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Efficient allocations and Equilibria with short-selling and Incomplete Preferences

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Abstract

This article reconsiders the theory of existence of efficient allocations and equilibria when consumption sets are unbounded below under the assumption that agents have incomplete preferences. It is motivated by an example in the theory of assets with short-selling where there is risk and ambiguity. Agents have Bewley's incomplete preferences. As an inertia principle is assumed in markets, equilibria are individually rational. It is shown that a necessary and sufficient condition for the existence of an individually rational efficient allocation or of an equilibrium is that the relative interiors of the risk adjusted sets of probabilities intersect. The more risk averse, the more ambiguity averse the agents, the more likely is an equilibrium to exist. The paper then turns to incomplete preferences represented by a family of concave utility functions. Several definitions of efficiency and of equilibrium with inertia are considered. Sufficient conditions and necessary and sufficient conditions are given for the existence of efficient allocations and equilibria with inertia.

Keywords: Uncertainty, risk, risk adjusted prior, no arbitrage, equilibrium with short-selling, incomplete preferences, equilibrium with inertia.

JEL Classification: C62, D50, D81,D84,G1.

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1 Introduction

The issue of existence of an equilibrium for finite markets with short-selling is an old problem first considered in the early seventies by Grandmont [8], Hart [11] and Green [9] and reconsidered later by Hammond [10] and Page [13], [14]. In these early papers, investors were assumed to hold a single probabilistic belief (homogeneous or heterogeneous) and be risk averse von Neumann-Morgenstern (vNM) utility maximizers. Two sufficient conditions for the existence of an equilibrium were given:

1. a condition which expresses that investors are sufficiently similar in their beliefs and risk aversions so that there exists a non empty set of prices (*the no-arbitrage prices*) for which no agent can make costless unbounded utility nondecreasing purchases,

2. a collective absence of arbitrage condition, which requires that investors do not engage in mutually compatible, utility nondecreasing trades.

These conditions have been generalized to abstract economies and are known as the *existence of no-arbitrage prices* condition (see Werner [17]) and the *no unbounded utility arbitrage* condition (NUBA) (see Page [14]). They were shown to be equivalent under adequate hypotheses. Other sufficient conditions were given. For a review of the subject in finite dimension, see Allouch et al [1], Dana et al [5], Page [13],[14]. All this trend of literature assumes complete preferences.

This paper extends the previous theory to incomplete preferences represented by families of concave utility functions. Such incomplete preferences include Bewley's [2] and Rigotti and Shannon [15] incomplete preferences. Under this representation assumption, it is easy to define and characterize the concepts of no-arbitrage prices and no unbounded utility arbitrage. Weak and strong concepts of efficiency and equilibria are defined. As in the case of complete preferences, existence of a no-arbitrage price is shown to be equivalent to NUBA and to be a sufficient condition for the existence of weakly efficient allocations and weak equilibria. Under further adequate assumptions (for example strict concavity and monotonicity of utilities), it is shown to be necessary and sufficient for the existence of efficient allocations and equilibria.

As Page [14] and Werner [17], the paper is motivated by an example in the theory of assets with short-selling. Agents are assumed not to have enough information to quantify uncertainty by a single probability, hence each agent has a set of priors. Agents are further assumed to have risk averse Bewley's [2] (or Rigotti and Shannon [15]) incomplete preferences. Under standard conditions on utility indices (strict concavity and increasingness) and sets of priors (convexity and compactness), it is shown that a necessary and sufficient for the existence of an individually rational efficient allocation or equilibrium is that the relative interiors of the agents' sets of risk adjusted probabilities intersect or that agents do not engage in mutually compatible trades that have non negative expectations with respect to their risk adjusted probabilities. The first condition generalizes the overlapping expectations given by Grandmont [8], Green [9] and Hammond [10] for the case of single belief. The second generalizes the NUBA type of condition given by Hart [11]. In a vNM framework, Hart [11] further shows that the more risk averse the agents, the more likely is an equilibrium to exist. In the Bewley's setting, it is shown that the more risk averse, the more ambiguity averse the agents, the more likely is an equilibrium to exist.

The example is also shown to have an interesting feature: the arbitrage concepts coincide with those of a Gilboa-Schmeidler's agent with same sets of priors and utility indices. Hence the condition that the relative interiors of the sets of risk adjusted probabilities intersect is also necessary and sufficient for the existence of an equilibrium in a model with Gilboa-Schmeidler's preferences (with same sets of priors and utility indices). This may seem at odds with the literature on efficiency or equilibria with Bewley's incomplete preferences (see Bewley's [2], Rigotti and Shannon [15] or Dana and Riedel [6]) which highlights the differences with using maxmin complete preferences. However it is not as a different issue is addressed, that of existence of equilibrium with short selling.

