Economic Consequences of Nth-Degree Risk Increases and Nth-Degree Risk Attitudes

Elyès Jouini, Clotilde Napp, Diego Nocetti

Economic Consequences of Nth-Degree Risk Increases and Nth-Degree Risk Attitudes*

Elyès Jouini — Ceremade
Clotilde Napp — CNRS-DRM
Diego Nocetti — Clarkson University

August 3, 2013

Abstract

We study comparative statics of Nth-degree risk increases within a large class of problems that involve bidimensional payoffs and additive or multiplicative risks. We establish necessary and sufficient conditions for unambiguous impact of Nth-degree risk increases on optimal decision making. We develop a simple and intuitive approach to interpret these conditions : novel notions of directional Nth-degree risk aversion that are characterized via preferences over lotteries

1 Introduction

Consumers select how much to save, how much to invest in different assets, how long to work, and how much to spend on medical care under a great degree of uncertainty. Firms invest large amounts of money in risky endeavors. Policy-makers allocate scarce resources to projects with highly uncertain returns (e.g. environmental and health care projects). Two natural questions that arise in these and other similar problems are the following: 1) If a decision maker faces a riskier environment, under what conditions will he or she increase or decrease the optimal level of exposure to the

*The authors thank the participants of the conference "Risk and choice: A conference in honor of Louis Eeckhoudt" (Toulouse, July, 2012) as well as those of EGRIE conference (Palma, September, 2012) and of the Risk Seminar (Paris-Dauphine, January 2013) and in particular Christophe Courbage, Harris Schlesinger and Bertrand Villeneuve. Special thanks to Louis Eeckhoudt who discussed the paper and provided very useful hints and comments as well as to two anonymous referees who permitted us to greatly improve the paper in particular by bringing our attention to the possibility to generalize the increasing risk results to a stochastic dominance framework. The first two authors gratefully acknowledge the financial support of the ANR (RISK project) as well as of the Risk Foundation (Groupama Chair).

risk? and 2) What is the interpretation of these conditions, which frequently involve establishing the sign of two or more high-order partial derivatives of the payoff function? The focus of the early literature analyzing these questions was on establishing the impact of a fairly narrow class of risk changes –mostly, increases in risk as defined by Rothschild and Stiglitz (1970)– and on interpreting the resulting conditions on the payoff function within the specific context of the problem being analyzed. More recently, the work of Louis Eeckhoudt has advanced our knowledge about these important questions in at least two fundamental ways. First, Eeckhoudt and colleagues have evaluated the impact of a larger class of risk increases –namely, Nth-degree stochastically dominated shifts (Jean, 1980) and Nth-degree risk increases as defined by Ekern (1980)– in different frameworks, including the classical problems of precautionary saving (Eeckhoudt and Schlesinger, 2008), precautionary labor supply (Chiu and Eeckhoudt, 2010), portfolio choice (Chiu et al., 2012), and production under output-price uncertainty (Chiu et al., 2012).[1] Second, Eeckhoudt and colleagues have introduced a new approach to characterize risk attitudes via preferences over pairs of simple lotteries (Eeckhoudt and Schlesinger, 2006, Eeckhoudt et al., 2007, Eeckhoudt et al., 2009, Denuit et al., 2010a, and Chiu et al., 2012). In particular, we now know that the signs of the derivatives of the utility function are closely tied to a preference to disaggregate harms across lottery outcomes. In this way, this new approach has established a link between the decision maker's attitudes towards risk (i.e. preferences over simple lotteries) and his or her optimal response to changes in risk.

The focus of the recent work and much of the earlier literature has been on determining unambiguous comparative statics results for a given set of individuals. In this type of exercise, it is established that all expected utility maximizers within a given set (i.e. a set of utility functions) will select the optimal response to an increase in risk (or will rank lotteries) in a particular unambiguous manner. An alternative line of inquiry, taken on this paper, is to determine the maximal set of expected utility maximizers for which an increase in risk induces a particular behavior. This alternative type of exercise not only establishes unambiguous comparative statics results for a given set of individuals, but it also reveals how a particular behavioral response to a risk increase is related to the preferences of the decision maker. It is only by performing this type of analysis that behavior in different settings can be pinned down to a maximal set of individuals and, in particular, that the optimal response to changes in risk in a specific problem can be linked *directly* to the decision

[1] Other work evaluating economic consequences of Nth-degree risk increases includes Baiardi and Menegatti's (2011) analysis of the trade-off between (dirty) production and environmental-quality and Courbage and Rey's (2012) analysis of the optimal allocation of a national health budget.

maker's choice between lottery pairs. In this paper, we use this latter approach to study a large class of problems that involve 1) a bivariate utility function, 2) a linear constraint that links the two attributes that enter the decision maker's payoff function, and 3) Nth-degree stochastically dominated shifts, with particular emphasis on increases in Nth-degree risk *à la* Ekern (1980).

We begin the analysis in the next section by revisiting Ekern's (1980) results. Following the methods of Rothschild and Stiglitz (1970), Ekern (1980) characterized Nth-degree risk increases. We prove a dual result along the lines of Denuit et al. (1999). This result is the main ingredient for characterizing necessary and sufficient conditions for unambiguous comparative statics of risk within our framework and also for the interpretation of our results.

In Section 3 we perform the comparative statics analysis within the mentioned framework and for two different scenarios, one in which the risk is additive and another one in which the risk is multiplicative in the decision variable. In each case, we provide necessary and sufficient conditions for an unambiguous impact of Nth-degree risk increases and we compare our conditions with existing conditions in the literature. As an illustration, we show that no decision maker that views the attributes as goods (i.e. with a positive marginal utility) will always increase the level of the decision variable when faced with a first-degree multiplicative-risk increase. Similarly, no decision maker with diminishing marginal utility (i.e. risk averse) will always increase the level of the decision variable when faced with a Rothschild-Stiglitz multiplicative-risk increase. While these results are intuitive in the classical portfolio choice problem, we show that they hold much more generally within our framework and that they generalize to increases in Nth-degree risk.

Section 4 develops a simple and intuitive interpretation of the previously obtained conditions. We propose concepts of additive and multiplicative directional Nth-degree risk aversion; these are characterized via preferences for harms disaggregation across outcomes of 50-50 bivariate lotteries. The harms we consider involve unidimensional increases in Nth-degree risk together with bidimensional non-stochastic shifts in the attributes. These concepts of directional Nth-degree risk aversion include as special cases the concept of prudence analyzed by Eeckhoudt and Schlesinger (2006) and the concept of cross-prudence analyzed by Eeckhoudt et al. (2007). For a fixed level of one of the attributes, the lotteries we study are isomorphic to a set of lotteries studied by Eeckhoudt et al. (2009) in the case of additive risks (generalized by Denuit et al. 2010a to the multivariate case) and to the lotteries studied by Chiu et al. (2012) in the case of multiplicative risks. Our main contribution is to characterize the preference for harms disaggregation (i.e. to establish necessary and sufficient conditions). Specifically, we characterize the situations where Nth-degree changes in risk have an unambiguous impact on the optimal decision in terms of Nth-degree preference for harms disaggregation.

