Linear Transforms, Values and Least Square Approximation for Cooperation Systems

Ulrich Faigle, Michel Grabisch

UNIVERSITÉ PARIS 1 – PANTHÉON - SORBONNE –

Documents de Travail du
Centre d'Economie de la Sorbonne

CES Working Papers

Linear Transforms, Values and Least Square Approximation for Cooperation Systems

Ulrich FAIGLE, Michel GRABISCH

2014.10

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://centredeconomiesorbonne.univ-paris1.fr/
ISSN : 1955-611X

Linear Transforms, Values and Least Square Approximation for Cooperation Systems

Ulrich FAIGLE* Michel GRABISCH†

January 9, 2014

Abstract

We study linear properties of TU-games, revisiting well-known issues like interaction transforms, the inverse Shapley value problem and the concept of semivalues and least square values. We embed TU-games into the model of cooperation systems and influence patterns, which allows us to introduce linear operators on games in a natural way. We focus on transforms, which are linear invertible maps, relate them to bases and investigate many examples (Möbius transform, interaction transform, Walsh transform, *etc.*). In particular, we present a simple solution to the inverse problem in its general form: Given a linear value Φ and a game v, find all games v' such that $\Phi(v) = \Phi(v')$. Generalizing Hart and Mas-Colell's concept of a potential, we introduce general potentials and show that every linear value is induced by an appropriate potential. We furthermore develop a general theory of allocations with a quadratic optimality criterion under linear constraints, obtaining results of Charnes *et al.*, and Ruiz *et al.* and others as special cases. We prove that this class of allocations coincides exactly with the class of all linear values.

Keywords: Cooperation system, cooperative game, basis, transform, inverse problem, potential, linear value, semivalue
JEL Classification: C71

*Mathematisches Institut, Universität zu Köln, Weyertal 80, 50931 Köln, Germany. Email: faigle@zpr.uni-koeln.de

†Corresponding author. Paris School of Economics, University of Paris I, 106-112, Bd. de l'Hôpital, 75013 Paris, France. Tel. (33) 144-07-82-85, Fax (33)-144-07-83-01. Email: michel.grabisch@univ-paris1.fr

1 Introduction

It is well known that finite TU-games with n players, and more generally set functions on a set N of $n = |N|$ elements, form a 2^n-dimensional vector space with particular attention usually being paid to two special bases: the set of unanimity games ζ_S, $S \in 2^N$, defined by $\zeta_S(T) = 1$ if and only if $T \supseteq S$, and 0 otherwise, and the bases of identity games δ_S, $S \in 2^N$, defined by $\delta_S(T) = 1$ if and only if $T = S$, and 0 otherwise. The coefficients of a game in the basis of unanimity games is known as the *Harsanyi dividends* [16], or the *Möbius inverse* [22], also called *Möbius transform* [10], of the game.

But also other transforms (namely, invertible linear operators) have been proposed and studied in the literature (*e.g.*, the interaction transform [9], the Walsh transform [28], which is also known as Fourier transform, *etc.*). So far, these transforms, although recognized to be important in discrete mathematics, are not very well known in the game theory community. In fact, it seems that the linear properties of TU-games–while often used indirectly–have not yet been fully exploited in game theoretic research in their own right. To the best of our knowledge, the obvious correspondence between bases and linear transforms has never been addressed explicitly, for example. As a consequence, the famous "inverse problem" for TU-games, which asks for a description all games v' having the same Shapley value as a given game v, has been solved in a somewhat tedious way (see Kleinberg and Weiss [18] or Dragan [4]). We solve the inverse problem for general linear values in Section 3.2.

In a similar vein, recall that allocations for cooperative games have been studied with respect to certain extremal properties, which has led to the notions of so-called *least square values* and *semivalues*, exhibiting, *e.g.*, the Shapley and the Banzhaf value as particular instances (see, *e.g.*, Charnes *et al.* [1], Ruiz *et al.* [23], Dragan [6]). We develop a general theory for allocations with quadratic optimality properties under linear constraints in Section 5 and Section 6 that enables us to show that this class of allocations coincides exactly with the class of all linear values.

In our present study of linear properties of game theoretic concepts it is convenient (and natural) to embed TU-games into the context of *cooperation systems* and *influence patterns* of coalitions. We introduce this model in Section 2. Linear transforms and their interplay with bases are investigated in Section 3 and illustrated with fundamental examples and applications.

Hart and Mas-Collel [17] have given an interpretation of the Shapley value as a *potential value* which provides an interesting link to the (non-cooperative!)

potential games of Monderer and Shapley [20]. It turns out that a comprehensive framework for linear potentials as generalizations of transforms exists in the context of cooperation systems. We show in Section 4 how this general framework allows us to exhibit every linear value as a potential value, which suggests a much closer tie between cooperative and non-cooperative game theory than has until now been realized.

2 Cooperation systems

Let N by a finite set of $n = |N|$ players and denote by $\mathcal{N} = 2^N$ the collection of all subsets (or *coalitions*) $S \subseteq N$. A *cooperation system* is a pair (N, F), where $F : \mathcal{N} \times \mathcal{N} \to \mathbb{R}$ is a map that reflects for every pair (S, T) of coalitions, the amount $F(S, T)$ of *influence* S exerts on T. We refer to F as the *influence pattern* of (N, F). A *valuation* on (N, F) is a map $v : \mathcal{N} \to \mathbb{R}$ that assigns to every coalition S a value $v(S)$. In the terminology of classical cooperative game theory, a valuation v with $v(\emptyset) = 0$ is the *characteristic function* of a *cooperative TU game* (N, v).

The valuations v form the 2^n-dimensional vector space $\mathbb{R}^{\mathcal{N}}$, while the possible influence patterns F define the vector space $\mathbb{R}^{\mathcal{N}\times\mathcal{N}}$. Mathematically, one may think of a valuation $v \in \mathbb{R}^{\mathcal{N}}$ as a parameter vector, indexed by the $S \in \mathcal{N}$. $F \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ corresponds to a matrix $[f_{ST}]$ with rows and columns indexed by the members of $\mathcal{N}$ and coefficients $f_{ST} = F(S, T)$.

2.1 Examples

Example 2.1 (Counting pattern) *Let $C = [c_{ST}] \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ be the pattern defined by*

$$c_{ST} = |S \cap T|.$$

Here the individual players act independently. The influence of a coalition S on another coalition T depends only on the number of players in S that are also members of T.

Example 2.2 (Parity pattern) *Let $\Pi \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ be the pattern with coefficients*

$$\pi_{ST} = (-1)^{|S\cap T|}.$$

Π is called the parity pattern. Two coalitions S and T exert a positive ("+1") or negative ("−1") influence on each other depending on whether they have an even

or an odd number of players in common. Π *is closely related to the Hadamard and Walsh transform of* $\mathbb{R}^{\mathcal{N}}$ *(see Section 3.1 below).*

Example 2.3 (Containment pattern) *Let* $Z \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ *be the pattern with the coefficients*

$$\zeta_{ST} = \begin{cases} 1, & \text{if } S \subseteq T \\ 0, & \text{otherwise.} \end{cases}$$

Z *is the containment pattern, where a coalition* S *is thought to be able to exert an influence on any coalition* T *containing it. For a given* $S \in \mathcal{N} \setminus \emptyset$*, the valuation* v *defined by* $v(T) = \zeta_{ST}$*,* $T \in \mathcal{N}$*, is known as a unanimity game.*

Example 2.4 (Influence in voting) *In [13], influence is studied in terms of voting dynamics as follows. Let* S *be the set of 'yes'-voters at time* t *and let* p_{ST} *be the probability for* T *to be the set of 'yes'-voters at time* $t+1$*. The transition* $S \to T$ *is the result of a round of discussion among voters. So* p_{ST} *is a measure for the influence the constellation* S *exerts on the constellation* T*. Here, the pattern matrix* $P = [p_{ST}]$ *is row-stochastic and defines a Markov process on* $\mathcal{N}$*.*

2.2 Influence spaces, bases and additive games

Associate with the influence pattern $F = [f_{ST}]$ the 2^n *influence functions* f_S with values

$$f_S(T) = f_{ST} = F(S,T)$$

and define the *influence space* $\mathcal{F}$ of F as the collection of all linear combinations of influence functions:

$$\mathcal{F} = \{v \in \mathbb{R}^{\mathcal{N}} \mid v = \sum_{S\in\mathcal{N}} \lambda_S f_S, \lambda_S \in \mathbb{R}\}.$$

$\mathcal{F}$ is a subspace of $\mathbb{R}^{\mathcal{N}}$ and corresponds to the row space of the matrix F.