The paper is organized as follows: section 2 deals with the example and provides an existence theorem while section 3 deals with its generalization.

2 An example

2.1 Bewley preferences

We consider a standard Arrow-Debreu model of complete contingent security markets with short selling. There are two dates, 0 and 1. At date 0, there is uncertainty about which state s from a state space $\Omega = \{1, ..., k\}$ will occur at date 1. At date 0, agents who are uncertain about their future endowments trade contingent claims for date 1. The space of contingent claims is the set of random variables from$\Omega \rightarrow \mathbb{R}$. The random variable X which equals x_1 in state 1, x_2 in state 2 and x_k in state k, is identified with the vector in $X \in \mathbb{R}^k$, $X = (x_1, \dots, x_k)$. Let $\triangle = \{\pi \in \mathbb{R}^k_+ : \sum_{s=1}^k \pi_s = 1\}$ be the probability simplex in $\mathbb{R}^k$. Let $\text{int}\triangle = \{\pi \in \triangle,\ \pi_s > 0, \forall\ s\}$. For $A \subseteq \triangle$, $\text{int}A = \{p \in A\ \ |$

$\exists$ a ball $B(p,\varepsilon)$ s.t. $B(p,\varepsilon) \cap$ int $\triangle \subseteq A\}$. For a given $\pi \in \triangle$, we denote by $E_\pi(X) := \sum_{l=1}^{k} \pi_l x_l$ the expectation of X. Finally, for a given price $p \in \mathbb{R}^k$, $p \cdot X := \sum_{l=1}^{k} p_l x_l$, the price of X.

There are m agents indexed by $i = 1, \ldots, m$. Agent i has an endowment $E^i \in \mathbb{R}^k$ of contingent claims. Let $(E^i)_{i=1}^m$ be the m-tuple of endowments and $E = \sum_{i=1}^m E^i$ be the aggregate endowment. We assume that agent i has a convex compact set of priors $P^i \subseteq$int$\triangle$ and an incomplete Bewley preference relation $\succeq$ over $\mathbb{R}^k$ defined by

$$X \succeq^i Y \iff E_\pi(u^i(X)) \geq E_\pi(u^i(Y)), \ \forall\, \pi \in P^i \tag{1}$$

where $u^i : \mathbb{R} \to \mathbb{R}$ is a strictly concave, increasing differentiable utility index fulfilling $u^i(0) = 0$. The associated strict preference is $X \succ^i Y$ if $X \succeq^i Y$ and $E_\pi(u^i(X)) > E_\pi(u^i(Y))$ for some $\pi \in P^i$.

2.2 Individual and collective absence of arbitrage

In this subsection, we define and characterize the useful vectors of a Bewley preference relation of type (1). They are the directions such that trading at any positive scale makes the agent better off. Our main result is that they coincide with those of a Gilboa-Schmeidler utility defined by (u, P) (see (2) below) .

We then recall the concepts of no-arbitrage prices and of collective absence of arbitrage (NUBA). As these concepts only depend on useful vectors, we obtain that two economies with Bewley preferences or Gilboa-Schmeidler utilities with same indices and sets of priors have same sets of no-arbitrage prices and same NUBA condition.

2.2.1 Useful vectors

To simplify notations, in this subsection, the agent's index is omitted. We consider an agent described by a pair (u, P) of a utility index and a set of priors. For $\pi \in P$, let

$$\widehat{P}_\pi(X) = \{Y \in \mathbb{R}^k \mid E_\pi(u(Y)) \geq E_\pi(u(X))\}$$

be the set of contingent claims preferred to X for the utility $E_\pi(u(.))$ and $R_\pi(X) = \{W \in \mathbb{R}^k \mid E_\pi(u(X + \lambda W)) \geq u(X), \ \ \forall\, \lambda \in \mathbb{R}_+\}$ be its asymptotic cone. As $E_\pi(u(.))$ is concave, $R_\pi(X)$ is independent of X and denoted R_π. Taking $X = 0$, we obtain:

$$R_\pi = \{W \in \mathbb{R}^k \mid E_\pi(u(\lambda W)) \geq u(0) = 0, \ \forall\, \lambda \in \mathbb{R}_+\}$$

Let

$$\widehat{P}(X) = \{Y \in \mathbb{R}^k \mid E_\pi(u(Y)) \geq E_\pi(u(X)), \ \ \forall\, \pi \in P\}$$

be the set of contingent claims preferred to X for the Bewley's preference defined by (1) and $R(X)$ be its asymptotic cone. From Rockafellar's [16] corollary 8.3.3, $R(X) = \cap_{\pi \in P} R_\pi(X) = \cap_{\pi \in P} R_\pi$. Hence it is independent of X and denoted R.