We provide some concluding remarks in Section 5. The proofs for all the results in the paper in terms of Nth-degree risk increases are collected in Appendix A. In Appendix B, we establish and prove analogous results for more general Nth-degree stochastically dominated shifts.

2 A Preliminary Result

Let us start by revisiting the notion of increases in risk proposed by Ekern (1980).

Definition 1 (Ekern) *Let $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ denote two random variables with values in $[0, B]$. For $i = 1, 2$, we denote by $F_{\widetilde{\alpha}_i}^{[1]}$ the distribution functions and, for $k = 1, 2, ...$ we define the functions $F_{\widetilde{\alpha}_i}^{[k+1]}$ on $\mathbb{R}_+$ by*

$$F_{\widetilde{\alpha}_i}^{[k+1]}(x) = \int_0^x F_{\widetilde{\alpha}_i}^{[k]}(t)dt \text{ for } x \in \mathbb{R}_+.$$

We say that $\widetilde{\alpha}_2$ is an increase in Nth-degree risk over $\widetilde{\alpha}_1$, and we denote it by $\widetilde{\alpha}_2 \succcurlyeq_N \widetilde{\alpha}_1$, if $F_{\widetilde{\alpha}_2}^{[N]}(x) \geq F_{\widetilde{\alpha}_1}^{[N]}(x)$ for all $x \in [0, B]$ where the inequality is strict for some x and $F_{\widetilde{\alpha}_2}^{[k]}(B) = F_{\widetilde{\alpha}_1}^{[k]}(B)$ for $k = 1, ..., N$.

For example, an increase in 2nd-degree risk coincides with Rothschild and Stiglitz's (1970) mean preserving increase in risk, while an increase in 3rd-degree risk coincides with a mean and variance preserving increase in risk that Menezes et al. (1980) labeled 'increase in downside risk'. An increase in 4th-degree risk is equivalent to what Menezes and Wang (2005) call an 'increase in outer risk'.

Ekern (1980) characterizes increases in Nth-degree risk: he establishes that $\widetilde{\alpha}_2$ is an increase in Nth-degree risk over $\widetilde{\alpha}_1$ (as defined above) if and only if[2] $E[q(\widetilde{\alpha}_2)] > E[q(\widetilde{\alpha}_1)]$ for all N times continuously differentiable real valued function q such that $(-1)^N q^{(N)} > 0$ where $q^{(N)} = \frac{d^N q}{d\alpha^N}$. In particular, this means that Ekern (1980) shows that if a function q is such that $(-1)^N q^{(N)} > 0$ then $E[q(\widetilde{\alpha}_2)] > E[q(\widetilde{\alpha}_1)]$ for all pair $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ such that $\widetilde{\alpha}_2 \succcurlyeq_N \widetilde{\alpha}_1$. The following Lemma, established by Denuit et al. (1999) in a somewhat different form, extends this last result by also showing the

[2] It is more common to define Nth-degree risk increases in terms of a utility function $u(\alpha)$ as follows: $\widetilde{\alpha}_2$ is an increase in Nth-degree risk over $\widetilde{\alpha}_1$ if and only if $E[u(\widetilde{\alpha}_2)] < E[u(\widetilde{\alpha}_1)]$ for all N times continuously differentiable real valued utility function u such that $(-1)^{(N+1)} u^{(N)} > 0$. Our equivalent characterization is somewhat more natural when dealing with the effect of risk increases on optimally chosen variables (i.e. on marginal utility).

reverse implication.[3] It characterizes the set of N times continuously differentiable functions for which $E\left[q(\widetilde{\alpha}_2)\right] \geq E\left[q(\widetilde{\alpha}_1)\right]$ for all pair $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ where $\widetilde{\alpha}_2$ is an increase in Nth-degree risk over $\widetilde{\alpha}_1$ and shows that these functions are exactly those such that $(-1)^N q^{(N)} \geq 0$.[4]

Lemma 1 (Denuit, De Vylder, and Lefevre) *Let q be a given real valued function that is N times continuously differentiable on $\mathbb{R}_+$. The following are equivalent.*

1. *For all pair $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ such that $\widetilde{\alpha}_2 \succcurlyeq_N \widetilde{\alpha}_1$, we have $E\left[q(\widetilde{\alpha}_2)\right] \geq E\left[q(\widetilde{\alpha}_1)\right]$.*

2. *For all $x \geq 0$, we have $(-1)^N q^{(N)}(x) \geq 0$.*

This result pins down how expected utility maximizers in a given set rank increases in risk and it also reveals how a particular behavioral response to a risk increase is related to the preferences of the decision maker. As an illustration, for $N = 2$, we retrieve the classical result that a risk averse agent (i.e. an agent who dislikes mean preserving spreads) is an agent whose utility function is concave. Lemma 1 will prove to be essential in Sections 3 and 4.

Remark. Lemma 1 can be generalized to Nth-degree stochastically dominated shifts. Such shifts do not restrict the first $N-1$ moments of the distributions to be equal. Following Jean (1980), and using the notation in Definition 1, $\widetilde{\alpha}_2$ is dominated by $\widetilde{\alpha}_1$ in the sense of Nth-degree stochastic dominance, and we denote it by $\widetilde{\alpha}_2 \succcurlyeq_{NSD} \widetilde{\alpha}_1$, if $F_{\widetilde{\alpha}_2}^{[N]}(x) \geq F_{\widetilde{\alpha}_1}^{[N]}(x)$ for all $x \in [0, B]$, where the inequality is strict for some x, and $F_{\widetilde{\alpha}_2}^{[k]}(B) \geq F_{\widetilde{\alpha}_1}^{[k]}(B)$ for $k = 1, ..., N-1$. Lemma 1 then applies to Nth-degree stochastic dominance shifts when 1) $\widetilde{\alpha}_2 \succcurlyeq_N \widetilde{\alpha}_1$ is replaced by $\widetilde{\alpha}_2 \succcurlyeq_{NSD} \widetilde{\alpha}_1$ and 2) the condition $(-1)^N q^{(N)}(x) \geq 0$ is replaced by the condition $(-1)^k q^{(k)}(x) \geq 0$ for $k = 1, ..., N$. As an illustration, for $N = 1, 2$, we retrieve the classical result that 1st-degree stochastic dominance is characterized by non-decreasing utility functions and 2nd-degree stochastic dominance is characterized by non-decreasing and concave utility functions. Appendix B provides a precise statement and a proof of these results.

[3] Ekern's (1980) result, but not Lemma 1, also implies that $\widetilde{\alpha}_2 \succcurlyeq_N \widetilde{\alpha}_1$ is necessary to have unanimity of ranking within the class of utility functions q such that $(-1)^N q^{(N)} > 0$, so the information provided by the two results is somewhat different. We thank an anonymous referee for pointing this out.

[4] Denuit et al.'s (1999) result is expressed in terms of maximal generators for s-convex orders and established for non-necessarily regular functions by the introduction of the concept of Nth degree divided difference. The authors remark that for regular functions the conditions on Nth-degree divided difference are equivalent to conditions on the Nth derivative. In Appendix A we propose a direct and simpler proof of the result when the functions are assumed to be regular.

