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Intertemporal equilibrium with production: bubbles and efficiency

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Abstract

We consider a general equilibrium model with heterogeneous agents, borrowing constraints, and exogenous labor supply. First, the existence of intertemporal equilibrium is proved even if the aggregate capitals are not uniformly bounded above and the production functions are not time invariant. Second, (i) we call by physical capital bubble a situation in which the fundamental value of physical capital is lower than its price, (ii) we say that the interest rates are low if the sum of interest rates is finite. We show that physical capital bubble is equivalent to a situation with low interest rates. Last, we prove that with linear technologies, every intertemporal equilibrium is efficient. Moreover, there is a room for both efficiency and bubble.

Keywords: Intertemporal equilibrium, physical capital bubble, efficiency, infinite horizon.

JEL Classification : C62, D31, D91, G10

1 Introduction

Following Becker, Bosi, Le Van and Seegmuller (2014), we consider a dynamic general equilibrium model with heterogeneous agents. However, our framework is different from their model in three points: (i) for simplicity, we consider exogenous labor suply, (ii) our technology is not stationary, (iii) aggregate capital stock is not necessarily uniformly bounded from above. Heterogeneous agents decide to invest and consume. If they invest in physical capital, this asset will not only give them return in term of consumption good

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at the next period but also give back a fraction of the same asset (after being depreciated). Agents cannot borrow.

Our first contribution is to prove the existence of intertemporal general equilibrium. To do it, we firstly prove the existence of equilibrium for each $T-$truncated economy. Hence, we have a sequence of equilibria which depend on T. We then prove that this sequence has a limit (for the product topology) which is an equilibrium for the infinite horizon economy.

We say that physical capital bubble occurs at equilibrium (for short, *bubble*) if the price of the physical asset is greater than its fundamental value. We say interest rates are low at equilibrium (for short, *low interest rates*) if the sum of returns on capital is finite. Our second contribution to prove that bubble is equivalent to low interest rates.
The no-bubble result in Becker, Bosi, Le Van and Seegmuller (2014) can be viewed as a particular case of our result. Indeed, in Becker, Bosi, Le Van and Seegmuller (2014), thanks to the concavity of a stationary technology, the aggregate capital stock is uniformly bounded, and then real return of the physical capital is uniformly bounded from below. Therefore, the sum of returns equals infinity. According to our result, the physical capital bubble is ruled out.
However, when we allow non-stationary production functions, there may be a bubble at equilibrium. To see the point, take linear production functions whose productivity at date t is denoted by a_t. At equilibrium, real return of physical capital at date t must be a_t. As mentioned above, there is a bubble if and only if $\sum\limits_{t=0}^{\infty} a_t < \infty$. We can now see clearly that there is a bubble if productivities decrease with sufficiently high speed.

Our third contribution is about the efficiency of intertemporal equilibrium. An intertemporal equilibrium is called to be efficient if its aggregate capital path is efficient in sense of Malinvaud (1953). We prove that with linear production functions, every intertemporal equilibrium is efficient. However, as we mentioned above, this efficient intertemporal equilibrium may have bubble if productivities decrease with sufficiently high speed. Therefore, we have both efficient and bubble at equilibrium with such technologies. Note that our result does not require any conditions about the convergence or boundedness of the capital path as in previous literature.

Related literature
(1) *On rational bubbles.* Tirole (1982) proved that there is no financial asset bubble in a rational expectation model without endowment. A survey on bubble in models with asymmetric information, overlapping generation, heterogeneous-beliefs can be found in Brunnermeier and Oehmke (2012). Doblas-Madrid Doblas-Madrid (2012) presented a model of speculative bubbles where rational agents buy an overvalued asset because given their private information, they believe they have a good chance of reselling at a profit to a greater fool. Martin and Ventura (1953), Ventura (2012) did not define bubble as we do. Instead, they defined bubble as a short-lived asset.

(2) *On the efficiency of a capital path.* Malinvaud (1953) introduced the concept of efficiency of a capital path and give a sufficient condition of the efficiency: $\lim\limits_{t\to\infty} p_t K_t = 0$,

where (p_t) is a sequence of competitive prices, (K_t) is the capital path.[1] Following Malinvaud, Cass (1972) considered capital path which is uniformly bounded from below. Under the concavity of a stationary production function and some mild conditions, he proved that a capital path is inefficient if and only if the sum (over time) of future values of a unit of physical capital is finite. Cass and Yaari (1971) given a necessary and sufficient condition for a consumption plan (C) to be efficient, which can be stated that the limit inferior of differences between the present value of any consumption plan and the plan (C) is negative.
Our paper is also related to Becker and Mitra (2012) where they proved that a Ramsey equilibrium is efficient if the most patient household is not credit constrained from some date. However, their result is based on the fact that consumption of each household is uniformly bounded from below. In our paper, we do not need this condition. Instead, the efficient capital path in our model may converge to zero. Mitra and Ray (2012) studied the efficiency of a capital path with nonconvex production technologies and examined whether the Phelps-Koopmans theorem is valid. However, their results are no longer valid without the convergence or the boundedness of capital paths.