Notation 1 *In matrix notation, we think of a parameter vector* v *as a row vector (and thus of its transpose* v^T *as a column vector). For example, we write*

$$\mathcal{F} = \{vF \mid v \in \mathbb{R}^{\mathcal{N}}\} = \mathbb{R}^{\mathcal{N}} F.$$

Lemma 2.1 (Basis lemma) *Equality* $\mathcal{F} = \mathbb{R}^{\mathcal{N}}$ *holds if and only if the* 2^n *influence functions* f_S *are linearly independent and hence form a basis of* $\mathbb{R}^{\mathcal{N}}$*.*

$\diamond$

It is easy to see (see Ex. 2.5 below) that the 2^n influence functions ζ_S of the containment pattern Z are linearly independent and hence form a basis of $\mathbb{R}^{\mathcal{N}}$, known as the *basis of unanimity games*. The inverse pattern is given by the *Möbius matrix* $M = Z^{-1}$ with the coefficients (Rota [22])

$$\mu_{ST} = \begin{cases} (-1)^{|T \setminus S|} & \text{if } S \subseteq T \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

On the other hand, the influence functions c_S of the counting pattern C are *not* linearly independent. Indeed, setting $\zeta_i = \zeta_{\{i\}}$, we have

$$\zeta_i = c_{\{i\}} \quad \text{and} \quad c_S = \sum_{i \in S} \zeta_i \quad \text{for all } i \in N \text{ and } S \in \mathcal{N} \setminus \emptyset. \tag{2}$$

It follows that the n influence functions ζ_i form a basis of the influence space $\mathcal{C}$ of the counting pattern C. In other words, $\mathcal{C}$ is the vector space of all characteristic functions v of the form

$$v = \sum_{i \in N} v(\{i\})\zeta_i \quad \text{or} \quad v(S) = \sum_{i \in S} v(\{i\}) \;\; \forall S \subseteq N.$$

So $\mathcal{C}$ is the n-dimensional space of all *additive* cooperative games on N.

Example 2.5 *Let $F = [f_{ST}]$ be such that for all $S, T \in \mathcal{N}$, one has $f_{SS} \neq 0$ and $f_{ST} = 0$ unless $S \subseteq T$. Labeling the rows and columns of F so that S precedes T whenever $S \subset T$ exhibits F as (upper) triangular with non-zero diagonal. So the 2^n influence functions f_S are seen to be linearly independent (see Grabisch et al. [11] and Denneberg and Grabisch [2] for a detailed treatment of this type of matrix in the context of interaction).*

2.3 Linear values

Recall in game theoretic terminology, that a *linear value* is a linear map $\Phi : \mathbb{R}^{\mathcal{N}} \to \mathbb{R}^N$ that evaluates the strength, power or reward of player $i \in N$ relative to the valuation v as $\Phi_i(v)$. A linear value Φ can be considered as a mapping Φ from $\mathbb{R}^{\mathcal{N}}$ to $\mathcal{C}$, the influence space of the counting pattern, assigning to any valuation v an additive game Φ^v defined by

$$\Phi^v(S) = \sum_{i \in S} \Phi_i(v) \quad \text{for any } S \in \mathcal{N} \setminus \emptyset \tag{3}$$

with the convention $\Phi^v(\emptyset) = 0$. A classical example is the Shapley's [26] value Φ^{Sh}, defined by

$$\Phi_i^{\text{Sh}}(v) = \sum_{S \subseteq N} \frac{(n - |S|)!(|S| - 1)!}{n!} \big(v(S) - v(S \setminus i)\big) \qquad (i \in N). \tag{4}$$

3 Bases and linear transforms

Consider a cooperation system (N, F) with a full-dimensional influence space $\mathcal{F} = \mathbb{R}^{\mathcal{N}}$. Hence, by Lemma 2.1, F is invertible, and the rows f_S, $S \in \mathcal{N}$, of F form a basis of $\mathbb{R}^{\mathcal{N}}$, yielding for any $v \in \mathbb{R}^{\mathcal{N}}$ a unique representation

$$v = \sum_{S \in \mathcal{N}} w_S f_S,$$

or in matrix notation $v = wF$ with w the row-vector $[w_S]_{S \in \mathcal{N}}$, and consequently $w = vF^{-1}$. Following some tradition, one may view the mapping $v \mapsto w$ as a *transform*, namely a linear and invertible operator on $\mathbb{R}^{\mathcal{N}}$ with F^{-1} as its standard matrix representation (see below for such examples). Letting Ψ be any such transform $v \mapsto \Psi^v$, the above observations yield a one-to-one correspondence between bases and transforms:

Lemma 3.1 (Equivalence between bases and transforms) *For every basis F of influence functions f_S, there is a (unique) transform Ψ such that for any $v \in \mathbb{R}^{\mathcal{N}}$,*

$$v = \sum_{S \in \mathcal{N}} \Psi^v(S) f_S, \tag{5}$$

whose inverse Ψ^{-1} is given by $v \mapsto (\Psi^{-1})^v = \sum_{T \in \mathcal{N}} v(T) f_T = vF$.

Conversely, to any transform Ψ corresponds a unique basis F such that (5) holds, given by $f_S = (\Psi^{-1})^{\delta_S}$, where δ_S is the identity game with $\delta_S(T) = 1$ if $S = T$ and 0 otherwise. $\diamond$

For further reference, note that the vector space $\mathbb{R}^{\mathcal{N}}$ is euclidean with the scalar product

$$\langle v, w \rangle = \sum_{S \in \mathcal{N}} v(S) w(S). \tag{6}$$

3.1 Examples

We give a number of examples for illustration.

The Möbius transform. This is the transform m associated with the basis of unanimity games (*i.e.*, the containment pattern Z):

$$v(T) = \sum_{S \in \mathcal{N}} m^v(S)\zeta_S(T) = \sum_{S \subseteq T} m^v(S), \quad \text{for all } T \subseteq N. \tag{7}$$

The coefficients $m^v(S)$ are also known as the *Harsanyi dividends* of v [16]. Using the coefficients of the inverse matrix in (1), one obtains an explicit formula for the Harsanyi dividends:

$$m^v(S) = \sum_{T \in \mathcal{N}} v(T)\mu_{TS} = \sum_{T \subseteq S} (-1)^{|S \setminus T|} v(T). \tag{8}$$

The commonality transform. The *commonality (*or *co-Möbius) function* [25, 11]) is the operator $v \mapsto \check{m}^v$ on $\mathbb{R}^{\mathcal{N}}$, where

$$\check{m}^v(S) = \sum_{T \supseteq N \setminus S} (-1)^{n-|T|} v(T) = \sum_{T \subseteq S} (-1)^{|T|} v(N \setminus T) \quad \text{for all } S \in \mathcal{N}. \tag{9}$$