3 Optimal Decision and Increasing Risk

The problem that we analyze follows closely the setup in Dardanoni (1988). The decision maker has an increasing, strictly concave and infinitely differentiable two-dimensional utility function $U(y,z)$ defined for y and z positive. As usual, we denote by $U^{(i,j)}(y,z)$ the $(i,j)^{th}$ cross partial derivative of the utility function. Uncertainty is described by a probability space $(\Omega, \mathcal{F}, P)$ where Ω describes the set of possible states of the world, $\mathcal{F}$ is the set of measurable events and P is a probability measure. The decision maker has the possibility to buy a quantity x of an asset that has a random payoff $\tilde{\mu} \geq 0$ in terms of the second attribute at a (deterministic) cost $p > 0$ in terms of the first attribute. The initial endowment of the decision maker in terms of the first attribute is deterministic and denoted by K. The initial endowment in terms of the second attribute is random and denoted by $\tilde{\alpha} \geq 0$. Both random variables $\tilde{\alpha}$ and $\tilde{\mu}$ are assumed to be bounded above. The decision maker's problem $P_{U,K,p}(\tilde{\alpha}, \tilde{\mu})$ is then the following

$$Max_x E\left[U\left(K - xp, x\tilde{\mu} + \tilde{\alpha}\right)\right]. \tag{1}$$

The first-order necessary condition and the second-order sufficient condition for optimality, assuming an interior solution, are given by

$$E\left[g\left(x, \tilde{\alpha}, \tilde{\mu}\right)\right] = 0 \tag{2}$$

$$E\left[g_x\left(x, \tilde{\alpha}, \tilde{\mu}\right)\right] < 0 \tag{3}$$

with $g(x,\alpha,\mu) = -pU^{(1,0)}(K - xp, x\mu + \alpha) + \mu U^{(0,1)}(K - xp, x\mu + \alpha)$ and where $g_x(x,\alpha,\mu) = p^2 U^{(2,0)}(K - xp, x\mu + \alpha) - 2p\mu U^{(1,1)}(K - xp, x\mu + \alpha) + \mu^2 U^{(0,2)}(K - xp, x\mu + \alpha)$. By strict concavity of U, this last quantity is always negative. The first order condition is then necessary and sufficient.

In the analysis that follows we will alternatively need the following Inada-type conditions:

Assumption A1 The utility function U is such that $\lim_{z \to 0} \frac{U^{(0,1)}(y,z)}{U^{(1,0)}(y,z)} = \infty$ and $\lim_{z \to \infty} \frac{U^{(0,1)}(y,z)}{U^{(1,0)}(y,z)} = 0$.

Assumption A2 The utility function U is such that $z\frac{U^{(0,1)}(y,z)}{U^{(1,0)}(y,z)}$ is unbounded.

It is classical to assume that $\lim_{z \to 0} U^{(0,1)}(y,z) = \infty$ and $\lim_{z \to \infty} U^{(0,1)}(y,z) = 0$ which means that the second attribute is necessary for the agent's survival and that the agents approach satiation for large quantities of that attribute. Assumption A1 is a little bit stronger and limits the possibilities for substitutability between the two attributes. A low level of the second attribute increases the marginal utility for that attribute faster than for the other one and satiation with respect to the second attribute does not mean satiation with respect to the first one. For example, Assumption A1 is satisfied by homothetic utility functions satisfying the classical Inada conditions in the direction of both attributes, i.e. $\lim_{z\to 0} U^{(0,1)}(y,z) = \infty$, $\lim_{z\to\infty} U^{(0,1)}(y,z) = 0$, $\lim_{y\to 0} U^{(1,0)}(y,z) = \infty$ and $\lim_{y\to\infty} U^{(1,0)}(y,z) = 0$. With a separable utility function $U(y,z) = u(y) + v(z)$, Assumption A1 is equivalent to the classical Inada conditions on v.

Assumption A2 means that U must satisfy either $\lim_{z\to 0} z\frac{U^{(0,1)}(y,z)}{U^{(1,0)}(y,z)} = \infty$ (which is a stronger condition than the first condition in Assumption A1) or $\lim_{z\to\infty} z\frac{U^{(0,1)}(y,z)}{U^{(1,0)}(y,z)} = \infty$. For example, with a separable utility function, Assumption A2 can be rewritten as $zv'(z)$ is unbounded. This is the case for all CRRA functions except log.

The problem $P_{U,K,p}$ has found many important applications, including

- The 2-date optimal saving model with either time-non-separable utility (e.g. Leland, 1968, Sandmo, 1970) or time-separable utility (e.g. Kimball, 1990, Eeckhoudt and Schlesinger, 2008, Chiu et al., 2012) and uncertainty surrounding the rate-of-return on saving or the future income endowment.

- The canonical portfolio problem with one risky asset and one risk-free asset (e.g. Rothschild and Stiglitz, 1971, Chiu et al., 2012).[5]

- The optimal allocation of income to medical expenditures and consumption of non-medical goods (e.g. Dardanoni and Wagstaff, 1990), and the optimal allocation of a national health budget (Courbage and Rey, 2012), when either the return on medical expenditures or the consumer's health status is uncertain.

- The trade-off between leisure and consumption, with wage income or non-wage income risks (e.g. Block and Heineke, 1973, Tressler and Menezes, 1980, Chiu and Eeckhoudt, 2010).

- The trade-off between (dirty) production and environmental-quality, with uncertainty surrounding the damages that the productive activity generates or the level of environmental quality itself (e.g. Baiardi and Menegatti, 2011).

[5] The canonical portfolio problem arises as a special case of this framework in which the attributes are perfect substitutes (setting $U(y,z) = v(y+z)$ and redefining the variables), making it, in essence, a univariate problem. Assumption A2 is always satisfied in this setting.

- The private provision of public goods under uncertainty surrounding the contributions of others (Sandler et al., 1987, Keenan et al., 2006).

- The leisure/production trade-off of an entrepreneur facing price uncertainty in a competitive environment (Rothschild and Stiglitz, 1971, Chiu et al., 2012).

In some of the above cases, like in the 2-date optimal saving model or the canonical portfolio problem, the asset is a classical financial asset. In other cases, the asset corresponds to the mechanism that transforms money into health (health care) or leisure into money (labor) or production into environmental quality, etc.

The objective in these articles has been the evaluation of how increases in risk affect the optimal value of the choice variable. While the earlier literature focused almost exclusively on 2nd-degree stochastic dominance shifts and, in particular, on mean preserving increases in risk as defined by Rothschild and Stiglitz (1970), a number of recent papers have evaluated the effect of Nth-degree stochastically dominated shifts and, in particular, of increases in Nth-degree risk as defined by Ekern (1980) (e.g. Eeckhoudt and Schlesinger, 2008, Chiu and Eeckhoudt, 2010, Baiardi and Menegatti, 2011, Chiu et al., 2012, Courbage and Rey, 2012). In the present paper, we will also focus on the general Nth-degree risk framework, considering separately the case where the endowment in terms of the second attribute $\tilde{\alpha}$ is random and the case where the asset's payoff $\tilde{\mu}$ is random. Analogous results for Nth-degree stochastic dominance shifts can be found in Appendix B.

