1^{\boldsymbol{\lambda}}(x) = \boldsymbol{\lambda}\mathbf{u}_1(x)$ and $\mathbf{u}_2^{\boldsymbol{\lambda}}(x) = \boldsymbol{\lambda} + (1-\boldsymbol{\lambda})\,\mathbf{u}_2(x)$. Then $\mathbf{u}_1^{\boldsymbol{\lambda}}(x)\,\mathbf{1}_{[a,w_0]}$ ($\mathbf{u}_2^{\boldsymbol{\lambda}}(x)\,\mathbf{1}_{[w_0,b]}$) is the rationalizing function over $[a, w_0]$ ($[w_0, b]$) checking: $\mathbf{u}_1^{\boldsymbol{\lambda}}(w_0) = 0$ and $\mathbf{u}_1^{\boldsymbol{\lambda}}(w_0) = \boldsymbol{\lambda}$ ($\mathbf{u}_2^{\boldsymbol{\lambda}}(w_0) = \boldsymbol{\lambda}$ and $\mathbf{u}_2^{\boldsymbol{\lambda}}(b) = 1$).Let $\mathbf{u}(x) = \mathbf{u}_1^{\boldsymbol{\lambda}}(x)\,\mathbf{1}_{[a,w_0]} + \mathbf{u}_2^{\boldsymbol{\lambda}}(x)\,\mathbf{1}_{[w_0,b]}$. Then $\mathbf{u}(.)$ is a rationalizing function over $[a, w_0]$ and over $[w_0, b]$, To be smooth, it must check the additional condition that its left-hand size derivative, calculated at point w_0 , is equal to the correesponding right-hand size one, *i.e.*

$\mathbf{u}_2^{\boldsymbol{\lambda}\prime}(w_0) = (1-\boldsymbol{\lambda})\,\mathbf{u}_2'(w_0)$

or, equivalently:

$\boldsymbol{\lambda} = \mathbf{u}_2'(\mathbf{w_0}) / (\mathbf{u}_1'(w_0) + \mathbf{u}_2'(w_0))$

The condition is clearly sufficient.

If x_{n+1} were greater than x_n, $\widetilde{w}_{1-p_{n+1}}^{x_{n+1},b}$ would exhibit first-order stochastic dominance over $\widetilde{w}_{p_n}^{a,x_n}$. Hence, x_{n+1} is lower than x_n and $\{x_n\}_{n\in\mathbb{N}}$ is a decreasing sequence. It is also bounded below by a. Consequently, it converges towards a limit $\ell \geq a$. Next, note that the two strongly equivalent lotteries $\widetilde{w}_{p_n}^{a,x_n}$ and $\widetilde{w}_{1-p_{n+1}}^{x_{n+1},b}$have the same expected utility, *i.e.*, we have:

$$p_n u(x_n) = p_{n+1} u(x_{n+1}) + (1 - p_{n+1}) \quad for \quad n = 0, 1, ... \tag{6}$$

and summing the members of the above equalities yields:

$$\pi u(w) = p_n u(x_n) + \sum_{i=1}^{n} (1 - p_i) \quad for \quad n = 1, 2, ...$$

The above equality implies $S_n \overset{def}{=} \sum_{i=1}^{n}(1-p_i) \leq \pi u(w)$. Since$\{S_n\}_{n\in\mathbb{N}^*}$ is an increasing sequence, it converges towards a limit $\Sigma \leq \pi u(w)$. As a consequence, $S_n - S_{n-1} = (1 - p_{n.}) \to 0$, *i.e.* $p_{n.} \to 1$. Moreover, since we have: $\widetilde{w}_{p_{n+1}}^{a,x_{n+1}} \prec \widetilde{w}_{1-p_{n+1}}^{x_{n+1},b} \sim \widetilde{w}_{p_n}^{a,x_n}$, the sequence of binary lotteries $\left\{\widetilde{w}_{p_n}^{a,x_n}\right\}_{n\in\mathbb{N}}$ is decreasing and converges towards $\widetilde{w}_1^{a,l} = \delta(l)$. Similarly, $\{\widetilde{w}_{1-p_n}^{x_n,b}\}_{n\in\mathbb{N}^*}$ converges towards $\widetilde{w}_0^{l,b} = \delta(l)$.