Setting $\tilde{v}(T) = (-1)^{|T|} v(N \setminus T)$, we see that $\tilde{v}$ is the Möbius transform of $\check{m}^v$:

$$\check{m}^v(S) = \sum_{T \subseteq S} \tilde{v}(T) \quad \text{holds for all } S \subseteq N.$$

So the inversion relation (8) allows us to conclude

$$\tilde{v}(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} \check{m}^v(T)$$

and hence

$$v(S) = (-1)^{n-|S|} \tilde{v}(N \setminus S) = \sum_{T \subseteq N \setminus S} (-1)^{|T|} \check{m}^v(T). \tag{10}$$

Put the other way round, we have

$$(\check{m}^{-1})^v(S) = \sum_{T \subseteq N \setminus S} (-1)^{|T|} v(T).$$

Applying Lemma 3.1, we immediately find the associated basic valuations f_T:

$$f_T(S) = \sum_{B \subseteq N \setminus S} (-1)^{|B|} \delta_T(B) = \begin{cases} (-1)^{|T|} & \text{if } S \cap T = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

The (Shapley) interaction transform [9]. The transform $v \mapsto I^v$ defined by

$$
\begin{aligned}
I^v(S) &= \sum_{T \subseteq N \setminus S} \frac{(n-t-s)!t!}{(n-s+1)!} \sum_{L \subseteq S} (-1)^{|S \setminus L|} v(T \cup L) \\
&= \sum_{K \subseteq N} \frac{|N \setminus (S \cup K)|!|K \setminus S|!}{(n-s+1)!} (-1)^{|S \setminus K|} v(K),
\end{aligned}
\tag{11}
$$

where $s = |S|$ and $t = |T|$, has the inverse relation

$$
v(S) = \sum_{K \subseteq N} \beta^{|K|}_{|S \cap K|} I^v(K), \tag{12}
$$

where

$$
\beta^l_k = \sum_{j=0}^{k} \binom{k}{j} B_{l-j} \qquad (k \leq l),
$$

and $B_0, B_1, \ldots$ are the Bernoulli numbers. The first values of β^l_k are given in Table 1. Using Lemma 3.1, we find that the corresponding basis consists of the 2^n

$k \setminus l$	0	1	2	3	4
0	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$
1		$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{30}$
2			$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{2}{15}$
3				0	$-\frac{1}{30}$
4					$-\frac{1}{30}$

Table 1: The coefficients β^l_k

valuations b^I_T with values

$$
b^I_T(S) = \beta^{|T|}_{|T \cap S|} \quad \text{for all } S \in \mathcal{N}. \tag{13}
$$

Note that $I^v(S)$ becomes the Shapley value of $i \in N$ if $S = \{i\}$ is a singleton:

$$
I^v(\{i\}) = \sum_{T \subseteq N \setminus i} \frac{(n-t-1)!t!}{n!} \big(v(T \cup i) - v(T)\big) = \Phi^{\mathrm{Sh}}_i(v). \tag{14}
$$

In this sense, the *Shapley interaction* transform $v \mapsto I^v$ *extends* the Shapley value.

The Banzhaf interaction transform [11]. The transform $v \mapsto I_B^v$ extends the Banzhaf value and is defined by

$$I_{\mathrm{B}}^v(S) = \left(\frac{1}{2}\right)^{n-s} \sum_{K \subseteq N} (-1)^{|S \setminus K|} v(K) \tag{15}$$

with inverse

$$(I_{\mathrm{B}}^{-1})^v(S) = \sum_{K \subseteq N} \left(\frac{1}{2}\right)^k (-1)^{|K \setminus S|} v(K). \tag{16}$$

The basis corresponding to the Banzhaf interaction transform consists of the 2^n valuations $b_T^{I_B}$ with values

$$b_T^{I_B}(S) = \sum_{K \subseteq N} \left(\frac{1}{2}\right)^k (-1)^{|K \setminus S|} \delta_T(K) = \left(\frac{1}{2}\right)^{|T|} (-1)^{|T \setminus S|}.$$

REMARK. There is a general theory of *interaction values* of cooperative games that includes in particular Owen's [21] local interaction values and their extension to arbitrary subsets by Grabisch and Roubens [12]. As it turns out, these values are linear and hence fall into the scope of the present model. We do not go into further details here but refer the reader to Faigle and Voss [8].

The Hadamard transform. The transform $v \mapsto H^v$ with

$$H^v(S) = \frac{1}{2^{n/2}} \sum_{K \subseteq N} (-1)^{|S \cap K|} v(K) \tag{17}$$

and is self-inverse. The corresponding basis is formed by the 2^n valuations b_T^H with

$$b_T^H(S) = \frac{1}{2^{n/2}} \sum_{K \subseteq N} (-1)^{|S \cap K|} \delta_T(K) = \frac{1}{2^{n/2}} (-1)^{|S \cap T|}.$$

Up to the factor $\frac{1}{2^{n/2}}$, we recognize the parity pattern. Hence the parity pattern forms a basis of $\mathbb{R}^{\mathcal{N}}$.

The Walsh basis. An important example of an orthogonal basis for $\mathbb{R}^{\mathcal{N}}$ arises from coding the coalitions $S \subseteq N$ by vectors $\chi^S \in \{-1, 1\}^N$ with component value $\chi_i^S = -1$ if and only if $i \notin S$. Specifically, consider the function

$$\pi_S(x) = \prod_{i \in S} x_i \qquad (x \in \mathbb{R}^N)$$

for every $S \subseteq N$, with the convention $\pi_\emptyset(x) = 1$ for all $x \in \mathbb{R}^N$. The choice $x = \chi^T$ yields

$$\pi_S(\chi^T) = (-1)^{|S \setminus T|} =: w_S(T) \qquad (T \in \mathcal{N}).$$

We prove that the associated *Walsh functions* w_S form an orthogonal basis.

Theorem 3.1 *For all $S, T \subseteq N$, one has*

$$\langle w_S, w_T \rangle = \begin{cases} 2^n, & \textit{if } S = T \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. For any coalitions $S, T \in \mathcal{N}$, consider their symmetric difference $D = S \Delta T = (S \setminus T) \cup (T \setminus S)$. Then we find

$$\langle w_S, w_T \rangle = \langle \pi_S, \pi_T \rangle = \sum_{x \in \{-1,1\}^N} x^S x^T = \sum_{x \in \{-1,1\}^N} x^{S \Delta T} = \sum_{x \in \{-1,1\}^N} x^D.$$

If $S = T$, $\langle w_S, w_T \rangle = 2^n$ is obvious. If $S \neq T$ (and hence $D \neq \emptyset$), we have:

$$\sum_{x \in \{-1,1\}^N} x^D = 2^{|N \setminus D|} \sum_{x \in \{-1,1\}^D} x^D = 2^{|N \setminus D|}(1-1)^{|D|} = 0.$$

◇

Corollary 3.1 *The Walsh functions w_S forms a basis of $\mathbb{R}^{\mathcal{N}}$.*

Proof. Because the Walsh functions are pairwise orthogonal, they form an independent subset of $\mathbb{R}^{\mathcal{N}}$. Since there are $2^n = \dim \mathbb{R}^{\mathcal{N}}$ Walsh functions, this independent set must be a basis.