3.1 Uncertainty Over the Endowment

Suppose that $\widetilde{\mu} = \mu$ is deterministic and remains unchanged while considering a change in the second attribute initial endowment from $\widetilde{\alpha}_1$ to $\widetilde{\alpha}_2$, where $\widetilde{\alpha}_2$ is an increase in Nth-degree risk over $\widetilde{\alpha}_1$. Furthermore, let x_1^* and x_2^* respectively denote the solutions of $P_{U,K,p}(\widetilde{\alpha}_1, \mu)$ and $P_{U,K,p}(\widetilde{\alpha}_2, \mu)$.

In order to provide some intuition for the results that follow, let us consider the classical 2-date precautionary saving problem with a separable utility function U of the form $U(y, z) = u(y) + v(z)$. In such a setting, it is well known that an agent raises his optimal saving when adding a zero-mean risk (to a deterministic second period initial endowment) if and only if his marginal utility of future consumption is convex, $v^{(3)} \geq 0$ (see e.g. Gollier, 2001). In our more general setting where U is not separable and where the initial endowment $\widetilde{\alpha}_1$ is not necessarily deterministic, the analysis should lead us to introduce conditions that involve cross derivatives as well as the cost $\frac{p}{\mu}$ of the second attribute in terms of the first attribute in order to reflect the trade-off between these two attributes. The following proposition provides a characterization of the utility functions for which any increase in Nth-degree risk increases the optimal level of the choice variable.

Proposition 1 *Let U be a given increasing, strictly concave and infinitely differentiable utility function satisfying Assumption A1. Let us consider p and μ as given. The following properties are equivalent:*

1. *For all initial endowment $(K, \widetilde{\alpha}_1)$, any increase in Nth-degree risk over the second attribute initial endowment from $\widetilde{\alpha}_1$ to $\widetilde{\alpha}_2$ increases the optimal level of the choice variable, i.e. $x_2^* \geq x_1^*$.*

2. *For all (y, z), we have $(-1)^N \left(-pU^{(1,N)}(y, z) + \mu U^{(0,N+1)}(y, z)\right) \geq 0$.*

Therefore, for example, for $\widetilde{\alpha}_1 = k_1 \in \mathbb{R}_+^*$, if $\widetilde{\alpha}_2$ is an increase in first-degree risk over $\widetilde{\alpha}_1$ of the form $\widetilde{\alpha}_2 = k_2$, for a positive constant $k_2 < k_1$, and if $U^{(1,1)} \geq 0$ then $x_2^* \geq x_1^*$ (remark that, by concavity of U, $U^{(0,2)}$ is nonpositive). As a second example, for $\widetilde{\alpha}_1 = k_1 \in \mathbb{R}_+^*$, if $\widetilde{\alpha}_2$ is an increase in second-degree risk over $\widetilde{\alpha}_1$ of the form $\widetilde{\alpha}_2 = k_1 + \widetilde{\varepsilon}$, where $\widetilde{\varepsilon}$ is a mean zero random variable, and if $U^{(1,2)} \leq 0$ and $U^{(0,3)} \geq 0$ then $x_2^* \geq x_1^*$.

If the problem under consideration corresponds to the classical 2-date precautionary saving problem with endowment risk and with a separable utility function U of the form $U(y, z) = u(y) + v(z)$, we obtain that $x_1^* \leq x_2^*$ for all pair $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ such that $\widetilde{\alpha}_1 \preccurlyeq_N \widetilde{\alpha}_2$ if and only if $(-1)^N v^{(N+1)}(z) \geq 0$. The "if" part of this result has been established by Eeckhoudt and Schlesinger (2008). We emphasize that our condition $(-1)^N v^{(N+1)}(z) \geq 0$ is necessary and sufficient.[6]

3.2 Uncertainty Over the Asset's Payoff

Suppose now that $\widetilde{\alpha} = \alpha$ is deterministic and remains unchanged while considering a change from $\widetilde{\mu}_1$ to $\widetilde{\mu}_2$, where $\widetilde{\mu}_2$ is an increase in Nth-degree risk over $\widetilde{\mu}_1$. Let x_1^* and x_2^* respectively denote the solutions of $P_{U,K,p}(\alpha, \widetilde{\mu}_1)$ and $P_{U,K,p}(\alpha, \widetilde{\mu}_2)$. Furthermore, it will be useful to define a measure of Nth-degree relative risk aversion (in attribute z) as follows (see e.g. Chiu et al. 2012)

$$R^N(y, z) \equiv -z \frac{U^{(0,N+1)}(y,z)}{U^{(0,N)}(y,z)}.$$

When the utility function $U(y, z)$ is separable we will write this function as $R^N(z)$, which for $N = 1$ corresponds to the standard measure of relative risk aversion and for $N = 2$ corresponds to the measure of relative prudence (Kimball, 1990).

[6] Similarly, in the context of problems with non-separable utility and an additive risk, Chiu and Eeckhoudt (2010) and Baiardi and Menegatti (2011) established the sufficiency of the condition in Proposition 1.

The following Proposition establishes the analog of Proposition 1 for an increase in multiplicative risk.

Proposition 2 *Let U be a given increasing, strictly concave and infinitely differentiable utility function satisfying Assumption A2. The following properties are equivalent:*

1. *For all initial endowment (K, α) and all asset cost and payoff $(p, \widetilde{\mu}_1)$ such that $x_1^* \geq 0$, any increase in Nth-degree risk over the asset's payoff from $\widetilde{\mu}_1$ to $\widetilde{\mu}_2$ increases the optimal level of the choice variable, i.e. $x_2^* \geq x_1^*$.*

2. *For all (y, z), we have $(-1)^N U^{(1,N)}(y, z) \leq 0$, $R^N(y, z) \leq N$ and $(-1)^N U^{(0,N)}(y, z) \geq 0$.*

The necessary and sufficient conditions in Proposition 2 appear to be similar to a number of results established previously in the context of different applications of our model.[7] There is, however, a crucial difference. Consider again the 2-date saving problem with separable utility, but now with rate-of-return risk. This problem was analyzed by Rothschild and Stiglitz (1971) in the context of increases in 2nd-degree risk and more recently by Eeckhoudt and Schlesinger (2008) for more general increases in risk. In both papers, the authors show that when future labor income is zero (in our setting $\alpha = 0$, so $z = x\mu$) and the consumer is risk averse ($v^{(2)}(z) < 0$), he will save more in response to an increase in 2nd-degree risk if $R^2(z) \geq 2$. As our proof makes it clear, this condition is indeed necessary and sufficient for an increase in savings when $\alpha = 0$. But Proposition 2 also shows that the assumption of zero labor income does not come without loss of generality. Once we consider the more general case, the necessary and sufficient conditions for an increase in savings become $R^2(z) \leq 2$ and $v^{(2)}(z) \geq 0$. In other words, any risk-averse consumer will *decrease* savings in response to higher risk for some initial endowment levels.[8]

In essence, this result corresponds to Rothschild and Stiglitz's (1971, p. 72) conclusion that "no risk averse investor will always increase his holdings of risky assets when their variability increases." Proposition 2 implies that this important result holds much more generally in the

[7] For example, Rothschild and Stiglitz (1971), Keenan et al. (2006), Eeckhoudt and Schlesinger (2008), Chiu and Eeckhoudt (2010), Baiardi and Menegatti (2011), Chiu et al. (2012), and Courbage and Rey (2012).