$$R = \{W \in \mathbb{R}^k \mid E_\pi(u(\lambda W)) \geq 0,\ \forall\, \lambda \in I\!R_+,\ \ \pi \in P\}$$

and is called the set of *useful vectors* for $\succeq$. For a given pair (u, P), let

$$V(X) = \min_{\pi \in P} E_\pi(u(X)) \tag{2}$$

be the Gilboa-Schmeidler's utility.

$$\{W \in \mathbb{R}^k \mid V(\lambda W) \geq 0,\ \forall\, \lambda \in I\!R_+\} = \{W \in \mathbb{R}^k \mid E_\pi(u(\lambda W)) \geq 0,\ \forall\, \lambda \in I\!R_+, \pi \in P\}$$

Hence it coincides with R. Furthermore given $C \in I\!R^k$ a reference point, the C-reference dependent ambiguity averse (RAA) utility, axiomatized by Mihm [12] is defined by

$$V_C(X) = \min_{\pi \in P} [E_\pi(u(X)) - E_\pi(u(C))] \tag{3}$$

This is a concave variational utility, hence the set of useful vectors is independent of X. As $V_C(C) = 0$, $R_{V_C} = \{W \in \mathbb{R}^k \mid V_C(C + \lambda W) \geq 0,\ \forall\, \lambda \in I\!R_+\}$. Equivalently

$$R_{V_C} = \{W \in \mathbb{R}^k \mid E_\pi(u(C + \lambda W)) \geq E_\pi(u(C)),\ \forall\, \lambda \in I\!R_+, \pi \in P\} = R$$

The previous discussion is summarized in the following lemma:

Lemma 1 *The set of useful vectors for a Bewley preference of type (1) coincides with the set of useful vectors for a Gilboa-Schmeidler preference (2) or of an RAA utility (3) for any reference point $C \in I\!R^k$.*

Let us recall a characterization of useful vectors proven in Dana and Le Van [4]. To this end, let

$$\widetilde{P} = \left\{ p \in \triangle \ \mid \exists\ \pi \in P,\ Z \in \mathbb{R}^k \text{ s. t. } p_s = \frac{\pi_s u'(z_s)}{E_\pi(u'(Z))},\ \forall\, s = 1, \ldots, k \right\} \tag{4}$$

be the set of risk adjusted probabilities. We have (see Dana and Le Van [4]) :

Lemma 2 $R = \{W \in \mathbb{R}^k \mid \ E_p(W) \geq 0,\ \textit{for all}\, p \in \tilde{P}\}$.

In the next two subsections, we introduce two concepts of absence of arbitrage, a concept of individual no-arbitrage and a concept of collective absence arbitrage. These concepts only depend on agents' useful vectors.

We first define the concept of no-arbitrage price (see Werner [17]).

Definition 1 *A price vector $p \in \mathbb{R}^k$ is a " no-arbitrage price" for agent i if $p \cdot W > 0$, for all $W \in R^i \backslash \{0\}$. A price vector $p \in \mathbb{R}^k$ is a " no-arbitrage price" for the economy if it is a no-arbitrage price for each agent.*

Let S^i denote the set of no-arbitrage prices for i. Using Lemma 2, Dana and Le Van [4] characterize S^i in terms of risk adjusted probabilities.

Lemma 3 *1. The set of no-arbitrage prices for agent i is $S^i =$ cone int $\widetilde{P}^i$ where $p \in$ int $\widetilde{P}^i$ if and only if $\exists\ \pi \in P^i \cap$ int $\triangle$, $Z \in \mathbb{R}^k$, $\forall s$, $a < u'(z_s) < b$ and $p_s = \frac{\pi_s u'(z_s)}{E_\pi(u'(Z))}$. The more risk averse, the more ambiguity averse the agent, the larger is S^i.*

2. The set of no-arbitrage prices for the economy is $\underset{i}{\cap} S^i =$ cone $\underset{i}{\cap}$ int $\widetilde{P}^i$. The more risk averse, the more ambiguity averse the agents, the larger is the set of no-arbitrage prices for the economy.

2.2.2 Collective absence of arbitrage

From now on, a *feasible trade* is an $m-$tuple $W^1, \dots, W^m$ with $W^i \in \mathbb{R}^k$ for all i and $\sum_i W^i = 0$. We recall the *no-unbounded-arbitrage* condition (NUBA) introduced by Page [13] which requires inexistence of unbounded utility nondecreasing feasible trades.

Definition 2 *The economy satisfies the NUBA condition if $\sum_i W^i = 0$ and $W^i \in R^i$ for all i, implies $W^i = 0$ for all i.*

From Lemma 2, we may characterize the NUBA condition .