(3) Another concept of efficiency is *constrained efficiency.* About the constrained efficiency in general equilibrium models with financial asset, see Kehoe and Levine (1993), Alvarez and Jermann (2000), Bloise and Pietro (2011). About the constrained efficiency in the neoclassical growth model, see Davila, Hong, Krusell and Rios-Rull (2012).

The remainder of the paper is organized as follows. Section 2 describes the model. In section 3, existence of equilibrium is proved. Section 4 studies physical capital bubble. Section 5 explores our results on the efficiency of equilibria. Conclusion will be presented in Section 6. Technical details are gathered in Appendix.

2 Model

We follow Becker, Bosi, Le Van and Seegmuller (2014), but we consider: (i) exogenous labor supply, (ii) non-stationary production functions.

Consumption good: at each period t, price of consumption is denoted by p_t and agent i consumes $c_{i,t}$ units of consumption good.

Physical capital: at time t, if agent i decides to buy $k_{i,t+1} \geq 0$ units of new capital, then at period $t+1$, after begin depreciated, agent i will receive $(1-\delta)k_{i,t+1}$ units of old capital and a return on capital $k_{i,t+1}$ at the rate r_{t+1}. Here, δ is the rate of capital depreciation rate.

[1] See Malinvaud (1953), Lemma 5, page 248.

Each household i takes sequences of prices $(p, r) = (p_t, r_t)_{t=0}^{\infty}$ as given and solves

$$(P_i(p,r)): \quad \max_{\left((c_{i,t},k_{i,t+1})_{i=1}^m\right)_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t})\right] \tag{1}$$

$$\text{subject to}: \quad k_{i,t+1} \geq 0 \tag{2}$$

$$p_t(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) \leq r_t k_{i,t} + \theta^i \pi_t(p_t, r_t), \tag{3}$$

where $(\theta^i)_{i=1}^m$ is the share of profit, $\theta^i \geq 0$ for all i and $\sum_{i=1}^m \theta^i = 1$.

Firm: For each period, there is a representative firm who takes prices (p_t, r_t) as given and maximizes its profit.

$$(P(r_t)): \quad \pi_t(p_t, r_t) := \max_{K_t \geq 0} \left[p_t F_t(K_t) - r_t K_t\right]$$

We write π_t instead of $\pi_t(p_t, r_t)$ if there is no confusion.

Definition 1. *A sequence of prices and quantities* $\left(\bar{p}_t, \bar{r}_t, (\bar{c}_{i,t}, \bar{k}_{i,t+1})_{i=1}^m, \bar{K}_t\right)_{t=0}^{+\infty}$ *is an equilibrium of the economy* $\mathcal{E} = \left((u_i, \beta_i, k_{i,0}, \theta_i)_{i=1}^m, F,\right)$ *if the following holds.*

(i) Price positivity: $\bar{p}_t, \bar{r}_t > 0$ *for* $t \geq 0$.

(ii) Market clearing: at each $t \geq 0$,

$$\textit{good} \quad : \quad \sum_{i=1}^m [\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1-\delta)\bar{k}_{i,t}] = F_t(\bar{K}_t) \tag{4}$$

$$\textit{capital} \quad : \quad \bar{K}_t = \sum_{i=1}^m \bar{k}_{i,t}. \tag{5}$$

(iii) Optimal consumption plans: for each i, $\left((\bar{c}_{i,t}, \bar{k}_{i,t+1})_{i=1}^m\right)_{t=0}^{\infty}$ *is a solution to problem* $(P_i(\bar{p}, \bar{r}))$.

(iv) Optimal production plan: for each $t \geq 0$, $(\bar{K}_t)$ *is a solution to problem* $(P(\bar{r}_t))$.

3 The existence of equilibrium

The following result proves that the feasible aggregate capital and the feasible consumption are bounded for the product topology.

Lemma 1. *Feasible individual and aggregate capitals and feasible consumptions are in a compact set for the product topology. Moreover, they are uniformly bounded if there exists* t_0 *and an increasing, concave function* G *such that: (i) for every* $t \geq t_0$ *we have* $F_t(K) \leq G(K)$ *for every* K*, (ii) there exists* $x > 0$ *such that* $G(y) + (1-\delta)y \leq y$ *for every* $y \geq x$.

Proof. Denote

$$\begin{aligned} D_0 := D_0(F_0,\delta,K_0) &:= F_0(K_0)+(1-\delta)K_0,\\ D_t := D_t((F_s)_{s=0}^t,\delta,K_0) &:= F_t(D_{t-1}((F_s)_{s=0}^{t-1},\delta,K_0))+(1-\delta)D_{t-1}((F_s)_{s=0}^{t-1},\delta,K_0),\ \forall t\geq 0. \end{aligned}$$

Then $\sum_{i=1}^m c_{i,t}+K_{t+1}\leq D_t$ for every $t\geq 0$.

We now assume t_0 and the function G (as in Lemma 1) exist. We are going to prove that $0\leq K_t\leq \max\{D_0,...,D_{t_0-1},x\}=:M$. Indeed, $K_t\leq D_{t-1}\leq M$ for every $t\leq t_0$. For $t\geq t_0$, we have

$$K_{t+1}=\sum_{i=1}^m k_{i,t+1} \quad\leq\quad G(K_t)+(1-\delta)K_t.$$

Then $K_{t_0+1}\leq G(K_{t_0})+(1-\delta)K_{t_0}\leq G(M)+(1-\delta)M\leq M$. Iterating the argument, we obtain $K_t\leq M$ for each $t\geq 0$.
Feasible consumptions are bounded because $\sum_{i=1}^m c_{i,t}\leq G(K_t)+(1-\delta)K_t$. □

We need the following assumptions.