[8] Note that the condition $R^N(y, z) \leq N$ is coupled with $(-1)^N U^{(0,N)}(y, z) \geq 0$ which, for $N = 2$, excludes risk averse consumers. In the special case that $\alpha = 0$ this additional condition is no longer necessary. Therefore, assuming risk aversion, $(-1)^2 U^{(0,2)}(y, z) < 0$, leads to a change in the direction of the inequality of the first condition, which becomes $R^2(y, z) \geq 2$, as established by Rothschild and Stiglitz (1971) and Eeckhoudt and Schlesinger (2008) in the context of the 2-date saving problem with rate-of-return risk.

context of problem $P_{U,K,p}(\alpha,\tilde{\mu})$ and generalizes the conclusion by stating that no agent for which $(-1)^N U^{(0,N)}(y,z) < 0$ will always increase the demand for the asset in problem $P_{U,K,p}(\alpha,\tilde{\mu})$ when facing an increase in Nth-degree risk.

As another illustration, consider the static labor supply problem analyzed by Chiu and Eeckhoudt (2010). In this context y represents leisure, the choice variable x is labor supply, μ represents wage income, and α represents non-wage income. Chiu and Eeckhoudt (2010) show that the conditions $(-1)^N U^{(1,N)}(y, x\mu+\alpha) \leq 0$ and $(-1)^N \left(x\mu U^{(0,N+1)}(y, x\mu+\alpha) + NU^{(0,N)}(y, x\mu+\alpha)\right) \geq 0$ are sufficient for an increase in Nth-degree risk in wage income to increase the supply of labor. Note that the second condition does not correspond to our second condition in Proposition 2. Indeed, the derivatives are not taken at the same point and the two conditions coincide only for $\alpha = 0$. In fact, if we divide by $U^{(0,N)}(y, x\mu+\alpha)$ the condition introduced by Chiu and Eeckhoudt (2010) we obtain a condition on the concept of proportional N-th degree relative risk aversion, which is defined as $-z\frac{U^{(0,N+1)}(y,z+\alpha)}{U^{(0,N)}(y,z+\alpha)}$. Instead, our condition relies on the concept of N-th degree relative risk aversion, which is more natural and easier to interpret. Our Proposition implies, in particular, that imposing the Chiu and Eeckhoudt (2010) condition on the proportional N-th degree relative risk aversion for all α is equivalent to imposing a condition on the usual concept of N-th degree relative risk aversion as well as imposing $(-1)^N U^{(0,N)}(y,z) \geq 0$. This means that Proposition 2 of Chiu and Eeckhoudt (2010) cannot be applied in order to characterize the situations where the supply of labor is increased in response to a risk increase *at all initial endowment levels* α. Indeed, the authors assume that $(-1)^N U^{(0,N)}(y,z) \leq 0$, which is not consistent with the necessary condition $(-1)^N U^{(0,N)}(y,z) \geq 0$. This implies, for instance, that a mean preserving spread in wage income cannot always increase the supply of labor for a consumer with diminishing marginal utility of consumption.

These results then suggest that, when considering a risk increase on the asset's payoff, it may be more natural to analyze the conditions under which the agent decreases his level of exposure to the asset's risk. While Proposition 2 characterized the situations where any increase in risk on the asset's payoff implies an increase in the choice variable, the following result characterizes the situations where any increase in risk on the asset's payoff implies a decrease in the choice variable.

Corollary 1 *Let U be a given increasing, strictly concave and infinitely differentiable utility function satisfying Assumption A2. The following properties are equivalent:*

1. *For all initial endowment (K,α) and all asset cost and payoff $(p,\widetilde{\mu}_1)$ such that $x_1^* \geq 0$, any*

increase in Nth-degree risk over the asset's payoff from $\widetilde{\mu}_1$ to $\widetilde{\mu}_2$ decreases the optimal level of the choice variable, i.e. $x_1^* \geq x_2^*$.

2. *For all (y,z), we have $(-1)^N U^{(1,N)}(y,z) \geq 0$, $R^N(y,z) \leq N$ and $(-1)^N U^{(0,N)}(y,z) \leq 0$.*

For example, if we consider the 2-date optimal saving problem with rate-of-return risk and a separable utility function, we obtain that any increase in N-th degree risk leads to a decrease of the optimal saving (i.e. $x_2^* \leq x_1^*$ for all $\widetilde{\mu}_1 \preccurlyeq_N \widetilde{\mu}_2$) if and only if, for all z, $R^N(z) \leq N$ with $(-1)^N v^{(N)}(z) \leq 0$, which corresponds to the conditions established by Eeckhoudt and Schlesinger (2008) in a model without labor income ($\alpha = 0$).[9] Again, we remark that our conditions are necessary and jointly sufficient and that they hold for all (y,z). Still in a one dimensional setting, we retrieve the following well-known results in the context of the classical portfolio choice problem (see e.g. Rothschild and Stiglitz, 1971, Hadar and Seo, 1990, Gollier, 2001, p. 61):

- A risk averse agent decreases his optimal demand for the risky asset at all wealth levels and when facing an increase in 1st-degree risk in the asset's payoff if the degree of relative risk aversion is no greater than one, $R^1(z) \leq 1$ (or equivalently, $0 \leq R^1(z) \leq 1$ since the utility function is assumed to be nondecreasing and concave).

- A risk averse agent decreases his optimal demand for the risky asset at all wealth levels and when facing an increase in 2nd-degree risk in the asset's payoff if the degree of relative prudence is no greater than two, $R^2(z) \leq 2$ (or equivalently, $0 \leq R^2(z) \leq 2$ when the utility function is assumed to be prudent).

In summary, this section establishes precisely the necessary and sufficient conditions for unambiguous comparative statics of changes in risk in a large class of problems. Our next objective is to develop a simple and intuitive approach to interpret these conditions.

4 Lottery Choices and Optimal Exposure to Risk

In this section we show that the optimal response to changes in risk can be characterized via preferences over particular classes of lottery pairs. For any two bidimensional lotteries A and B,

[9] Suppose, for example, that relative risk aversion is constant: $v(z) = (1-\gamma)^{-1} z^{1-\gamma}$. Then, if $\gamma < 1$ an Nth-degree risk increase in the rate-of-return will decrease savings. For $\gamma > 1$, an Nth-degree risk increase in the rate-of-return will increase savings for some wealth levels and it will decrease savings for other wealth levels.

we use the notation $A \succ B$ to denote the individual's preference relation "lottery A is preferred to lottery B." We first consider the case of *additive* risks and then evaluate the case of *multiplicative* risks.