Corollary 1 *NUBA is equivalent to: there exists no feasible trade $W^1, \dots, W^m$ with $W^i \neq 0$ for some i that fulfills $E_\pi(W^i) \geq 0$ for all i and $\pi \in \widetilde{P}^i$.*

2.3 Existence of efficient allocations and equilibria

2.3.1 Concepts in equilibrium theory

We next recall standard concepts in equilibrium theory.
Given the allocation of initial endowments $(E^i)_{i=1}^m$, an allocation $(X^i)_{i=1}^m \in (\mathbb{R}^k)^m$ is attainable (or feasible) if $\sum_{i=1}^m X^i = E$. The set of B-individually rational attainable allocations $A((E^i)_{i=1}^m)$ is defined by

$$A((E^i)_{i=1}^m) = \left\{ (X^i)_{i=1}^m \in (\mathbb{R}^k)^m \ \mid \ \sum_{i=1}^m X^i = E \ \text{ and } \ X^i \succeq^i E^i, \ \forall\, i \right\}.$$

Definition 3 *Given* $(E^i)_{i=1}^m$*, an attainable allocation* $(X^i)_{i=1}^m$ *is* B*-efficient if there does not exist* $(X'^i)_{i=1}^m$ *attainable such that* $X'_i \succeq^i X_i$ *for all* i *with a strict inequality for some* i*. It is weakly* B*-efficient if there does not exist* $(X'^i)_{i=1}^m$ *attainable such that* $E_\pi(u^i(X'^i)) > E_\pi(u^i(X^i))$*,* $\forall\, \pi \in \mathcal{P}^i$*,* $\forall\, i$*. It is individually rational (weakly)* B*-efficient if it is (weakly)* B*-efficient and* $X^i \succeq^i E^i$ *for all* i*.*

Since u^i is strictly concave and strictly increasing and P^i is compact for all i, $(X^i)_{i=1}^m$ is B-efficient if and only if $(X^i)_{i=1}^m$ is weakly B-efficient (see Lemma 4 and Remark 3 below).

Definition 4 *A pair* $(X^*, p^*) \in A((E^i)_{i=1}^m) \times \mathbb{R}^k \backslash \{0\}$ *is a (weak)* B*-equilibrium with inertia if*

1. *for each agent* i *and* $X^i \in \mathbb{R}^k$*,* $X^i \succ^i X^{i*}$ *(*$E_\pi(u^i(X^i)) > E_\pi(u^i(X^{i*}))$*,* $\forall\, \pi \in \mathcal{P}^i$*,* $\forall\, i$*) implies* $p^* \cdot X^i > p^* \cdot X^{i*}$*,*
2. *for each agent* i*,* $p^* \cdot X^{i*} = p^* \cdot E^i$*.*

Since u^i is strictly concave for all i, it may easily be verified that (X^*, p^*) is a B-equilibrium with inertia if and only if it is a weak B-equilibrium with inertia. Note that at an equilibrium with inertia, the allocation is individually rational.

2.3.2 Existence of efficient allocations and equilibria

The following theorem fully characterizes existence of individually rational B-efficient allocations as well as B-equilibria with inertia.

Theorem 1 *The following assertions are equivalent:*

1. *there exists a no-arbitrage price, equivalently* $\bigcap_i \mathrm{int}\, \widetilde{P}^i \neq \emptyset$*,*
2. *there exists no feasible trade* $W^1, \ldots, W^m$ *with* $W^i \neq 0$ *for some* i *and* $E_\pi(W^i) \geq 0$ *for all* $\pi \in \widetilde{P}^i$ *and for all* i*,*
3. *the set of* B*-individually rational attainable allocations is compact,*
4. *there exists a* B*-individually rational efficient allocation,*
5. *there exists a* B*-equilibrium with inertia.*

Furthermore any equilibrium price is a no-arbitrage price.

Proof: Since $X^i \succeq^i E^i$ is equivalent to $V_{E^i}(X) \geq V_{E^i}(E^i) = 0$, the set of individually rational attainable allocations for the economy with Bewley's preferences coincides with the set of individually rational attainable allocations for the economy with RAA utilities with reference points $(E^i)_{i=1}^m$. Applying Dana and Le Van [4] to these utilities, we obtain the equivalence between 1, 2 and 3.

Let us prove that 1 implies 4. Applying Dana and Le Van [4] to RAA utilities with reference points $(E^i)_{i=1}^m$, we obtain the existence of an individually rational efficient allocation $(\bar{X}'^i)_{i=1}^m$ for RAA utilities. Let us show that it is B-weakly efficient. Suppose not. Then there exists $(X'^i)_{i=1}^m$ attainable such that $E_\pi(u^i(X'^i)) > E_\pi(u^i(\bar{X}^i))$ for all i and $\pi \in \mathcal{P}^i$ but then $V_{E^i}(X'^i) > V_{E^i}(\bar{X}^i)$ for all i contradicting the efficiency of $(\bar{X}'^i)_{i=1}^m$ for RAA utilities. Therefore $(\bar{X}'^i)_{i=1}^m$ is a B-individually rational weakly efficient allocation, hence a B-individually rational efficient allocation.