◇

Let us find the corresponding transform $v \mapsto W^v$. By Lemma 3.1, the inverse transform is immediate:

$$(W^{-1})^v(S) = \sum_{T \subseteq N} v(T)(-1)^{|T \setminus S|}.$$

The direct transform can be discerned by solving the linear system

$$v(S) = \sum_{T \subseteq N} W^v(T)(-1)^{|T \setminus S|} \quad (S \in \mathcal{N}),$$

or by simply noticing that $w_T(S) = 2^{|T|} b_T^{I_B}(S)$, which yields

$$v(S) = \sum_{T \subseteq N} I_B^v(T) b_T^{I_B}(S) = \sum_{T \subseteq N} W^v(T) w_T(S)$$

and hence the components of W^v as

$$W^v(T) = \left(\frac{1}{2}\right)^{|T|} I_B^v(T) \quad \text{for all } T \in \mathcal{N}.$$

Relation between the Hadamard basis and Walsh basis. These two bases are related as follows:

$$b_T^H(S) = b_S^H(T) = \frac{1}{2^{n/2}}(-1)^{|S \cap T|} = \frac{1}{2^{n/2}}(-1)^{|S \setminus (N \setminus T)|} = \frac{1}{2^{n/2}} w_S(N \setminus T).$$

Hence the two bases are equal up to a multiplicative constant and a reordering of the coordinates of the vectors. It follows that also the Hadamard basis is orthogonal (and, therefore, the parity pattern yields an orthogonal basis), and in addition orthonormal.

3.2 The inverse problem

In game theory, the following "inverse problem" is well known: for a given linear value Φ and game v, find all games v' such that

$$\Phi(v) = \Phi(v') \quad \text{or, equivalently,} \quad \Phi(v - v') = 0.$$

This problem was solved[1] by Kleinberg and Weiss [18] for the Shapley value by exhibiting a basis for the associated *null space* or *kernel*:

$$\ker(\Phi) = \{v \in \mathbb{R}^{\mathcal{N}} \mid \Phi_i(v) = 0 \; \forall i \in N\}.$$

Our linear analysis provides an immediate particular solution for the Shapley inverse problem and similar ones (*e.g.*, Banzhaf) if a transform that extends the value in question is available.

[1] See also Yokote et al. [29] for a recent work on this topic, and Dragan [6], who solved this problem for the Shapley value [4] and later for all semivalues [5] in a simpler way than Kleinberg and Weiss.

Let us illustrate the method with the Shapley value Φ^{Sh}. Since the Shapley interaction transform $v \mapsto I^v$ extends Φ^{Sh}, the relations (12) and (14) yield for any valuation $v \in \mathbb{R}^{\mathcal{N}}$,

$$v = \sum_{S \in \mathcal{N}} I^v(S) b_S^I = \sum_{i \in N} \Phi_i^{\text{Sh}}(v) b_{\{i\}}^I + \sum_{|S| \neq 1} I^v(S) b_S^I,$$

which implies

$$v \in \ker(\Phi^{\text{Sh}}) \quad \Longleftrightarrow \quad v = \sum_{|S| \neq 1} I^v(S) b_S^I$$

i.e.,

$$\ker(\Phi^{\text{Sh}}) = \Big\{ \sum_{|S| \neq 1} \lambda_S b_S^I \mid \lambda_S \in \mathbb{R} \Big\}. \tag{18}$$

Example 3.1 *The representation (18) for valuations $v \in \ker(\Phi^{\text{Sh}})$ yields*

$$v(S) = \sum_{T \subseteq N, |T| \neq 1} \lambda_T \beta_{|S \cap T|}^{|T|} \quad \textit{for all } S \subseteq N.$$

In the case $N = \{1, 2, 3\}$, one thus obtains

$$\begin{aligned}
v(\emptyset) &= \lambda_\emptyset + \frac{1}{6}(\lambda_{12} + \lambda_{13} + \lambda_{23}) \\
v(1) &= \lambda_\emptyset - \frac{1}{3}\lambda_{12} - \frac{1}{3}\lambda_{13} + \frac{1}{6}\lambda_{23} + \frac{1}{6}\lambda_{123} \\
v(2) &= \lambda_\emptyset - \frac{1}{3}\lambda_{12} + \frac{1}{6}\lambda_{13} - \frac{1}{3}\lambda_{23} + \frac{1}{6}\lambda_{123} \\
v(3) &= \lambda_\emptyset + \frac{1}{6}\lambda_{12} - \frac{1}{3}\lambda_{13} - \frac{1}{3}\lambda_{23} + \frac{1}{6}\lambda_{123} \\
v(12) &= \lambda_\emptyset + \frac{1}{6}\lambda_{12} - \frac{1}{3}\lambda_{13} - \frac{1}{3}\lambda_{23} - \frac{1}{6}\lambda_{123} \\
v(13) &= \lambda_\emptyset - \frac{1}{3}\lambda_{12} + \frac{1}{6}\lambda_{13} - \frac{1}{3}\lambda_{23} - \frac{1}{6}\lambda_{123} \\
v(23) &= \lambda_\emptyset - \frac{1}{3}\lambda_{12} - \frac{1}{3}\lambda_{13} + \frac{1}{6}\lambda_{23} - \frac{1}{6}\lambda_{123} \\
v(123) &= \lambda_\emptyset + \frac{1}{6}(\lambda_{12} + \lambda_{13} + \lambda_{23}).
\end{aligned}$$

The preceding idea can be refined to yield a method that allows us to construct a basis for the null space of an arbitrary linear value $\Phi : \mathbb{R}^{\mathcal{N}} \to \mathbb{R}^N$.

Let $k = \dim \Phi(\mathbb{R}^{\mathcal{N}}) \leq n$ be the dimension of Φ and recall the well-known dimension formula for linear maps:

$$\dim \ker \Phi = \dim \mathbb{R}^{\mathcal{N}} - \dim \Phi(\mathbb{R}^{\mathcal{N}}) = 2^n - k. \tag{19}$$

In the case $k = 0$, one has $\ker \Phi = \mathbb{R}^{\mathcal{N}}$. So any basis of $\mathbb{R}^{\mathcal{N}}$ solves the inverse problem for Φ. Let us therefore investigate the non-trivial situation $k \geq 1$.

Select a basis $E = \{e_1, \ldots, e_k\}$ for the range $\Phi(\mathbb{R}^{\mathcal{N}})$ of Φ as well as k arbitrary valuations $b_1, \ldots, b_k \in \mathbb{R}^{\mathcal{N}}$ such that

$$\Phi(b_i) = e_i \quad (i = 1, \ldots, k).$$

Lemma 3.2 *The set* $\{b_1, \ldots, b_k\}$ *of valuations is linearly independent.*

Proof. Suppose the statement is false and there are non-trivial scalars λ_i such that

$$\sum_{i=1}^{k} \lambda_i b_i = 0 \in \mathbb{R}^{\mathcal{N}} \quad \text{and hence} \quad \Phi(\sum_{i=1}^{k} \lambda_i b_i) = 0 \in \mathbb{R}^{N}.$$

The linearity of Φ then implies $0 = \sum_{i=1}^{k} \lambda_i \Phi(b_i) = \sum_{i=1}^{k} \lambda_i e_i$, which contradicts the independence of the set $E \subseteq \mathbb{R}^N$, however.

$\diamond$

By Lemma 3.2 and basic facts from linear algebra, $\{b_1, \ldots, b_k\}$ may be completed to a basis

$$B = \{b_1, \ldots, b_k, b_{k+1}, \ldots, b_{2^n}\}$$

for the domain $\mathbb{R}^{\mathcal{N}}$. Moreover, the basis E for the range $\Phi(\mathbb{R}^{\mathcal{N}})$ guarantees, for each $b_j \in B$, unique scalars $\epsilon_1^{(j)}, \ldots, \epsilon_k^{(j)}$ such that

$$\Phi(b_j) = \sum_{i=1}^{k} \epsilon_i^{(j)} e_i = \sum_{i=1}^{k} \epsilon^{(j)} \Phi(b_i).$$

Because Φ is linear, the valuations $b_j^{\Phi} = b_j - \sum_{i=1}^{k} \epsilon^{(j)} b_i$ are in $\ker \Phi$:

$$\Phi(b_j^{\Phi}) = \Phi(b_j) - \Phi(\sum_{i=1}^{k} \epsilon^{(j)} b_i) = 0.$$

We have thus arrived at a solution of the inverse problem.