Assumption (H1): For each i, the utility function u_i of agent i is strictly increasing, strictly concave, continuous differentiable, and $u(0)=0, u'(0)=\infty$.

Assumption (H2): $F_t(\cdot)$ is continuously differentiable, strictly increasing, concave, the input is essential ($F_t(0)=0$) and $F_t(\infty)=\infty$.

Assumption (H3): $\delta\in(0,1)$ and $k_{i,0}>0$ for every i.[2]

Assumption (H4): For each i, utility of agent i is finite

$$\sum_{t=0}^{\infty}\beta_i^t u_i(D_t)<\infty.$$

3.1 Existence of equilibrium in $\mathcal{E}^T$

We define $T-$ truncated economy $\mathcal{E}^T$ as $\mathcal{E}$ but there are no activities from period $T+1$ to the infinity, i.e., $c_{i,t}=k_{i,t}=0$ for every $i=1,\dots,m$, and for every $t\geq T+1$.

In the economy $\mathcal{E}^T$, agent i takes sequences of prices $(p,r)=(p_t,r_t)_{t=0}^T$ as given and maximizes his intertemporal utility by choosing consumption and investment levels.

$$\begin{aligned} (P_i(p,r)):\quad & \max_{(c_{i,t},k_{i,t+1})_{t=0}^T}\Big[\sum_{t=0}^T\beta_i^t u_i(c_{i,t})\Big] && (6)\\ \text{subject to:}\quad & k_{i,t+1}\geq 0, && (7)\\ \text{(budget constraints)}\quad & p_t(c_{i,t}+k_{i,t+1}-(1-\delta)k_{i,t})\leq r_t k_{i,t}+\theta^i\pi_t, && (8) \end{aligned}$$

[2] Becker, Bosi, Le Van and Seegmuller (3) weekens H3 by assuming $\sum_{i=1}^m k_{i,0}>0$ because of positive labor income.

where $k_{i,T} = 0$.

Then we define the bounded economy $\mathcal{E}_b^T$ as $\mathcal{E}^T$ but all variables are bounded in the following bounded sets:

$$\begin{aligned}(c_{i,t})_{t=0}^{T} &\in \mathcal{C}_i := [0, B_c]^{T+1}\\ (k_{i,t})_{t=1}^{T+1} &\in \mathcal{K}_i := [0, B_k]^T\\ K := (K_t)_{t=1}^{T+1} &\in \mathcal{K} := [0, B]^T,\end{aligned}$$

where $B_c > \max_t F_t(B) + (1-\delta)B$, $B > mB_k$.

Proposition 1. *Under Assumptions* $(H1) - (H3)$*, there exists an equilibrium for* $\mathcal{E}_b^T$.

Proof. See Appendix. □

Proposition 2. *An equilibrium of the economy* $\mathcal{E}_b^T$ *is also an equilibrium of the unbounded economy* $\mathcal{E}^T$.

Proof. Similar to the one in Becker, Bosi, Le Van and Seegmuller (2014). □

3.2 Existence of equilibrium in $\mathcal{E}$

Theorem 1. *Under Assumptions (H1)-(H4), there exists an equilibrium.*

Proof of Theorem 1. We have shown that for each $T \geq 1$, there exists an equilibrium for the economy $\mathcal{E}^T$. We denote by $(\bar{p}^T, \bar{r}^T, (\bar{c}_i^T, \bar{k}_i^T)_{i=1}^m, \bar{K}^T)$ an equilibrium of $T-$ truncated economy $\mathcal{E}^T$.
We can normalize by setting $\bar{p}_t^T + \bar{r}_t^T = 1$ for every $t \leq T$.
We see that

$$0 < \bar{c}_{i,t}^T, \bar{K}_t^T \quad \leq \quad D_t.$$

Without loss of generality, we can assume that

$$\begin{aligned}&(\bar{p}^T, \bar{r}^T, (\bar{c}_i^T, \bar{k}_i^T)_{i=1}^m, \bar{K}^T)\\ &\xrightarrow{T\to\infty} (\bar{p}, \bar{r}, (\bar{c}_i, \bar{k}_i)_{i=1}^m, \bar{K}) \quad \text{(for the product topology).}\end{aligned}$$

We are going to prove that: (i) all markets clear, (ii) at each date t, $\bar{K}_t$ is a solution to the firm's maximization problem, (iii) $\bar{r}_t > 0$ for each $t \geq 0$, (iv) $(\bar{c}_i, \bar{k}_i)$ is a solution to the maximization problem of agent i for each $i = 1, \ldots, m$, (v) $\bar{p}_t > 0$ for each t. Consequently, we obtain that $(\bar{p}, \bar{r}, (\bar{c}_i, \bar{k}_i)_{i=1}^m, \bar{K})$ is an equilibrium for the economy $\mathcal{E}$.

(i) By taking the limit of market clearing conditions for the truncated economy, we obtain the market clearing conditions for the economy $\mathcal{E}$.