Let us next show that 4 implies 2. Let $(\bar{X}^i)_{i=1}^m$ be a B-efficient allocation. Suppose that there exists a feasible trade $W^1, \ldots, W^m$ with $W^i \in R^i$ for all i and $W^i \neq 0$ for some i. We then have $E_\pi(u^i(\bar{X}^i + tW^i)) \geq E_\pi(u^i(\bar{X}^i))$, for all $\pi \in \mathcal{P}^i$ for all i and as u^i is strictly concave, $E_\pi(u^i(\bar{X}^i + tW^i)) > E_\pi(u^i(\bar{X}^i))$ for all i such that $W^i \neq 0$ and $\pi \in \mathcal{P}^i$. The allocation $(\bar{X}^i + tW^i)_{i \in I}$ being feasible, this contradicts the B-efficiency of $(\bar{X}^i)_{i=1}^m$. Hence 1-4 are equivalent.

Finally let us show that 1 is equivalent to 5. Let us first remark that if (X^*, p^*) is an equilibrium with inertia, then p^* is a no arbitrage price. Indeed as u^i is strictly concave, if W^i is useful and $W^i \neq 0$, then $E_\pi(u^i(X^{*i} + tW^i)) > E_\pi(u^i(X^{*i}))$, $\forall\, \pi \in \mathcal{P}^i$, hence $p^* \cdot W^i > 0$. Hence if there exists an equilibrium, there exists a no-arbitrage price and 5 implies 1. Conversely if 1 holds true, then from Dana and Le Van [4], the economy with variational utilities $(V_{E^i})_{i \in I}$ has an equilibrium (X^*, p^*). Let us show that (X^*, p^*) is a weak B-equilibrium with inertia. Indeed if $E_\pi(u^i(X^i)) > E_\pi(u^i(X^{*i}))$, $\forall\, \pi \in \mathcal{P}^i$, then $V_{E^i}(X) > V_{E^i}(X^{*i})$, hence $p^* \cdot X^i > p^* \cdot X^{*i}$. Furthermore as the family of utilities $(V_{E^i})_{i \in I}$ is strictly concave, if $X^{*i} \neq E^i$, then $V_{E^i}(X^{*i}) > V_{E^i}(E^i)$, hence $X^{*i} \succ^i E^i$ proving that (X^*, p^*) is a weak Bewley equilibrium with inertia, hence a Bewley equilibrium with inertia. ■

Remark 1 *When stating theorem 1, we have assumed that all agents had Bewley's incomplete preferences. From Dana and Le Van [4], we could have assumed that all agents had Gilboa-Schmeidler's utilities. A more general result is true : agent i may either have a Bewley's incomplete preference or a Gilboa-Schmeidler's utility or an RAA utility with any reference point with utility index u^i and sets of priors $\mathcal{P}^i$ or any utility with useful vectors R^i. Indeed if agent i has a Bewley's incomplete preference, one can consider the fictitious agent with an RAA utility with reference point E^i, utility index u^i and priors $\mathcal{P}^i$. From Dana and Le Van [4], we obtain the existence of an equilibrium for the ficti-*

tious economy. As in the proof of 1 implies 4, any equilibrium of the fictitious economy is an equilibrium of the original economy.

Remark 2 *The strict concavity of the utility functions plays an important and subtle role. First, for most purposes, when $X \succ Y$, we may assume that $E_\pi(u(X)) > E_\pi(u(Y))$, $\forall\ \pi \in \mathcal{P}$. Second, strict concavity is used to prove the equivalence between weak B-equilibrium and B-equilibrium and weak B-efficiency and B-efficiency. Third, as u is strictly concave, for any useful vector $W \neq 0$ and any $\pi \in P$, the map $t \to E_\pi(u(X + tW))$ is strictly increasing. In theorem 1, this is used in 4 implies 2 and in the assertion that an equilibrium price is a no-arbitrage price.*

Remark 3 *In our model an efficient allocation exists iff $\bigcap_i int\ \widetilde{P}^i \neq \emptyset$ while a given attainable allocation $(X^i)_{i=1}^m$ is efficient iff the sets of risk-adjusted probabilities $\widetilde{P}^i(X^i)$ at X^i, $i = 1, \ldots, m$ intersect (see Rigotti and Shannon [15]).*

3 An abstract economy with incomplete preferences

3.1 The economy

Many agents in financial markets are not single individuals but groups of individuals. We learn from social choice that it is not possible to assign a complete preference satisfying reasonable assumptions to a group of individuals. A natural way to resolve this paradox is to allow for the possibility of incomplete preferences. In this section, we assume that each agent has a set of utilities, one for each member of the group. Two goods may be compared by the agent only if there is unanimity in the group on the choice.