Theorem 3.2 *Let* $B^\Phi = \{b_1, \ldots, b_k, b^\Phi_{k+1}, \ldots, b^\Phi_{2^n}\}$. *Then*

(i) B^Φ *is a basis for* $\mathbb{R}^\mathcal{N}$.

(ii) $B^\Phi_0 = \{b^\Phi_{k+1}, \ldots, b^\Phi_{2^n}\}$ *is a basis for* $\ker \Phi$.

Proof. Every b_j is a linear combination of vectors in B^Φ:

$$b_j = b^\Phi_j + \sum_{i=1}^{k} \epsilon_i^{(j)} b_i.$$

Because B generates $\mathbb{R}^\mathcal{N}$, also B^ϕ generates $\mathbb{R}^\mathcal{N}$. Because of $|B^\Phi| = 2^n$, B^ϕ is linearly independent and, therefore a basis, which proves (i).

We have seen that $B^\Phi_0 \subseteq \ker \Phi$ holds. Since $B^\Phi_0 \subseteq B^\Phi$ is linearly independent and $|B^\Phi_0| = 2^n - k = \dim \ker \Phi$, B^Φ_0 must be a basis of $\ker \Phi$.

◇

Application: The inverse Shapley value problem revisited. Our refined construction includes Dragan's [4] solution of the inverse problem for the Shapley value Φ^{Sh} as a straightforward special case. To see this, consider the basis $B = \{\zeta_S \mid S \in \mathcal{N}\}$ of $\mathbb{R}^\mathcal{N}$. For each $i \in N$, we have

$$\Phi^{\mathrm{Sh}}(\zeta_i) = e_i = \text{ ith unit vector in } \mathbb{R}^N,$$

which implies $\dim \ker \Phi^{\mathrm{Sh}} = 2^n - n$. For each coalition $S \neq \emptyset$, we have

$$\Phi^{\mathrm{Sh}}(\zeta_S) = \frac{1}{|S|}\sum_{i\in S} e_i = \Phi^{\mathrm{Sh}}(\frac{1}{|S|}\sum_{i\in S}\zeta_i) \quad \text{and thus} \quad \zeta_S^{\Phi^{\mathrm{Sh}}} = \zeta_S - \frac{1}{|S|}\sum_{i\in S}\zeta_i,$$

which yields the following set $B_0^{\Phi^{\mathrm{Sh}}}$ as a basis for the null space $\ker \Phi^{\mathrm{Sh}}$:

$$B_0^{\Psi^{\mathrm{Sh}}} = \{\zeta_\emptyset\} \cup \{\zeta_S^{\Phi^{\mathrm{Sh}}} \mid S \in \mathcal{N}, |S| \geq 2\}.$$

4 Potential functions and values

Given the cooperation system (N, F) with influence functions f_S, we associate with every $v \in \mathbb{R}^\mathcal{N}$ its F*-potential*

$$v^F = \sum_{S\in\mathcal{N}} v(S) f_S \quad (= vF \text{ in matrix notation}). \tag{20}$$

So the influence space $\mathcal{F}$ of F contains precisely the F-potentials:

$$\mathcal{F} = \{\sum_{S\in\mathcal{F}} \lambda_S f_S \mid \lambda_S \in \mathbb{R}\} = \{v^F \mid v \in \mathbb{R}^{\mathcal{N}}\}.$$

Note that $v \mapsto v^F$ is a linear operator on $\mathbb{R}^{\mathcal{N}}$ and is invertible (*i.e.*, every $v \in \mathbb{R}^{\mathcal{N}}$ is uniquely determined by its F-potential) if and only if the influence functions form a basis of $\mathbb{R}^{\mathcal{N}}$. In such a case, the potential corresponds to an inverse transform (and hence to a transform in its own right) in the sense of Section 3.

Potentials are closely related to linear values (see Section 2.3). Indeed, a pattern F gives rise to the (linear) *potential value* ∂^F, where

$$\partial_i^F(v) = v^F(N) - v^F(N \setminus i) \quad (i \in N).$$

Theorem 4.1 *Every linear value* Φ *arises as the potential value* ∂^F *relative to a suitable influence pattern* F.

Proof. Since $v \mapsto \Phi^v$ is a linear map, there is an influence pattern F such that $\Phi^v = vF = v^F$ holds for all $v \in \mathbb{R}^{\mathcal{N}}$. Hence we find for all $i \in N$,

$$\Phi_i(v) = \Phi^v(N) - \Phi^v(N \setminus i) = v^F(N) - v^F(N \setminus i) = \partial_i^F.$$

◇

We illustrate this and continue the analysis with the Shapley value.

4.1 The Shapley value

We rewrite (4) in a more general form, where the game is restricted to the coalitions T contained in $S \subseteq N$:

$$\Phi_i^{\mathrm{Sh}}(v, S) = \sum_{T\subseteq S} \frac{(s-t)!(t-1)!}{s!}(v(T) - v(T \setminus i)).$$

It is well-known and easy to see that for the influence functions ζ_U one has

$$\Phi_i^{\mathrm{Sh}}(\zeta_U, S) = \begin{cases} 1/u & \text{if } i \in U \subseteq S \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

In the case $i \notin U$, the second sum term in the expression for $\Phi^{\mathrm{Sh}}(\zeta_U, S)$ vanishes. So we find for any coalition $U \neq \emptyset$,

$$\sum_{T\subseteq S} \frac{(s-1)!(s-t)!}{t!}\zeta_U(T) = \begin{cases} 0 & \text{if } U \not\subseteq S \\ 1/u & \text{if } U \subseteq S. \end{cases}$$

Since $v \mapsto \Phi_i^{\text{Sh}}(v, S)$ is a linear map, the Shapley value (for player i) can be equivalently defined as the linear functional with property (21) for all coalitions U. Setting

$$P^v(S) = \sum_{T \subseteq S} \frac{(s-1)!(s-t)!}{t!} v(T), \tag{22}$$

we see:

Theorem 4.2 (Hart and Mas-Colell [17]) *For every cooperative game (N, v) and player $i \in N$ one has*

$$\Phi_i^{\text{Sh}}(v, N) = P^v(N) - P^v(N \setminus i).$$

Proof. By the linearity of $v \mapsto \Phi_i^{\text{Sh}}(v, N)$, it suffices to verify the Theorem for potentials of the form $v = \zeta_U$. If $i \in U$, we have $P^{\zeta_U}(N \setminus i) = 0$ and therefore

$$\Phi_i^{\text{Sh}}(\zeta_U, qqN) = 1/u = P^{\zeta_U}(N) - P^{\zeta_U}(N \setminus i).$$

If $i \notin U$, we have $P^{\zeta_U}(N \setminus i) = 1/u$ and thus

$$\Phi_i^{\text{Sh}}(\zeta_U, N) = 0 = P^{\zeta_U}(N) - P^{\zeta_U}(N \setminus i).$$

◇

Let $P = [p_{ST}]$ be the pattern with coefficients

$$p_{ST} = \begin{cases} 1 & \text{if } S = T = \emptyset \\ (t-1)!(s-t)!/s! & \text{if } T \subseteq S \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

Then Theorem 4.2 says that the Shapley value is also a potential value in our sense:

$$\Phi_i^{\text{Sh}}(v, N) = P^v(N) - P^v(N \setminus i) = v^P(N) - v^P(N \setminus i) = \partial^P(v). \tag{24}$$

In fact, the Shapley value is the "typical" linear value:

Theorem 4.3 *Let Φ be an arbitrary linear value on $\mathbb{R}^{\mathcal{N}}$. Then there exists a pattern G such that Φ arises as the Shapley value relative to G:*

$$\Phi_i(v, N) = \Phi_i^{\text{Sh}}(v^G, N) \quad \textit{for all } v \in \mathbb{R}^{\mathcal{N}},\ i \in N.$$

Proof. Notice that the pattern matrix P, given as in (23), admits an inverse P^{-1}. Indeed, arranging the rows and columns of P so that S always precedes T if $S \subset T$ holds, turns P into a triangular matrix with non-zero diagonal elements $p_{SS} \neq 0$.