(ii) Take $K \geq 0$ arbitrary. We have $\bar{p}_t^T F_t(K) - \bar{r}_t^T K \leq \bar{p}_t^T F_t(\bar{K}_t^T) - \bar{r}_t^T \bar{K}_t^T$. Let T tend to infinity, we obtain that $\bar{p}_t F_t(K) - \bar{r}_t K \leq \bar{p}_t F_t(\bar{K}_t) - \bar{r}_t \bar{K}_t$. Therefore, the optimality of $\bar{K}_t$ is proved.

(iii) If $\bar{r}_t = 0$ then $\bar{p}_t = 1$ (since $\bar{r}_t^T + \bar{p}_t^T = 1$). The optimality of $\bar{K}_t$ implies that $\bar{K}_t = \infty$. This is a contradiction, because we have $\bar{K}_t = \lim_{t\to\infty} \bar{K}_t^T \leq D_t < \infty$.

(iv) First, we give some notations. For each i and t, we define $B_i^T(\bar{p},\bar{r})$ and $C_i^T(\bar{p},\bar{r})$ as follows

$$\begin{aligned} B_i^T(\bar{p},\bar{r}) &:= \big\{(c_{i,t},k_{i,t+1})_{t=0}^T \in \mathbb{R}_+^{T+1}\times\mathbb{R}_+^{T+1} : \text{(a) } k_{i,T+1}=0, \text{(b) } \forall t=0,\dots,T,\\ &\qquad k_{i,t+1}>0, \quad \bar{p}_t[c_{i,t}+k_{i,t+1}-(1-\delta)k_{i,t}] < \bar{r}_t k_{i,t} + \theta^i\pi_t(\bar{p}_t,\bar{r}_t)\big\}\\ C_i^T(\bar{p},\bar{r}) &:= \big\{(c_{i,t},k_{i,t+1})_{t=0}^T \in \mathbb{R}_+^{T+1}\times\mathbb{R}_+^{T+1} : \text{(a) } k_{i,T+1}=0, \text{(b) } \forall t=0,\dots,T,\\ &\qquad k_{i,t+1}\geq 0, \quad \bar{p}_t[c_{i,t}+k_{i,t+1}-(1-\delta)k_{i,t}] \leq \bar{r}_t k_{i,t} + \theta^i\pi_t(\bar{p}_t,\bar{r}_t)\big\}. \end{aligned}$$

Since $\bar{r}_t > 0$ for every t, it is easy to prove that $B_i^T(\bar{p},\bar{r}) \neq \emptyset$.

Let (c_i,k_i) be an feasible allocation of the problem $P_i(\bar{p},\bar{r})$. We have to prove that $\sum\limits_{t=0}^{\infty}\beta_i^t u_i(c_{i,t}) \leq \sum\limits_{t=0}^{\infty}\beta_i^t u_i(\bar{c}_{i,t})$.

We define $(c'_{i,t},k'_{i,t+1})_{t=0}^T$ as follows: $c'_{i,t} = c_{i,t}$ for every $t\leq T$, $=0$ if $t>T$; $k'_{i,t+1} = k_{i,t+1}$ for every $t \leq T-1$, $=0$ if $t \geq 0$. We see that $(c'_{i,t},k'_{i,t+1})_{t=0}^T$ belongs to $C_i^T(\bar{p},\bar{r})$. Since $B_i^T(\bar{p},\bar{r})\neq\emptyset$, there exists a sequence $\Big((c^n_{i,t},k^n_{i,t+1})_{t=0}^T\Big)_{n=0}^{\infty} \in B_i^T(\bar{p},\bar{r})$ with $k^n_{i,T+1}=0$, and this sequence converges to $(c'_{i,t},k'_{i,t+1})_{t=0}^T$ when n tends to infinity. We have

$$\bar{p}_t(c^n_{i,t}+k^n_{i,t+1}-(1-\delta)k^n_{i,t}) < \bar{r}_t k^n_{i,t} + \theta^i\pi_t(\bar{p}_t,\bar{r}_t)$$

We can chose $s_0 > T$, high enough, such that: for every $s \geq s_0$, we have

$$\bar{p}^s_t(c^n_{i,t}+k^n_{i,t+1}-(1-\delta)k^n_{i,t}) < \bar{r}^s_t k^n_{i,t} + \theta^i\pi_t(\bar{p}^s_t,\bar{r}^s_t).$$

It means that $(c^n_{i,t},k^n_{i,t+1})_{t=0}^T \in C_i^T(\bar{p}^s,\bar{r}^s)$. Therefore, we get $\sum\limits_{t=0}^{T}\beta_i^t u_i(c^n_{i,t}) \leq \sum\limits_{t=0}^{s}\beta_i^t u_i(\bar{c}^s_{i,t})$.

Let s tend to infinity, we obtain $\sum\limits_{t=0}^{T}\beta_i^t u_i(c^n_{i,t}) \leq \sum\limits_{t=0}^{\infty}\beta_i^t u_i(\bar{c}_{i,t})$.

Let n tends to infinity, we have $\sum\limits_{t=0}^{T}\beta_i^t u_i(c_{i,t}) \leq \sum\limits_{t=0}^{\infty}\beta_i^t u_i(\bar{c}_{i,t})$ for every T.