More precisely, we consider an economy with m agents and d goods. Agent i has consumption space $\mathbb{R}^d$, endowment $E^i \in \mathbb{R}^d$ and incomplete (or complete) convex preferences over $\mathbb{R}^d$, defined by a family $\mathcal{U}^i : \mathbb{R}^d \to \mathbb{R}$ of concave utility functions : $X^i \in \mathbb{R}^d$ is preferred to $Y^i \in \mathbb{R}^d$ by agent i, denoted $X^i \succeq^i Y^i$ if $u_i(X^i) \geq u_i(Y^i)$ for every $u_i \in \mathcal{U}^i$. The associated strict preference is $X \succ^i Y$ if $X \succeq^i Y$ and $u^i(X) > u^i(Y)$ for some $u^i \in \mathcal{U}^i$. Let $E = \sum_{i=1}^m E^i$ be the aggregate endowment.

We assume that for every i, the utilities in $\mathcal{U}^i$ are everywhere finite, concave, and there is a topology on $\mathcal{U}^i$ which makes it compact and such that the evaluation map $u \in \mathcal{U}^i \mapsto u(X)$ is continuous for every $X \in \mathbb{R}^d$. We assume that $u(0) = 0$ for all $u \in \mathcal{U}^i$ and all i. Let us give two examples of such families.

For notational simplicity, we drop the subscript i.

Example 1
Let T be a compact subset of $I\!R^p$ which can be interpreted as a subset of parameters and $\mathcal{U} = \{u : I\!R^d \times I\!R^p \to I\!R\}$, u continuous on $I\!R^d \times T$ and $u(.,t)$ concave for every $t \in T$.

Example 2
$\mathcal{U}$ is a set of concave functions $I\!R^d \to I\!R$, closed for the topology of uniform convergence on compact subsets and uniformly bounded on each ball of $I\!R^d$. $\mathcal{U}$ is then k_R Lipschitz on the ball of radius R for some constant k_R which depends on the radius. From Ascoli's theorem, $\mathcal{U}$ is then compact for the topology of uniform convergence on comp2act subsets.

An allocation $(X^i)_{i\in I} \in I\!R^{dm}$ is feasible if $\sum_i X^i = E$. The set of individually rational attainable allocations $A((E^i)_{i=1}^m)$ is defined as in 2.3

3.2 Arbitrage concepts

We briefly redefine for abstract incomplete preferences, the concepts which were defined in section 2.2. Let

$$\widehat{P^i}(X) = \{Y \in \mathbb{R}^k \mid u(Y) \geq u(X), \ \ \forall\, u \in \mathcal{U}^i\} = \cap_{u\in\mathcal{U}^i}\{Y \in \mathbb{R}^k \mid u(Y) \geq u(X)\}$$

be the preferred set at X by i and $R^i(X)$ be its asymptotic cone. As the utilities $u \in \mathcal{U}^i$ are concave, $R^i(X)$

is independent of X and denoted R^i. It is called the set of *useful vectors* for $\succeq^i$. From Rockafellar's [16] corollary 8.3.3,

$$R^i = \{W \in I\!R^d \mid u(\lambda W) \geq 0, \ \forall\, \lambda \in I\!R_+, \ u \in \mathcal{U}^i\ \}.$$

As discussed in the previous section, for any $C \in I\!R^d$, R^i is also the set of useful vectors for any complete preference represented by a utility of the form $V_{E^i}(X) = \min_{u\in\mathcal{U}^i}[u(X) - u(E^i)]$. Note that this utility is well defined under our assumptions.

A price vector $p \in \mathbb{R}^d$ is a " no-arbitrage price" for agent i if $p \cdot W > 0$, for all $W \in R^i\backslash\{0\}$. Let S^i denote the set of no-arbitrage prices for i. Then $S^i = -\mathrm{int}(R^i)^0$ (where $(R^i)^0 = \{p \in \mathbb{R}^d \ \mid \ p \cdot X \leq 0, \ \text{ for all } \ X \in R^i\}$). A price vector $p \in \mathbb{R}^k$ is a " no-arbitrage price" for the economy if it is a no-arbitrage price for each agent. A price vector $p \in \mathbb{R}^k$ is a no-arbitrage price for the economy if and only if $p \in \cap_i S^i = -\cap_i \mathrm{int}(R^i)^0$.