4.1 Additive Risks

Consider two attributes with nonnegative initial quantities y and z, and imagine a lottery in which with a 50 percent chance risk $\widetilde{\alpha}_1$ is added to z and with a 50 percent chance risk $\widetilde{\alpha}_2$ is added to z, where $\widetilde{\alpha}_2$ is an increase in Nth-degree risk over $\widetilde{\alpha}_1$. Denote this lottery by $[(y, z + \widetilde{\alpha}_1); (y, z + \widetilde{\alpha}_2)]$. Now consider the following location experiment: the consumer is told that she must accept the bundle $(x_1\rho_y, x_1\rho_z)$ in tandem with one of the lottery outcomes of her choice and the bundle $(x_2\rho_y, x_2\rho_z)$ in tandem with the other lottery outcome, where $(x_1, x_2, \rho_y, \rho_z)$ are constants and $x_2 > x_1$; to which outcome will she affect each bundle?

The preceding question consists in evaluating the following pair of lotteries,

$$\begin{aligned} A_a &= \left[\left(y + x_2\rho_y, z + x_2\rho_z + \widetilde{\alpha}_2\right); \left(y + x_1\rho_y, z + x_1\rho_z + \widetilde{\alpha}_1\right)\right] \\ B_a &= \left[\left(y + x_1\rho_y, z + x_1\rho_z + \widetilde{\alpha}_2\right); \left(y + x_2\rho_y, z + x_2\rho_z + \widetilde{\alpha}_1\right)\right]. \end{aligned} \tag{4}$$

Clearly, the answer depends on the magnitude and the direction of (ρ_y, ρ_z). We therefore propose the following definition,

Definition 2 *We say that preferences display Nth-degree risk aversion in the direction of (ρ_y, ρ_z) if, for all $(y, z, x_1, x_2) \in \mathbb{R}^4_+$ such that $x_2 > x_1$ and for all pair of random variables $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ such that $\widetilde{\alpha}_2$ is an increase in Nth-degree risk over $\widetilde{\alpha}_1$, we have $A_a \succ B_a$.*

To understand these concepts it might be useful to consider a few special cases that were analyzed by Eeckhoudt and Schlesinger (2006) and Eeckhoudt et al. (2007).[10] Suppose first that $\rho_y = 0$, $\rho_z > 0$ and $N = 1$ or 2, in which case, we retrieve the univariate definition of risk aversion and of prudence in attribute z as in Eeckhoudt and Schlesinger (2006). Still with $N = 1$ or 2 and taking $\rho_y > 0$ and $\rho_z = 0$, we recover Eeckhoudt et al.'s (2007) definitions of correlation aversion and of cross-prudence. As explained by Eeckhoudt et al. (2007), in all of these cases the consumer

[10] Eeckhoudt et al. (2007) restrict themselves to the case $x_1 = 0$. All our results could be generalized further, along the lines of Eeckhoudt et al.'s (2009) results in a univariate framework, by considering the case in which x_1 is a first order stochastically dominated shift over x_2.

views the risks as 'mutually aggravating,' so she prefers to disaggregate the harms across outcomes of the lotteries. Similarly, for general N and general directions, our lottery ordering generalizes these concepts by capturing a preference to disaggregate harms across lottery outcomes, where the harms are represented by an Nth-degree risk increase ($\widetilde{\alpha}_2 \succcurlyeq_N \widetilde{\alpha}_1$) and by a shift from x_2 to $x_1 < x_2$ in the direction of $\left(\rho_y, \rho_z\right)$ (notice that we refer to the bundle $\left(x_1\rho_y, x_1\rho_z\right)$ as a harm, in the direction of $\left(\rho_y, \rho_z\right)$, relative to the bundle $\left(x_2\rho_y, x_2\rho_z\right)$ even though the latter bundle would be perceived as more favorable in the usual sense when ρ_y and ρ_z are negative).

The following proposition establishes precisely what this lottery preference means in an expected utility framework.

Proposition 3 *Let U be a given increasing, strictly concave and infinitely differentiable utility function on $\mathbb{R}^2_+$. The preferences represented by U display Nth-degree risk aversion in the direction of $\left(\rho_y, \rho_z\right)$ if and only if $(-1)^N f^{(0,N)}(y,z) \geq 0$ where f is defined on $\mathbb{R}^2_+$ by $f(y,z) = \rho_y U^{(1,0)}(y,z) + \rho_z U^{(0,1)}(y,z)$.*

There are, in fact, two effects that determine the consumer's bivariate preference for harm disaggregation. The condition for risk aversion in the direction of $\left(\rho_y, \rho_z\right)$ can be decomposed into 1) a 'Nth-degree risk aversion in z effect', captured by the term $\rho_z U^{(0,N+1)}$, and 2) a 'Nth-degree correlation aversion' effect, captured by the term $\rho_y U^{(1,N)}$.

Generalizing the work of Eeckhoudt et al. (2009) (i.e. the case with $\rho_y = 0$ and $\rho_z > 0$ and for a fixed value of y), Denuit et al. (2010a) established the "if" part of this Proposition for the case $\rho_y > 0$ and $\rho_z > 0$.[11] The contribution of our Proposition is twofold. First, and most obvious, our notion of directional Nth-degree risk aversion is more encompassing. More importantly, by making use of Lemma 1, the proposition characterizes the set of expected utility maximizers that display Nth-degree risk aversion in the direction of $\left(\rho_y, \rho_z\right)$. This characterization enables us to establish a direct link between lottery choices and optimal exposure to risk, as we do in the following corollary.

Corollary 2 *Let U be a given increasing, strictly concave, infinitely differentiable utility function on $\mathbb{R}^2_+$, satisfying Assumption A1. Let us consider $(p, \mu) \in \mathbb{R}^2_+$ as given. The following properties are equivalent:*

[11] Eeckhoudt et al. (2009) showed that if $\widetilde{\alpha}_2 \succcurlyeq_N \widetilde{\alpha}_1$ and $\tilde{x}_1 \succcurlyeq_M \tilde{x}_2$, then the 50-50 lottery $[\widetilde{\alpha}_2 + \tilde{x}_1, \widetilde{\alpha}_1 + \tilde{x}_2]$ is an (N+M)th-degree risk increase over $[\widetilde{\alpha}_2 + \tilde{x}_2, \widetilde{\alpha}_1 + \tilde{x}_1]$ in the sense of Ekern (1980), i.e. the second lottery is preferred by all decision makers with $(-1)^{N+M} U^{(0,N+M)}(y,z) \leq 0$. When $\rho_y = 0$ and $\rho_z > 0$, and for a fixed value of y, our Proposition deals with the special case in which $M = 1$. Denuit et al (2010a) proved a multivariate version of Eeckhoudt et al. (2009)'s results.

1. *For all initial endowment* $(K, \widetilde{\alpha}_1)$*, any increase in Nth-degree risk over the second attribute initial endowment from* $\widetilde{\alpha}_1$ *to* $\widetilde{\alpha}_2$ *increases the optimal level of the choice variable, i.e.* $x_2^* \geq x_1^*$.

2. *The preferences represented by* U *display Nth-degree risk aversion in the direction of* $(-p, \mu)$.