Let T tend to infinity, we obtain $\sum\limits_{t=0}^{\infty}\beta_i^t u_i(c_{i,t}) \leq \sum\limits_{t=0}^{\infty}\beta_i^t u_i(\bar{c}_{i,t})$.

(v) p_t is strictly positive thanks to the strict increasingness of the utility functions.

□

4 Physical asset bubble

Let $\Big(p_t, r_t, (c_{i,t},k_{i,t})_{i=1}^m, K_t\Big)_{t=0}^{+\infty}$ be an equilibrium. Without loss of generality, we assume that $p_t = 1$ for every t.

Lemma 2. *For each t, we have*

$$1 = (1-\delta+\frac{r_{t+1}}{p_{t+1}})\gamma_{t+1} \tag{9}$$

where $\gamma_{t+1} := \max\limits_{i\in\{1,\dots,m\}} \dfrac{\beta_i u'_i(c_{i,t+1})}{u'_i(c_{i,t})}$.

Proof. Firstly, we write all FOCs for the economy $\mathcal{E}$. Denote by $\lambda_{i,t}$ the multiplier with respect to the budget constraint of agent i and by μ_{t+1} the multiplier with respect to the borrowing constraint (i.e., $k^T_{i,t+1} \geq 0$) of agent i.

$$\beta_i^t u_i'(c_{i,t}) = \lambda_{i,t} p_t$$
$$\lambda_{i,t} p_t = \lambda_{i,t+1}(r_{t+1} + p_{t+1}(1-\delta)) + \mu_{i,t+1}$$
$$\mu_{i,t+1} k_{i,t+1} = 0.$$

Therefore, we have $\dfrac{p_{t+1}}{r_{t+1} + p_{t+1}(1-\delta)} \geq \dfrac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}$ for every i.
Since $K_t > 0$ at equilibrium, there exists i such that $k_{i,t+1} > 0$. For such agent, we have $\mu_{i,t+1} = 0$. Thus, $\lambda_{i,t} p_t = \lambda_{i,t+1}(r_{t+1} + p_{t+1}(1-\delta))$. Consequently, we get (9) □

Definition 2. *We define the discount factor of the economy from initial date to date t as follows*

$$Q_0 := 1, \quad Q_t := \prod_{s=1}^{t} \gamma_s, \quad t \geq 1. \tag{10}$$

According to Lemma 2, we have $Q_t = (1 - \delta + \frac{r_{t+1}}{p_{t+1}})Q_{t+1}$ for every $t \geq 0$. As a consequence, we can write

$$\begin{aligned}
1 &= (1-\delta + \frac{r_1}{p_1})Q_1 = (1-\delta)Q_1 + \frac{r_1}{p_1}Q_1 \\
&= (1-\delta)(1-\delta+\frac{r_2}{p_2})Q_2 + \frac{r_1}{p_1}Q_1 = (1-\delta)^2 Q_2 + (1-\delta)\frac{r_2}{p_2}Q_2 + \frac{r_1}{p_1}Q_1 \\
&= \cdots \\
&= (1-\delta)^T Q_T + \sum_{t=1}^{T} (1-\delta)^{t-1}\frac{r_t}{p_t}Q_t.
\end{aligned}$$

Interpretation. In this model, physical capital is viewed as a long-lived asset.

1. At date 1, one unit (from date 0) of this asset will give $(1-\delta)$ units of physical capital and $\frac{r_1}{p_1}$ units of consumption good as its dividend.
2. At date 2, $(1-\delta)$ units of physical capital will give $(1-\delta)^2$ units of physical capital and $(1-\delta)\frac{r_2}{p_2}$ units of consumption good ...

Therefore, the fundamental value of physical capital at date 0 can be defined by

$$FV_0 = \sum_{t=1}^{\infty} (1-\delta)^{t-1} \frac{r_t}{p_t} Q_t.$$

Definition 3. *We say that there is a capital asset bubble if physical capital's price is greater that its fundamental value, i.e.,* $1 > \sum\limits_{t=1}^{\infty} (1-\delta)^{t-1}\frac{r_t}{p_t}Q_t$.

We can see that there is a bubble on capital asset if and only if $\lim\limits_{t\to\infty} (1-\delta)^t Q_t > 0$.

Definition 4. *We say that interest rates are low at equilibrium if*

$$\sum_{t=1}^{\infty} \frac{r_t}{p_t} < \infty. \tag{11}$$

Otherwise, we say that interest rates are high.

We now state our main result in this section.

Proposition 3. *There is a bubble if and only if interest rates are low*

Proof. According (9), we see that $Q_t = (1 - \delta + \frac{r_{t+1}}{p_{t+1}})Q_{t+1}$. Hence, we have

$$\begin{aligned} 1 &= (1-\delta+\frac{r_1}{p_1})Q_1 = (1-\delta+\frac{r_1}{p_1})(1-\delta+\frac{r_2}{p_2})Q_2 \\ &= \ldots = Q_T \prod_{t=1}^{T}(1-\delta+\frac{r_t}{p_t}) = Q_T(1-\delta)^T \prod_{t=1}^{T}[1+\frac{r_t}{(1-\delta)p_t}]. \end{aligned}$$

Consequently, $\lim\limits_{t\to\infty}(1-\delta)^t Q_t > 0$ if and only if $\prod\limits_{t=1}^{\infty}[1+\frac{r_t}{(1-\delta)p_t}] < +\infty$. This condition is equivalent to

$$\sum_{t=1}^{\infty} \frac{r_t}{p_t} < \infty. \tag{12}$$

It means that interest rates are low. □

We point out some consequences of Proposition 3.