We now show that $\ell = a$. To see this, assume $\ell > a$. Then, since $\widetilde{w}_{p_n}^{a,x_n} \succ \delta(l)$, there exists a binary lottery $\widetilde{w}_{p_n}^{a,x_n^*}$ such that $l < x_n^* < x_n$, and $\widetilde{w}_{p_n}^{a,x_n^*} \sim \delta(l)$. Let x_{n+1}^* and p_{n+1}^* be defined by $\widetilde{w}_{1-p_{n+1}^*}^{x_{n+1}^*,b} \approx \widetilde{w}_{p_n}^{a,x_n^*}$. Since $\{\widetilde{w}_{1-p_n}^{x_n,b}\}_{n\in\mathbb{N}^*}$ converges towards $\delta(l)$, there exists an integer N, such that $m \geq N \Rightarrow l \leq x_m < x_{n+1}^*$ and $p_m \geq p_{n+1}^*$. This implies that $\widetilde{w}_{1-p_{n+1}^*}^{x_{n+1}^*,b}$ should be preferred to the $\widetilde{w}_{1-p_m}^{x_m,b}$s and, consequently, that $\delta(l)$ should be preferred to the $\widetilde{w}_{1-p_m}^{x_m,b}$s, that contradicts the fact that $\{\widetilde{w}_{1-p_n}^{x_n,b}\}_{n\in\mathbb{N}}$ is decreasing and converges towards $\delta(l)$. Hence $\ell = a$

and $\{S_n\}_{n\in\mathbb{N}}$ converges towards $\Sigma = \pi u(w)$. As a consequence, equality **(Eq. 11)** is checked.

Proof of Proposition 10.

First, note that $n.e.u.$ utility functions could be normalized differently. Indeed the normalization conditions $\mathbf{u}(a) = 0$ and $\mathbf{u}(b) = 1$, play no particular role in the above analysis. Hence it is equivalent to state that $\mathbf{u}(x)$ is a rationalizing function with the standard normalization conditions or that $\boldsymbol{v}(.)$ is also a rationalizing function checking the following conditions $\boldsymbol{v}(a) = \underline{\boldsymbol{u}} \neq 0$ and $\boldsymbol{v}(b) = \overline{\boldsymbol{u}} \neq 1$, *iff.*$\boldsymbol{v}(x) = \underline{\boldsymbol{u}} + (\overline{\boldsymbol{u}} - \underline{\boldsymbol{u}})\,\mathbf{u}(x)$.

Now let: $\mathbf{u}_1^{\boldsymbol{\lambda}}(x) = \boldsymbol{\lambda}\mathbf{u}_1(x)$ and $\mathbf{u}_2^{\boldsymbol{\lambda}}(x) = \boldsymbol{\lambda} + (1-\boldsymbol{\lambda})\,\mathbf{u}_2(x)$. Then $\mathbf{u}_1^{\boldsymbol{\lambda}}(x)\,\mathbf{1}_{[a,w_0]}$ ($\mathbf{u}_2^{\boldsymbol{\lambda}}(x)\,\mathbf{1}_{[w_0,b]}$) is the rationalizing function over $[a, w_0]$ ($[w_0, b]$) checking: $\mathbf{u}_1^{\boldsymbol{\lambda}}(w_0) =$ 0 and $\mathbf{u}_1^{\boldsymbol{\lambda}}(w_0) = \boldsymbol{\lambda}$ ($\mathbf{u}_2^{\boldsymbol{\lambda}}(w_0) = \boldsymbol{\lambda}$ and $\mathbf{u}_2^{\boldsymbol{\lambda}}(b) = 1$).Let $\mathbf{u}(x) = \mathbf{u}_1^{\boldsymbol{\lambda}}(x)\,\mathbf{1}_{[a,w_0]} + \mathbf{u}_2^{\boldsymbol{\lambda}}(x)\,\mathbf{1}_{[w_0,b]}$. Then $\mathbf{u}(.)$ is a rationalizing function over $[a, w_0]$ and over $[w_0, b]$, To be smooth, it must check the additional condition that its left-hand size derivative, calculated at point w_0 , is equal to the correesponding right-hand size one, *i.e.*

$\mathbf{u}_2^{\boldsymbol{\lambda}\prime}(w_0) = (1-\boldsymbol{\lambda})\,\mathbf{u}_2'(w_0)$

or, equivalently:

$\boldsymbol{\lambda} = \mathbf{u_2'}(\mathbf{w_0}) \,/\, (\mathbf{u}_1'(w_0) + \mathbf{u}_2'(w_0))$

The condition is clearly sufficient.