By Theorem 4.1, Φ arises as the potential value relative to some pattern F. Letting $G = FP^{-1}$, we thus obtain for all $i \in N$,

$$v^F(N) - v^F(N \setminus i) = (v^G)^P(N) - (v^G)^P(N \setminus i) = \Phi_i^{\text{Sh}}(v^G, N).$$

$\diamond$

5 Least square approximation and linear maps

If the influence pattern $F \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ generates an influence space $\mathcal{F}$ of dimension $k = \dim \mathcal{F} < 2^n$, a valuation $v \in \mathbb{R}^{\mathcal{N}}$ will generally not be representable as a linear combination of influence functions, *i.e.*, v is not an F-potential. In this case, we look for a best F-potential $\hat{v} \in \mathcal{F}$ that approximates v. We measure the quality of an approximation by a quadratic criterion. That is, we assume to be given a weight matrix $W = [w_{ST}] \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ and a weight vector $c \in \mathbb{R}^{\mathcal{N}}$ and want to solve the problem

$$\min_{u\in\mathcal{F}} \sum_S \sum_T w_{ST}(v(S) - u(S))(v(T) - u(T)) + \sum_S c_S(v(S) - u(S)). \quad (25)$$

For any given (and hence fixed) $v \in \mathbb{R}^{\mathcal{N}}$, problem (25) is equivalent with

$$\min_{u\in\mathcal{F}} \sum_S \sum_T w_{ST}u(S)u(T) - \sum_S \tilde{c}_S u(S), \quad (26)$$

where $\tilde{c}_S = c_S + 2\sum_T w_{ST}v(T)$. A further simplification is possible by choosing a basis $B = \{b_1, \ldots, b_k\}$ for $\mathcal{F}$ with the correspondence

$$x = (x_1, \ldots, x_k) \in \mathbb{R}^k \quad \longleftrightarrow \quad u = \sum_{i=1}^k x_i b_i \in \mathcal{F}.$$

With respect to B, problem (26) becomes

$$\min_{x\in\mathbb{R}^k} \sum_{i=1}^k \sum_{j=1}^k q_{ij}x_i x_j - \sum_{i=1}^k \overline{c}_i x_i \quad (27)$$

with the coefficients

$$q_{ij} = \sum_S \sum_T w_{ST} b_i(S) b_j(T) \quad \text{and} \quad \overline{c}_i = \sum_S \tilde{c}_S b_i(S).$$

Example 5.1 *Assume* $w_{ST} = 1$ *if* $S = T$ *and* $w_{ST} = c_S = 0$ *otherwise. Then problem (25) is the least square approximation problem*

$$\min_{u \in \mathcal{F}} \|v - u\|^2 \quad \big(= \sum_S (v(S) - u(S))^2 \big).$$

The corresponding problem (27) has the parameters

$$q_{ij} = \sum_S b_i(S) b_j(S) \quad \textit{and} \quad \overline{c}_i = 2 \sum_S v(S) b_i(S).$$

Notice that the parameter vector $\overline{c}$ *in this example depends linearly on* v*.*

5.1 Optimality and linearity

There is no loss of generality in the symmetry assumption $q_{ij} = q_{ji}$ in (27) (otherwise we simply replace the coefficients by $\overline{q}_{ij} = (q_{ij} + q_{ji})/2$ and obtain the same problem). However, the problem will generally not have an optimal solution nor does an optimal solution need to be uniquely determined. Let us therefore make the additional assumption

(PD) The matrix $Q \in \mathbb{R}^{k \times k}$ with the coefficients q_{ij} is positive definite (*i.e.* $xQx^T = \sum_{ij} q_{ij} x_i x_j > 0$ holds for all $x \neq 0$)

Under this assumption, the optimization problem can be solved even under linear side constraints. (Recall that we assume parameter vectors x to be row vectors in our matrix notation. So the transpose x^T denotes a column vector.)

Theorem 5.1 *Assume that* $Q \in \mathbb{R}^{k \times k}$ *is positive definite,* $A \in \mathbb{R}^{k \times m}$ *an arbitrary matrix and* $g \in \mathbb{R}^m$ *a vector. Then the optimization problem*

$$\min_{x \in \mathbb{R}^k} xQx^T - xc^T \quad \textit{s.t.} \quad xA \geq g \tag{28}$$

either has no solution or a uniquely determined optimal solution x^**.*

Proof. If it well-known (see, *e.g.*, Faigle *et al.* [7]) that the positive matrix Q can be written as $Q = C^T C$, where C is an invertible matrix. Setting $y = xC^T$, $\overline{c}^T = (C^{-1})^T c^T$ and $\overline{A} = (C^{-1})^T A$, we arrive at the equivalent problem

$$\min_{y \in \mathbb{R}^k} \|y\|^2 - y\overline{c}^T \quad \text{s.t.} \quad y\overline{A} \geq g. \tag{29}$$

Assuming that (at least) one solution exists, choose $R > \|\overline{c}\|$ so large that

$$P_R = \{y \in \mathbb{R}^k \mid \|y\| \leq R, y\overline{A} \geq g\} \neq \emptyset.$$

Then an optimal solution of (29) must lie in P_{2R} because for any $z \in \mathbb{R}^k$, one has

$$\begin{array}{rcl} \|z\| > 2R & \Longrightarrow & zz^T - z\overline{c}^T \ > \ \|z\|^2/2 \ > \ 2R^2 \\ \|z\| \ \leq \ R & \Longrightarrow & zz^T - z\overline{c}^T \ \leq \ 2R^2. \end{array}$$

Since P_{2R} is compact the (continuous) objective function of (29) attains an optimal solution y^* on P_{2R}.

Let $z^* = \|y^*\|^2 - y^*\overline{c}^T$ be the corresponding optimal value and suppose there is another optimal solution $\tilde{y}$. Then $u = (y^* + \tilde{y})/2$ is a feasible solution with value

$$\begin{array}{rcl} uu^T - u\overline{c}^T & = & \frac{1}{4}(\|y^*\|^2 + \|\tilde{y}\|^2 + 2y^*\tilde{y}^T) - \frac{1}{2}y^*\tilde{c}^T - \frac{1}{2}\tilde{y}\overline{c}^T \\ & = & z^* - \frac{1}{4}\|y^* - \tilde{y}\|^2 \ < \ z^*, \end{array}$$

which contradicts the optimality of y^*. So y^* is the only optimal solution.