Corollary 1. *(Becker, Bosi, Le Van and Seegmuller (2014))*
Assume that $F_t = F$ for every t, F is strictly increasing, strictly concave. Then there is no bubble at equilibrium.

Proof. Since F is strictly increasing and strictly concave, aggregate capital stock is uniformly bounded, i.e., there exists $0 < K < \infty$ such that $K_t \leq K$. Consequently, $\frac{r_t}{p_t} = F'(K_t) > F'(K) > 0$ for every t. This implies that $\sum\limits_{t=1}^{\infty} \frac{r_t}{p_t} = \infty$. According to Proposition 3, there is no bubble. □

Corollary 2. *Assume that $F_t(K) = a_t K$ for each t. Then there is a bubble at equilibrium if and only if $\sum\limits_{t=1}^{\infty} a_t < \infty$.*

Proof. This is a direct consequence of Proposition 3 . □

This result shows that if the productivity decrease to zero with high speed, a bubble in physical capital will appear.

5 On the efficiency of equilibria

In this section, we study the efficiency of intertemporal equilibrium. Following Malinvaud (1953), we define the efficiency of a capital path as follows.

Definition 5. *Let F_t be a production function, δ be capital depreciation rate. A feasible path of capital is a positive sequence $(K_t)_{t=0}^{\infty}$ such that $0 \leq K_{t+1} \leq F_t(K_t) + (1-\delta)K_t$ for every $t \geq 0$ and K_0 is given.*
A feasible path is efficient if there is no other feasible path (K'_t) such that

$$F_t(K'_t) + (1-\delta)K'_t - K'_{t+1} \geq F_t(K_t) + (1-\delta)K_t - K_{t+1}$$

for every t with strict inequality for some t.

Here, aggregate feasible consumption at date t is defined by $C_t := F_t(K_t) + (1-\delta)K_t - K_{t+1}$.

Definition 6. *We say that an intertemporal equilibrium is efficient if its aggregate feasible capital path (K_t) is efficient.*

Our main result in this section requires some intermediate steps. First, we have as in (Malinvaud (1953)).

Lemma 3. *An equilibrium is efficient if $\lim\limits_{t\to\infty} Q_t K_{t+1} = 0$.*

Proof. Let (K'_t, C'_t) be a feasible sequence. We have just to show that

$$\liminf_{T\to+\infty} \sum_{t=0}^{T} Q_t \left(C_t - C'_t\right) \geq 0. \tag{13}$$

It is enough to prove that feasibility and first-order conditions imply

$$\sum_{t=0}^{T} Q_t \left(C_t - C'_t\right) \geq -Q_T K_{T+1} \tag{14}$$

Let us prove inequality (14). We have

$$\begin{aligned}
\Delta_T &\equiv \sum_{t=0}^{T} Q_t \left(C_t - C'_t\right) \\
&= \sum_{t=0}^{T} Q_t \left[F_t\left(K_t\right) - F_t\left(K'_t\right) + (1-\delta)\left(K_t - K'_t\right) - \left(K_{t+1} - K'_{t+1}\right)\right] \\
&\geq \sum_{t=0}^{T} Q_t \left[F'_t\left(K_t\right)\left(K_t - K'_t\right)\right] + (1-\delta)\left(K_t - K'_t\right)\right] - \sum_{t=0}^{T} Q_t \left(K_{t+1} - K'_{t+1}\right) \\
&= \sum_{t=0}^{T} Q_t \left(1 - \delta + \frac{r_t}{p_t}\right)\left(K_t - K'_t\right) - \sum_{t=0}^{T} Q_t \left(K_{t+1} - K'_{t+1}\right)
\end{aligned}$$

By noticing that $K_0 = K'_0$ and $Q_{t+1}\left(1-\delta+\frac{r_{t+1}}{p_{t+1}}\right) - Q_t = 0$, we then get:

$$\begin{aligned}
\Delta_T &\geq \sum_{t=1}^{T} Q_t\left(1-\delta+\frac{r_t}{p_t}\right)(K_t - K'_t) - \sum_{t=0}^{T} Q_t\left(K_{t+1}-K'_{t+1}\right)\\
&= \sum_{t=0}^{T-1}\left[Q_{t+1}\left(1-\delta+\frac{r_{t+1}}{p_{t+1}}\right) - Q_t\right]\left(K_{t+1}-K'_{t+1}\right) - Q_T\left(K_{T+1}-K'_{T+1}\right)\\
&\geq \sum_{t=0}^{T-1}\left[Q_{t+1}\left(1-\delta+\frac{r_{t+1}}{p_{t+1}}\right) - Q_t\right]\left(K_{t+1}-K'_{t+1}\right) - Q_T K_{T+1}\\
&= -Q_T K_{T+1}.
\end{aligned}$$

□

We also have the transversality condition of each agent.