An $m-$tuple $(W^1, \dots, W^m) \in \mathbb{R}^{dm}$ is a feasible trade if $\sum_i W^i = 0$.
A trade $W \in \mathbb{R}^d \backslash \{0\}$ is a *half-line* for a utility $u : \mathbb{R}^d \to \mathbb{R}$ if there exists $X \in \mathbb{R}^d$ such that $u(X + \lambda W) = u(X)$ for all $\lambda \geq 0$. As u is concave, when it has no half-line, then $u(X + \lambda W) > u(X)$ for every $X \in \mathbb{R}^d$ and $\lambda > 0$ and $W \neq 0$.

3.3 Efficiency and equilibrium concepts

Definition 5 *1. A feasible allocation $(X^i)_{i=1}^m$ is efficient if there does not exist $(X'^i)_{i=1}^m$ feasible such that $X'^i \succeq^i X^i$ for all i with a strict inequality for some i. It is weakly efficient if there does not exist $(X'^i)_{i=1}^m$ feasible such that $u^i(X'^i) > u^i(X^i),\ \forall\ u_i \in \mathcal{U}^i,\ \forall\ i$. It is individually rational (weakly) efficient if it is (weakly) efficient and $X^i \succeq^i E^i$ for all i.*

2. *A pair $(X^*, p^*) \in A((E^i)_{i=1}^m) \times I\!R^d \backslash \{0\}$ is an (weak) equilibrium with inertia if*
   - *(a) for any i and $X^i \in I\!R^d$, $X^i \succ^i X^{*i}$ ($u^i(X^i) > u^i(X^{*i})$ for every $u^i \in \mathcal{U}^i$) implies $p^* \cdot X^i > p^* \cdot X^{*i}$,*
   - *(b) for any i, $p^* \cdot X^{*i} = p^* \cdot E^i$.*

A *common increasing direction* is a trade $e \in I\!R^d$ such that $e \cdot p > 0$ for every $(X, u) \in I\!R^d \times \mathcal{U}^i,\ p \in \partial u(X)$ and every i. When e is a common increasing direction, $u(X - te) < u(X),\ u(X + te) > u(X)$ for every $t \geq 0,\ X \in I\!R^d,\ u \in \mathcal{U}^i$ and i.

Lemma 4 *Let u be strictly concave for every $u \in \mathcal{U}^i$ and i.*

1. *A pair $(X^*, p^*) \in A((E^i)_{i=1}^m) \times I\!R^d \backslash \{0\}$ is an equilibrium with inertia if and only if it is a weak equilibrium with inertia*
2. *If agents have a common increasing direction, then an attainable allocation $(X^i)_{i=1}^m$ is efficient if and only if it is weakly efficient.*

Proof: The proof of the first assertion is omitted. To prove the second, clearly, if $(X^i)_{i=1}^m$ is efficient, it is weakly efficient. Let us show that weak efficiency implies efficiency. Assume that $(X^i)_{i=1}^m$ is a weakly efficient allocation and that it is not efficient. W.l.o.g. assume that there exists a feasible allocation $(Y^i)_{i=1}^m$ such that $Y^1 \succ^1 X^1$ and $Y^i \succeq^i X^i,\ i \neq 1$. By considering $\frac{(Y^1 + X^1)}{2}$ instead of Y^1, we may assume that $u(Y^1) > u(X^1),\ \forall\ u \in \mathcal{U}^1$. For a given $\varepsilon > 0$, let

$$\mathcal{V}_\varepsilon = \left\{ u \in \mathcal{U}^1 \mid u(Y^1 - e\varepsilon) > u(X^1) \right\}$$

$\mathcal{V}_\varepsilon$ is open, since the evaluation maps are continuous. Let us show that $\cup_\varepsilon \mathcal{V}_\varepsilon = \mathcal{U}^1$. Let $u \in \mathcal{U}^1$. Since u is continuous, there exists ε_u such that $u(Y^1 - e\varepsilon) >$

$u(X^1)$ for every $\varepsilon \leq \varepsilon_u$. Hence $\cup_\varepsilon \mathcal{V}_\varepsilon = \mathcal{U}^1$. As $\mathcal{U}^1$ is compact, there exists a finite subcovering of $\mathcal{U}^1$ by $(\mathcal{V}_{\varepsilon_i})$. Let $\varepsilon = \min_i \varepsilon_i$ and $\varepsilon' = \frac{\varepsilon}{m-1}$. We then have

$$Y^1 - e\varepsilon \succ^1 X^1 \text{ and } Y^i + e\varepsilon' \succ^i X^i,\ i \neq 1$$

contradicting the weak efficiency of X. ■

Remark 4 *It is easy to verify that if (X^*, p^*) is a weak equilibrium with inertia, then X^* is weakly efficient and that, if agents have a common increasing direction and (X^*, p^*) is an equilibrium with inertia, then X^* is efficient. In other words, the first welfare theorem holds true and is straightforward. Carlier and Dana [3] provide general conditions on the families of utilities under which a weakly efficient allocation is a weak equilibrium with transfer. Hence a weak form of the second welfare theorem holds true. Under these conditions, when the utilities are strictly concave and have a common increasing direction (as it is the case in the Bewley model of section 2), weak equilibria with inertia coincide with equilibria with inertia, weakly efficient allocations are efficient and any efficient allocation is an equilibrium with transfer.*