◇

It is furthermore well-known (see, *e.g.*, Faigle *et al.* [7]) that the optimal solutions of problem (28) are characterized by satisfying the Karush-Kuhn-Tucker (KKT) conditions

$$\begin{array}{rcrcl} 2xQ & - & zA^T & = & c \\ xA & & & \geq & g \\ & & z & \geq & 0. \end{array} \tag{30}$$

Assume now that $c : \mathbb{R}^{\mathcal{N}} \to \mathbb{R}^k$ and $g : \mathbb{R}^{\mathcal{N}} \to \mathbb{R}^m$ are linear functions such that the KKT-system (30) has a solution $(x^*(v), z^*(v))$ for every $v \in \mathbb{R}^{\mathcal{N}}$. By Theorem 5.1, $x^*(v)$ is the unique optimal solution of (28). Moreover, the linearity of c and g entails the positive homogeneity and additivity of the map $v \mapsto x^*(v)$:

$$x^*(\lambda v) = \lambda x^*(v) \qquad \text{for all } v \in \mathbb{R}^{\mathcal{N}}, \lambda \geq 0 \tag{31}$$

$$x^*(v + w) = x^*(v) + x^*(w) \quad \text{for all } v, w \in \mathbb{R}^{\mathcal{N}}. \tag{32}$$

The nonnegativity restriction of the scalar λ in (31) is necessary in the presence of inequality restrictions $xA \geq g$. In the case of equality restrictions $xA = g$, there is no sign restriction on z in (30) and a linear operator $v \mapsto x^*(v)$ results.

Corollary 5.1 *Let $c : \mathbb{R}^{\mathcal{N}} \to \mathbb{R}^k$ and $g : \mathbb{R}^{\mathcal{N}} \to \mathbb{R}^m$ be linear maps such that $xA = g(v)$ always has a solution. Then the extremal problem*

$$\min_{x \in \mathbb{R}^k} xQx^T - xc(v)^T \quad \text{s.t.} \quad xA = g(v) \tag{33}$$

has a unique optimal solution $x^*(v)$. Moreover, $v \mapsto x^*(v)$ is a linear operator.

$\diamond$

The linear maps $v \mapsto x^*(v)$ that arise in the manner of Corollary 5.1 are not special. In fact, *every* linear operator $\phi : \mathbb{R}^{\mathcal{N}} \to \mathbb{R}^k$ arises that way (see Ex. 5.2).

Example 5.2 *Let $C \in \mathbb{R}^{k \times \mathcal{N}}$ be a matrix such that $\phi(v) = vC$ for all $v \in \mathbb{R}^{\mathcal{N}}$. Clearly, $x^* = vC$ is the unique optimal solution of*

$$\min_{x \in \mathbb{R}^k} \|x - vC\|^2 \quad \longleftrightarrow \quad \min_{x \in \mathbb{R}^k} xQx^T - 2xc^T(v), \tag{34}$$

with $Q = I$ the (positive definite) identity matrix and $c(v) = vC$.

6 Least square values and semivalues

We have seen that every linear value can be interpreted as arising from a general least square approximation problem (see Section 5.1 above). Special cases of least square approximation problems have received considerable attention in the literature and led to the concept of *least square values* and *semivalues* arising from their optimal solutions. Take, for example, the approximation problem

$$\min_{x \in \mathbb{R}^N} \sum_{S \in \mathcal{N}} \alpha_S (v(S) - x(S))^2 \quad \text{s.t.} \quad x(N) = v(N) \tag{35}$$

that asks for the best (α-weighted) least square approximation of a valuation v by an additive game $u \in \mathcal{C}$ of the form

$$u = \sum_{i \in N} x_i \zeta_i \qquad (x \in \mathbb{R}^N)$$

under the additional constraint $x(N) = v(N)$, *i.e.*, that u should be efficient in terms of v: $u(N) = v(N)$. This problem has a long history and has been studied by many authors. Hammer and Holzman ([14], later published in [15]) studied both the above version and the unconstrained version with equal weights ($\alpha_S = 1$ $\forall S$), and proved that the solution of the unconstrained version is nothing else than the Banzhaf value $I_B^v(\{i\})$, $i \in N$. More general versions of the unconstrained problem were solved by Grabisch *et al.* [11], where the approximation is done on the space of at most k-additive games (*i.e.*, games whose Möbius transform vanishes for subsets of size greater than k)[2].

In 1988, Charnes *et al.* [1] gave a solution for the case with the coefficients α_S being *symmetric* (*i.e.*, $\alpha_S = \alpha_T$ whenever $|S| = |T|$) and all positive. As a particular case, they derived the Shapley value from the coefficient choice

$$\alpha_S = \alpha_s = \frac{(n-2)!}{(s-1)!(n-1-s)!} \qquad (s = |S|).$$

Assuming also symmetric weights, Ruiz *et al.* [23] call the optimal solution of (35) a *least square value*. They gave an axiomatization of this class of least square values, and showed that the efficient normalization of a semivalue[3] is a least square value.

REMARK. *Ruiz et al. [23] state that their problem has a unique optimal solution for any choice of weights (see Theorem 3 there). In this generality, however, the statement is not correct as neither the existence nor the uniqueness can be guaranteed. So additional assumptions on the weights must be made.*

Our general discussion of least square approximation above shows that problem (35) reduces to the problem

$$\min_{x\in\mathbb{R}^N} xQx^T - xc^T \quad \text{s.t.} \quad xA = g(v) \tag{36}$$

where the matrix Q has the coefficients

$$q_{ij} = \sum_S \alpha_S \zeta_i(S)\zeta_j(S) = \sum_{S\ni i,j} \alpha_S$$

and c has the coefficients

$$c_i = \sum_{S\ni i} \alpha_S v(S).$$

[2] see also Ding [3], and Marichal and Mathonet [19]

[3] A *semivalue* is defined as the expected value of the marginal contribution of players, relative to a symmetric probability distribution on the coalitions.

A is the unit vector, *i.e.* $A = (1, 1, \ldots, 1)^T$, and $g(v) = v(N)$.

Clearly, solutions of (36) always exist. Hence unique optimal solutions $x^*(v)$ (and thus a value $v \mapsto x^*(v)$) can be guaranteed whenever the matrix Q is positive definite.

It is easy to check that positive weights $\alpha_S > 0$ yield positive definiteness of Q. However, also other weight systems might result in a positive definite Q (see Ex. 6.2 below).

6.1 Regular weights

We say that the weights α_S are *regular* if the matrix Q has just two types of coefficients q_{ij}, *i.e.*, if there are real numbers p, q such that

$$q_{ij} = \begin{cases} q & \text{if } i = j \\ p & \text{if } i \neq q. \end{cases}$$

Example 6.1 *Assume that the weights α_S are symmetric and set $\alpha(|S|) = \alpha_S$. Then we find for all $i \neq j$*

$$q_{ij} = \sum_{s=2}^{n} \binom{n-2}{s-2} \alpha(s) \quad \textit{and} \quad q_{ii} = \sum_{s=1}^{n} \binom{n-1}{s-1} \alpha(s).$$

So $Q = [q_{ij}]$ is regular.

Lemma 6.1 *Let $Q = [q_{ii}] \in \mathbb{R}^{N \times N}$ be regular with $q = q_{ii}$ and $p = q_{ij}$ for $i \neq j$. Then Q is positive definite if and only if $q > p \geq 0$.*

Proof. For any $x \in \mathbb{R}^n$, we have after some algebra

$$xQx^T = (q - p) \sum_{i=1}^{n} x_i^2 + p\overline{x}^2$$

where $\overline{x} = \sum_{i=1}^{n} x_i$, which makes the claim of the Lemma obvious.

◇

Note that this allows for possibly negative symmetric coefficients in Ex. 6.1, as shown in the following example.