This result is quite intuitive. It establishes that, in response to an increase in risk, the consumer's optimal level of the choice variable (e.g. the level of savings, of labor supply, of medical care) will reflect her preferences towards harm disaggregation as defined in this paper. Consider, for instance, the case of a mean preserving spread and, to be concrete, the problem of precautionary saving with time-non-separable utility. As explained above, two effects operate. First, a consumer that is prudent (in second period consumption) would like to allocate a higher level of savings to second period consumption to mitigate the increase in risk. Second, a higher level of savings implies a lower level of first period consumption, which a cross-prudent consumer dislikes to match with the higher risk. As a result, the higher level of risk implies a higher level of savings if the prudence effect is stronger than the cross-prudence effect, i.e. $-pU^{(1,2)}(y, z) + \mu U^{(0,3)}(y, z) > 0$, where in this context $p = 1$ and μ is the non-stochastic rate of return. Equivalently, savings increase in response to the higher risk if the consumer displays 2nd-degree risk aversion in the direction of $(-p, \mu)$, so he or she always prefers the lottery $[(y - px_2, z + x_2\mu + \widetilde{\alpha}_2)\,; (y - px_1, z + x_1\mu + \widetilde{\alpha}_1)]$ over the lottery $[(y - px_1, z + x_1\mu + \widetilde{\alpha}_2)\,; (y - px_2, z + x_2\mu + \widetilde{\alpha}_1)]$, where $x_2 > x_1$, $\widetilde{\alpha}_2 \succcurlyeq_2 \widetilde{\alpha}_1$, and the attributes of the lotteries represent consumption at two different dates.

Similarly, consider a decrease in second period income with certainty. A consumer that is risk averse (in second period consumption) would like to mitigate this harm by increasing savings. The harm will then be present when first period consumption is lower, which a correlation averse individual dislikes. Savings will increase when the risk aversion effect is stronger than the correlation aversion effect, i.e. $-pU^{(1,1)}(y, z) + \mu U^{(0,2)}(y, z) < 0$. Equivalently, savings will increase if the consumer displays 1st-degree risk aversion in the direction of $(-p, \mu)$, so he or she always prefers the lottery $[(y - px_2, z + x_2\mu + \alpha_2)\,; (y - px_1, z + x_1\mu + \alpha_1)]$ over the lottery $[(y - px_1, z + x_1\mu + \alpha_2)\,; (y - px_2, z + x_2\mu + \alpha_1)]$, where $x_2 > x_1$ and $\alpha_2 < \alpha_1$.

Clearly, a similar intuition holds for all of the above-mentioned applications and for increases in risk of any degree. A decision maker that views risks as mutually aggravating would like to compensate the higher risk in attribute z with a higher level of the choice variable. Alternatively, he could compensate the higher risk by *reducing* the level of the choice variable and, as a result, by increasing the level of the other attribute. Whether the level of the choice variable increases or decreases in response to an increase in risk then depends on the relative strengths of the two opposite forces. Corollary 2 then shows that evaluating the strength of the two opposite forces is equivalent to establishing the sign of the function $f^{(0,N)}(y,z)$, or equivalently, characterizing the decision maker's preference over the lotteries A_a and B_a (with $\rho_y = -p$ and $\rho_z = \mu$).

4.2 Multiplicative Risks

As before, consider the following experiment: the consumer needs to locate $(x_1\rho_y, x_1\rho_z)$ and $(x_2\rho_y, x_2\rho_z)$ to each of the two (random) outcomes of the lottery $(y, z+\widetilde{\alpha}_1)$ and $(y, z+\widetilde{\alpha}_2)$ where $x_2 > x_1 \geq 0$ and where $\widetilde{\alpha}_2 \geq 0$ is an Nth-degree risk increase over $\widetilde{\alpha}_1 \geq 0$. Now, however, the effect of $x_1\rho_z$ and $x_2\rho_z$ is to scale the risks $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$. In particular, the consumer evaluates the following lotteries:

$$\begin{aligned} A_m &= \left[\left(y + x_2\rho_y, z + x_2\rho_z\widetilde{\alpha}_2\right); \left(y + x_1\rho_y, z + x_1\rho_z\widetilde{\alpha}_1\right)\right] \\ B_m &= \left[\left(y + x_1\rho_y, z + x_1\rho_z\widetilde{\alpha}_2\right); \left(y + x_2\rho_y, z + x_2\rho_z\widetilde{\alpha}_1\right)\right]. \end{aligned} \tag{5}$$

We propose the following definitions.

Definition 3 *We say that preferences display Nth-degree multiplicative-risk attraction (resp. aversion) in the direction of (ρ_y, ρ_z) if, for all (y, z, x_1, x_2) such that $x_2 > x_1 \geq 0$ and for all pairs of random variables $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ such that $\widetilde{\alpha}_2$ is an increase in Nth-degree risk over $\widetilde{\alpha}_1$, we have $A_m \succ B_m$ (resp. $B_m \succ A_m$).*

To understand the different forces at play in such preference ordering, consider first the case with $\rho_y = 0$ and a fixed value of y analyzed by Chiu et al. (2012). On the one hand, since $x_2 > x_1$, the higher risk in A_m is scaled up, which hurts a consumer that dislikes higher risks. On the other hand, $x_2 > x_1$ also implies that the distribution of $z + x_2\rho_z\widetilde{\alpha}_2$ is shifted upwards relative to $z + x_1\rho_z\widetilde{\alpha}_1$. As in the previous section, this implies that an individual that prefers to disaggregate harms would like to match this higher level of z with the higher risk, making A_m relatively more attractive than B_m. Therefore, as stated by Chiu et al. (2012), the choice of A_m over B_m will depend on the relative strengths of these two opposite effects. If we now allow ρ_y to differ from zero we have, as in the previous section, another effect that arises from the consumer's preference to match the higher risk with a higher level of the other attribute (y). If $\rho_y > 0$ this additional effect will make A_m more desirable than B_m for an individual that prefers to disaggregate harms, while the opposite will be true if $\rho_y < 0$. In the next, we will focus on the case $\rho_y \leq 0$ and this tends to make B_m more desirable than A_m for an individual that prefers to disaggregate harms.[12] It is in reference to this situation that we choose to call Nth-degree risk aversion the fact that B_m is preferred to A_m and to call Nth-degree risk-attraction the opposite behavior.

The following Proposition establishes precisely the different forces at play in an expected utility model and for general increases in Nth-degree risk.

Proposition 4 *Let U be a given increasing, strictly concave and infinitely differentiable utility function on $\mathbb{R}_+^2$ and let $\rho_y \leq 0$ and $\rho_z \geq 0$ be given. The preferences represented by U display Nth-degree multiplicative-risk attraction (resp. aversion) in the direction of $\left(\rho_y, \rho_z\right)$ if and only if $(-1)^N U^{(1,N)}(y,z) \leq 0$ (resp. ≥ 0), $(-1)^N U^{(0,N)}(y,z) \geq 0$ (resp. ≤ 0) and $R^N(y,z) \leq N$.*

Note that the characterizations depend on $\left(\rho_y, \rho_z\right)$ only through their signs. Therefore, we will say that the preferences represented by U display *Nth-degree multiplicative-risk aversion (attraction) in the direction of* $\mathbb{R}_- \times \mathbb{R}_+$ in order to say that they display *Nth-degree multiplicative-risk aversion (attraction) in the direction of some* $\left(\rho_y, \rho_z\right) \in \mathbb{R}_- \times \mathbb{R}_+$ or equivalently *in the direction of all* $\left(\rho_y, \rho_z\right) \in \mathbb{R}_- \times \mathbb{R}_+$. We have then the following immediate corollary.