3.4 Existence results

Proposition 1 *The following assertions are equivalent:*

1. *there exists a no-arbitrage price for the economy ($\bigcap_i \mathrm{int}\,(R^i)^0 \neq \emptyset$),*
2. *NUBA: $\sum_i W^i = 0$ and $W^i \in R^i$ for all i implies $W^i = 0$ for all i,*
3. *the set of individually rational attainable allocations is compact.*

Proof: As the set of useful vectors of $\succeq^i$ and V_{E^i} coincide and the set of individually rational attainable allocations for the economy with utilities $(V_{E^i})_i$ coincides with the set of individually rational attainable allocations for the economy with preferences $(\succeq^i)_{i \in I}$, the equivalence between 1, 2 and 3 follows from standard results on arbitrage with complete preferences. ■

Theorem 2 *Let any assertion of Proposition 1 hold true.Then*

1. *there exists an individually rational weakly efficient allocation,*
2. *there exists a weak equilibrium with inertia.*

Proof: It is also standard that any assertion of Proposition 1 implies the existence of an individually rational efficient allocation $(\bar{X}^i)_{i=1}^m$ or of an equilibrium

(X^*, p^*) for the economy with utilities $(V_{E^i})_i$. By the same proofs as in theorem 1, $(\bar{X}^i)_{i=1}^m$ is an individually rational *weakly efficient* allocation for the economy with preferences $(\succeq^i)_i$ and (X^*, p^*) is a weak equilibrium with inertia. ■

Theorem 3 *1. If u has no half-line for every $u \in \mathcal{U}^i$ and i, then the assertions of Proposition 1 and Theorem 2 are equivalent and any weak equilibrium price is a no-arbitrage price.*

2. If u is strictly concave for every $u \in \mathcal{U}^i$ and i, then

(a) the assertions of Proposition 1 and Theorem 2 are equivalent to the existence of an equilibrium

(b) If furthermore, agents have a common increasing direction, then the assertions of Proposition 1 and of Theorem 2 are equivalent to the existence of an individually rational efficient allocation.

Proof: Let us first remark that if u has no half-line for every $u \in \mathcal{U}^i$ and i, any weak equilibrium price is a no-arbitrage price. Indeed let (X^*, p^*) be a weak equilibrium with inertia. Then for any useful vector $W^i \neq 0$, $u(X^{*i} + tW^i) > u(X^{*i})$, $\forall\ u \in \mathcal{U}^i$, for any $t > 0$. Hence $p^* \cdot W^i > 0$ which proves that p^* is a no-arbitrage price for the economy. This shows that assertion 2 of Theorem 2 implies assertion 1 of Proposition 1.

As mentioned in remark 4, it is easy to verify that a weak equilibrium with inertia is weakly efficient. Hence assertion 2 of Theorem 2 implies assertion 1 of 2. To complete the proof that the assertions of Proposition 1 and Theorem 2 are equivalent, let us show that assertion 1 of Theorem 2 implies assertion 2 of Proposition 1. Let $(\bar{X}^i)_{i=1}^m$ be an individually rational weakly efficient allocation and suppose that there exists a feasible trade $W^1, \ldots, W^m$ with $W^i \in R^i$ for all i and $W^i \neq 0$ for some i. We have $u(\bar{X}^i + tW^i) \geq u(\bar{X}^i)$, $\forall\ u \in \mathcal{U}^i$, for all i, and $u(\bar{X}^i + tW^i) > u(\bar{X}^i)$, $\forall u \in \mathcal{U}^i$, for any i such that $W^i \neq 0$. The allocation $(\bar{X}^i + tW^i)_{i\in I}$ being feasible, this contradicts the weak efficiency of $(\bar{X}^i)_{i=1}^m$.

Finally, if u is strictly concave, then u has no half-line, hence the assertions of Proposition 1 and Theorem 2 are equivalent. From lemma 4, any weak equilibrium with inertia is an equilibrium with inertia. If furthermore, agents have a common increasing direction, from lemma 4, any weakly efficient allocation is efficient . ■

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