Example 6.2 *Let* $n = 3$. *We get* $p = \alpha_2 + \alpha_3$ *and* $q = \alpha_1 + 2\alpha_2 + \alpha_3$. *Letting* $\alpha > 0$, *the following vectors* $(\alpha_1, \alpha_2, \alpha_3)$ *lead to a positive definite matrix* Q*:*

$$(0, \alpha, 0), \quad (\alpha, 0, \alpha), \quad (0, \alpha, -\alpha), \text{ etc.}$$

For the remainder of this section, let $Q \in \mathbb{R}^{N \times N}$ be a regular matrix with parameters $q > p \geq 0$, $c \in \mathbb{R}^N$ a vector and $g \in \mathbb{R}$ a scalar. Setting $\mathbf{1} = (1, 1, \ldots, 1)$, the optimization problem

$$\min_{x \in \mathbb{R}^N} xQx^T - xc^T \quad \text{s.t.} \quad x\mathbf{1}^T = x(N) = g \tag{37}$$

has a unique optimal solution $x^* \in \mathbb{R}^N$. Moreover, there is a unique scalar $z^* \in \mathbb{R}$ such that (x^*, z^*) is the unique solution of the associated KKT-system

$$\begin{array}{rcrcl} 2xQ & - & z\mathbf{1} & = & c \\ x\mathbf{1}^T & & & = & g. \end{array} \tag{38}$$

Verifying the KKT-system, the proof of the following explicit solution formulas is straightforward.

Theorem 6.1 *If* Q *is regular, the solution* (x^*, z^*) *of the KKT-system (38) is:*

$$\begin{array}{rcl} z^* & = & (2(q + (n-1)p)g - C)/n \quad (\text{with } C = c\mathbf{1}^T = \sum_{i \in N} c_i) \\ x_i^* & = & (c_i + z^* - 2pg)/(2q - 2p) \quad (i \in N). \end{array}$$

If Q *is furthermore positive definite, then* x^* *is an optimal solution for (37).* $\diamond$

In the case of symmetric weights $\alpha(s)$, the formulas in Theorem 6.1 yield the formulas derived by Charnes *et al.* for problem (35). To demonstrate the scope of Theorem 6.1 let us look at the extremal problem[4] studied by Ruiz *et al.* [24]

$$\min_{x \in \mathbb{R}^N} \sum_{S \subseteq N} m_S d(x, S)^2 \text{ s.t. } x(N) = v(N), \tag{39}$$

where $m_S > 0$ and

$$d(x, S) = \frac{v(S) - x(S)}{|S|} - \frac{v(N \setminus S) = x(N \setminus S)}{n - |S|}.$$

[4]see also Sun *et al.* [27] for similar problems

Letting $\alpha_S = |S|^2(n-|S|)^2 n^2 m_S$, $v^*(S) = v(N) - v(S)$ and

$$\overline{v}(S) = \frac{(n-|S|)v(S) + |S|v^*}{n}$$

(and thus $n\overline{v}(N) = v(N)$), we find that problem (39) becomes

$$\min_{x\in\mathbb{R}^N} \sum_{S\subseteq N} \alpha_S(\overline{v}(S) - x(S))^2 \text{ s.t. } x(N) = n\overline{v}(N).$$

Because $v \mapsto \overline{v}$ and $v \mapsto g(v) = n\overline{v}(N)$ are linear mappings, it is clear from Corollary 5.1 that the optimal solutions of (39) yield an efficient linear value for any choice of parameters m_S such that the associated matrix Q is positive definite.

If furthermore the weights m_S (and hence the α_S) are symmetric, Q is regular and the optimal solution can be explicitly computed from the formulas of Theorem 6.1.

References

[1] A. Charnes, B. Golany, M. Keane, and J. Rousseau. Extremal principle solutions of games in characteristic function form: core, Chebychev and Shapley value generalizations. In J.K. Sengupta and G.K. Kadekodi, editors, *Econometrics of Planning and Efficiency*, pages 123–133. Kluwer Academic Publisher, 1988.

[2] D. Denneberg and M. Grabisch. Interaction transform of set functions over a finite set. *Information Sciences*, 121:149–170, 1999.

[3] G. Ding, R. Lax, J. Chen, P. Chen, and B. Marx. Transforms of pseudo-boolean random variables. *Discrete Applied Mathematics*, 158:13–24, 2010.

[4] I. Dragan. The potential basis and the weighted Shapley value. *Libertas Mathematica*, 11:139–150, 1991.

[5] I. Dragan. On the inverse problem for semivalues of cooperative TU games. *International Journal of Pure and Applied Mathematics*, 22(4):539–555, 2005.

[6] I. Dragan. The least square values and the Shapley value for cooperative TU games. *TOP*, 14:61–73, 2006.

[7] U. Faigle, W. Kern, and G. Still. *Algorithmic Principles of Mathematical Programming*. Springer, Dordrecht, 2002.

[8] U. Faigle and J. Voss. A system-theoretic model for cooperation, interaction and allocation. *Discrete Applied Mathematics*, 159:1736–1750, 2011.

[9] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.

[10] M. Grabisch. Set function over finite sets: transformations and integrals. In E. Pap, editor, *Handbook of Measure Theory*, pages 1381–1401. Elsevier Science Publ., 2002.

[11] M. Grabisch, J.-L. Marichal, and M. Roubens. Equivalent representations of set functions. *Mathematics of Operations Research*, 25(2):157–178, 2000.

[12] M. Grabisch and M. Roubens. An axiomatic approach to the concept of interaction among players in cooperative games. *Int. Journal of Game Theory*, 28:547–565, 1999.

[13] M. Grabisch and A. Rusinowska. A model of influence based on aggregation functions. *Mathematical Social Sciences*, to appear. DOI: 0.1016/j.mathsocsci.2013.07.003.

[14] P. L. Hammer and R. Holzman. On approximations of pseudo-Boolean functions. RUTCOR Research Report RRR 29-87, State University of New Jersey, 1987.

[15] P. L. Hammer and R. Holzman. On approximations of pseudo-Boolean functions. *ZOR - Methods and Models of Operations Research*, 36:3–21, 1992.

[16] J. C. Harsanyi. A simplified bargaining model for the n-person cooperative game. *International Economic Review*, 4:194–220, 1963.

[17] S. Hart and A. Mas-Colell. Potential, value and consistency. *Econometrica*, 57:589–614, 1989.

[18] N. L. Kleinberg and J. H. Weiss. Equivalent n-person games and the null space of the Shapley value. *Mathematics of Operations Research*, 10(2):233–243, 1985.

[19] J.-L. Marichal and P. Mathonet. Weighted Banzhaf power and interaction indexes through weighted approximations of games. *Eur. J. of Operations Research*, 211:352–358, 2011.

[20] D. Monderer and L.S. Shapley. Potential games. *Games and Economic Behavior*, 14:124–143, 1996.

[21] G. Owen. Multilinear extensions of games. *Management Sci.*, 18:64–79, 1972.

[22] G. C. Rota. On the foundations of combinatorial theory I. Theory of Möbius functions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 2:340–368, 1964.

[23] L. M. Ruiz, F. Valenciano, and J. M. Zarzuelo. The family of least-square values for transferable utility games. *Games and Economic Behavior*, 24:109–130, 1998.

[24] L. M. Ruiz, F. Valenciano, and J. M. Zarzuelo. Some new results on least square values for tu games. *TOP*, 6:139–158, 1998.

[25] G. Shafer. *A Mathematical Theory of Evidence*. Princeton Univ. Press, 1976.

[26] L. S. Shapley. A value for n-person games. In H. W. Kuhn and A. W. Tucker, editors, *Contributions to the Theory of Games, Vol. II*, number 28 in Annals of Mathematics Studies, pages 307–317. Princeton University Press, 1953.

[27] H. Sun, Z. Hao, and G. Xu. Optimal solutions for TU-games with decision approach. Preprint, Northwestern Polytechnical University, Xi'an, Shaanxi, China, 2013.

[28] J. Walsh. A closed set of normal orthogonal functions. *American Journal of Mathematics*, 45:5–24, 1923.

[29] K. Yokote, Y. Funaki, and Y. Kamijo. Linear basis to the Shapley value. Technical report, Waseda Economic Working Paper Series, 2013.