Lemma 4. *At any equilibrium, we have* $\lim_{t\to\infty} \beta_i^t u'_i(c_{i,t})k_{i,t+1} = 0$ *for every* i.

Proof. See Kamihigashi (2002). □

The following result shows the impact of borrowing constraints on the efficiency of an intertemporal equilibrium.

Lemma 5. *Consider an equilibrium. If there exists a date such that, from this date on, the borrowing constraints of agents are not binding at this equilibrium, then it is efficient.*

Proof. Assume that there exists t_0 such that $k_{i,t} > 0$ for every i and for every $t \geq t_0$. Then we have: for every $t \geq t_0$

$$\frac{Q_t}{Q_{t_0}} = \beta_i^{t-t_0}\frac{u'_i(c_{i,t})}{u'_i(c_{i,t_0})}.$$

According to Lemma 4, we have $\lim_{t\to\infty} \beta_i^t u'_i(c_{i,t})k_{i,t+1} = 0$. Then $\lim_{t\to\infty} Q_t k_{i,t+1} = 0$ for every i. This implies that $\lim_{t\to\infty} Q_t K_{t+1} = 0$. Therefore, this equilibrium is efficient. □

We now state our main finding in this section.

Proposition 4. *Assume that the production functions are linear. Then every equilibrium path is efficient.*

Proof. Since production functions are linear, profit equals to zero. Recall that we have $c_{i,t} > 0$ for every i and every t. This implies that $k_{i,t} > 0$ at equilibrium. According to Lemma 5, every equilibrium path is efficient. □

Corollary 2 and Proposition 4 indicate that with linear production functions, there exists an equilibrium the capital path of which is efficient and a bubble may arise at this equilibrium.

6 Conclusion

We build infinite-horizon dynamic deterministic general equilibrium models in which heterogenous agents invest in physical capital and consume. We proved existence of equilibrium in this model, even if technologies are not stationary and aggregate capital is not uniformly bounded.

We say there is a bubble of physical capital at equilibrium if the physical capital's price is greater than its fundamental value. We point out that bubble is equivalent to low interest rates. In particular, there is a bubble if productivities decrease with high speed.

With linear technologies, every intertemporal equilibrium is efficient. Interestingly, it is possible to have both bubble and efficient at equilibrium.

A Appendix: Existence of equilibrium for the truncated economy

Proof of Proposition 1. Denote $\Delta := \{z_0 = (p, r) : 0 \leq p_t, r_t \leq 1, p_t + r_t = 1 \quad \forall t = 0, \ldots, T\}$,

$$B_i(p,r) := \Big\{ \quad (c_i, k_i) \in \mathcal{C}_i \times \mathcal{K}_i \text{ such that } : \forall t = 0, \ldots, T \\ p_t(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) < r_t k_{i,t} + \theta^i \pi_t \Big\},$$

and

$$C_i(p,r) := \Big\{ \quad (c_i, k_i) \in \mathcal{C}_i \times \mathcal{K}_i \text{ such that } : \forall t = 0, \ldots, T \\ p_t(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) \leq r_t k_{i,t} + \theta^i \pi_t \Big\},$$

Denote by $\bar{B}_i(z_0)$ the closure of $B_i(z_0)$.

Lemma 6. *For every $(p, r) \in \mathcal{P}$, we have $B_i(p, q) \neq \emptyset$ and $\bar{B}_i(p, q) = C_i(p, q)$.*

Proof. We rewrite $B_i(p, r)$ as follows

$$B_i(p,r) := \Big\{(c_i, a_i) \in \mathcal{C}_i \times \mathcal{A}_i \text{ such that } : \forall t = 0, \ldots, T \\ 0 < p_t((1-\delta)k_{i,t} - c_{i,t} - k_{i,t+1}) + r_t k_{i,t} + \theta^i \pi_t \Big\}.$$

Since $(1-\delta)k_{i,0} > 0$, we can choose $c_{i,0} \in (0, B_c)$ and $k_{i,1} \in (0, B_k)$ such that

$$0 < p_0((1-\delta)k_{i,0} - c_{i,0} - k_{i,1}) + r_0 k_{i,0} + \theta^i \pi_0.$$

By induction, we see that $B_i(p, r)$ is not empty. □

Lemma 7. *$B_i(p, r)$ is a lower semi-continuous correspondence on $\mathcal{P} := \Delta^{T+1}$. And $C_i(p, r)$ is upper semi-continuous on $\mathcal{P}$ with compact convex values.*

Proof. Clearly, since $B_i(p, r)$ is empty and has an open graph. □

We define $\Phi := \Delta \times \prod_{i=1}^{m}(\mathcal{C}_i \times \mathcal{K}_i) \times \mathcal{K}$. An element $z \in \Phi$ is in the form $z = (z_i)_{i=0}^{m+1}$ where $z_0 := (p, r), z_i := (c_i, k_i)$ for each $i = 1, \ldots, m$, and $z_{m+1} = K$.

We now define correspondences. First, we define φ_0 (for additional agent 0)

$$\varphi_0 : \prod_{i=1}^{m}(\mathcal{C}_i \times \mathcal{K}_i) \times \mathcal{K} \to 2^{\Delta}$$

$$\varphi_0((z_i)_{i=1}^{m+1}) := \operatorname*{arg\,max}_{(p,r)\in\Delta} \Big\{ \sum_{t=0}^{T} p_t\big(\sum_{i=1}^{m} [c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}] - F_t(K_t)\big) + \sum_{t=0}^{T} r_t\big(K_t - \sum_{i=1}^{m} k_{i,t}\big) \Big\}.$$

For each $i = 1, \ldots, m$, we define

$$\begin{aligned} \varphi_i : \quad & \Delta \to 2^{\mathcal{C}_i \times \mathcal{K}_i} \\ \varphi_i(p, r) \quad := \quad & \operatorname*{arg\,max}_{(c_i,k_i)\in C_i(p,r)} \Big\{ \sum_{t=0}^{T} \beta_i^t u_i(c_{i,t}) \Big\}. \end{aligned}$$

For each $i = m + 1$, we define

$$\begin{aligned} \varphi_{m+1} : \quad & \Delta \to 2^{\mathcal{K}} \\ \varphi_i(p, r) \quad := \quad & \operatorname*{arg\,max}_{K\in\mathcal{K}} \Big\{ \sum_{t=0}^{T} p_t F_t(K_t) - r_t K_t \Big\}. \end{aligned}$$

Lemma 8. *φ_i is upper semi-continuous convex-valued correspondence for each $i = 0, 1, \ldots, m+1$.*

Proof. This is a direct consequence of the Maximum Theorem. □

According to the Kakutani Theorem, there exists $(\bar{p}, \bar{r}, (\bar{c}_i, \bar{k}_i)_{i=1}^{m}, \bar{K})$ such that

$$(\bar{p}, \bar{r}) \in \varphi_0((\bar{c}_i, \bar{k}_i)_{i=1}^{m}, \bar{K}) \tag{15}$$
$$(\bar{c}_i, \bar{k}_i) \in \varphi_i(\bar{p}, \bar{r}) \tag{16}$$
$$\bar{K} \in \varphi_{m+1}(\bar{p}, \bar{r}). \tag{17}$$

Denote by $\bar{X}_t := \sum_{i=1}^{m} [\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1-\delta)\bar{k}_{i,t}] - F_t(\bar{K}_t)$ and $\bar{Y}_t = \bar{K}_t - \sum_{i=1}^{m} \bar{k}_{i,t}$ the excess demands for goods and capital respectively. For every $(p, r) \in \Delta^{T+1}$, we have

$$\sum_{t=0}^{T} (p_t - \bar{p}_t)\bar{X}_t + \sum_{t=0}^{T} (r_t - \bar{r}_t)\bar{Y}_t \le 0. \tag{18}$$

By summing the budget constraints, for each t, we get

$$\bar{p}_t \bar{X}_t + \bar{r}_t \bar{Y}_t \le 0. \tag{19}$$

Hence, we have: for every $(p_t, r_t) \in \Delta$

$$p_t \bar{X}_t + q_t \bar{Y}_t \le \bar{p}_t \bar{X}_t + \bar{r}_t \bar{Y}_t \le 0. \tag{20}$$

Therefore, we have $\bar{X}_t, \bar{Y}_t \leq 0$, which implies that

$$\sum_{i=1}^{m} \bar{c}_{i,t} + \bar{k}_{i,t+1} \leq (1-\delta) \sum_{i=1}^{m} \bar{k}_{i,t} + F_t(\bar{K}_t) \tag{21}$$

$$\bar{K}_t \leq \sum_{i=1}^{m} \bar{k}_{i,t}. \tag{22}$$

Lemma 9. *$\bar{p}_t, \bar{r}_t > 0$ for $t = 0, \ldots, T$.*

Proof. If $\bar{p}_t = 0$ then $\bar{c}_{i,t} = B_c > (1-\delta)B + F_t(B)$. Therefore, we get $\bar{c}_{i,t} + \bar{k}_{i,t+1} > (1-\delta) \sum_{i=1}^{m} \bar{k}_{i,t} + F_t(\bar{K}_t)$ which is a contradiction. Hence, $\bar{p}_t > 0$.
If $\bar{r}_t = 0$, then the optimality of $\bar{K}$ implies that $K_t = B$. However, we have $\bar{k}_{i,t} \leq B_k$ for every i, t. Consequently, $\sum_{i=1}^{m} \bar{k}_{i,t} \leq mB_k < B = K_t$, contradiction to (22). Therefore, we get $\bar{r}_t > 0$. □

Lemma 10. $\sum_{i=1}^{m} \bar{k}_{i,t} = \bar{K}_t$ *and* $\sum_{i=1}^{m} [\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1-\delta)\bar{k}_{i,t}] = F(\bar{K}_t)$

Proof. Since prices are strictly positive and the utility functions are strictly increasing, all the budget constraints are binding and, summing them across the individuals, we get

$$\bar{p}_t \bar{X}_t + \bar{r}_t \bar{Y}_t = 0. \tag{23}$$

We know that $\bar{X}_t, \bar{Y}_t \leq 0$ and $\bar{p}_t, \bar{r}_t > 0$. Then, $\bar{X}_t = \bar{Y}_t = 0$. The optimality of $(\bar{c}_i, \bar{k}_i)$ and $\bar{K}$ comes from (16) and (17). □

□

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