Corollary 3 *Let U be a given increasing, strictly concave and infinitely differentiable utility function satisfying Assumption A2. The following properties are equivalent:*

1. *For all initial endowment (K, α) and all asset cost and payoff $(p, \widetilde{\mu}_1)$ such that $x_1^* \geq 0$, any increase in Nth-degree risk over the asset's payoff from $\widetilde{\mu}_1$ to $\widetilde{\mu}_2$ increases (resp. decreases) the optimal level of the choice variable, i.e. $x_2^* \geq x_1^*$ (resp. $x_1^* \leq x_2^*$).*

2. *The preferences represented by U display Nth-degree multiplicative-risk attraction (resp. aversion) in the direction of $\mathbb{R}_- \times \mathbb{R}_+$.*

In other words, an individual will always decrease the demand for the asset in problem $P_{U,K,p}(\alpha, \widetilde{\mu})$ when the asset's payoff is subject to an increase in Nth-degree risk if and only if he or she always selects lottery B_m over lottery A_m. Similarly, an individual will always increase the demand for the asset in problem $P_{U,K,p}(\alpha, \widetilde{\mu})$ when the asset's payoff is subject to an increase in Nth-degree risk if and only if he or she always selects lottery A_m over lottery B_m. However, as it is clear from Proposition 4, no individual for which $(-1)^N U^{(0,N)}(y,z) \leq 0$ (e.g. for $N = 2$, risk averse individuals) can

[12] The case with $\rho_y \leq 0$ and $\rho_z \geq 0$ is the relevant scenario to interpret the conditions found in Section 3.

also display Nth-degree multiplicative-risk attraction in the direction of $\mathbb{R}_- \times \mathbb{R}_+$. Therefore, no such individual will always prefer lottery A_m over lottery B_m, or equivalently, no such individual will always increase the demand for the asset in problem $P_{U,K,p}(\alpha, \widetilde{\mu})$. We emphasize again that the "only if" part of Proposition 4 is the critical ingredient to provide a direct link between lottery choices and the optimal exposure to risk.

As an example, suppose that $P_{U,K,p}(\alpha, \widetilde{\mu})$ represents the classic labor supply problem with labor income risk, and consider a 2nd-degree risk increase. Given the higher risk, prudent individuals would like to mitigate the harm by increasing the supply of labor. Doing so, however, scales up the risk, which all risk-averse consumers dislike. In addition, a higher supply of labor implies that the higher risk is coupled with a lower level of leisure, and this is disliked by all cross prudent individuals. Whether the supply of labor increases or decreases then depends on the relative strength of these different forces. For consumers that are both risk averse and cross prudent, the supply of labor will always decrease if the coefficient of 2nd-degree relative risk aversion (i.e. relative prudence) is smaller than two. Equivalently, these consumers will always prefer lottery B_m over lottery A_m, where the attributes of the lotteries represent leisure and consumption, $(\rho_y, \rho_z) = (-1, 1)$, and $\widetilde{\alpha}_2 \succcurlyeq_2 \widetilde{\alpha}_1$. For all other risk averse individuals, the higher risk will have an ambiguous effect on the supply of labor, and their preference over lotteries B_m and A_m will depend on the initial endowments of leisure and consumption.

Remark. As mentioned above, Chiu et al. (2012) analyzed the univariate case with $\rho_y = 0$ and y fixed. They conclude that (Theorem 2): (using our notation) given $(-1)^n U^{(0,n)}(y,z) \leq 0$ for $n = N, N+1$, then $A_m \succ (\prec) B_m$ if and only if $(-1)^N \left(xU^{(0,N+1)}(y, x+z) + NU^{(0,N)}(y, x+z)\right) \geq (\leq) 0$ for all $x \geq 0$. As it is clear from the proof of Proposition 4, the second part of this statement is equivalent to our results when $\rho_y = 0$. We remark that the "only if" part follows from Lemma 1. Furthermore, our proof also clarifies that the assumption $(-1)^N U^{(0,N)}(y,z) \leq 0$ is not consistent with the condition $(-1)^N \left(xU^{(0,N+1)}(y, x+z) + NU^{(0,N)}(y, x+z)\right) \geq 0$. In fact, as stated in Proposition 4, the condition $(-1)^N U^{(0,N)}(y,z) \geq 0$ is necessary for an individual to always prefer A_m over B_m in an expected utility framework.

5 Concluding Remarks

Given the ubiquitous presence of uncertainty in most economic decisions, it is not surprising that a large amount of research has been devoted towards understanding the economic consequences of changes in risk. The work of Professor Eeckhoudt has been of fundamental importance towards that goal, raising and answering new questions and also looking for new answers to classical questions. Professor Eeckhoudt's work has also stimulated a large amount of research in this area and this paper has been a result of such inspiration.

This paper complements Professor Eeckhoudt's work in two important ways. First, we establish the minimum set of necessary and sufficient conditions for unambiguous comparative statics of changes in risk in the large class of problems involving bidimensional consequences. Second, we link these conditions with more primitive attitudes towards risk in the form of preferences over simple lottery pairs. In particular, we show that making unambiguous statements about the ordering of a particular class of lottery pairs is equivalent to making unambiguous statements regarding the optimal response to changes in risk in the problems under consideration.

In this paper, we considered separately the case in which the risk is additive and the case in which the risk is multiplicative. It would be interesting to analyze the case in which both risks are present and to evaluate the impact on optimal choice of bivariate stochastic dominance shifts. In this case, again, the recent work of Eeckhoudt and colleagues on a characterization of multivariate stochastic dominance that is connected with a basic preference to disaggregate harms (e.g. Denuit et al. 2010a) and on the consequences of shifts in the dependence structure of two random variables (Denuit et al. 2010b, Denuit et al. 2011) may provide some hints on how to generalize our results of Sections 3 and 4.

Appendix A. Increases in Nth-degree Risk

Proof of Lemma 1

The fact that 2. implies 1. results directly from Ekern (1980). For the sake of completeness we rederive it. Let $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ be such that $\widetilde{\alpha}_2 \succcurlyeq_N \widetilde{\alpha}_1$. We have

$$\begin{aligned} E\left[q(\widetilde{\alpha}_2)\right] - E\left[q(\widetilde{\alpha}_1)\right] &= \int_0^B q(x) dF_{\widetilde{\alpha}_2}(x) - \int_0^B q(x) dF_{\widetilde{\alpha}_1}(x) \\ &= \sum_{k=1}^{N} (-1)^{k-1} q^{(k-1)}(B) \left[F_{\widetilde{\alpha}_2}^{[k]}(B) - F_{\widetilde{\alpha}_1}^{[k]}(B)\right] \\ &\quad + \int_0^B (-1)^N q^{(N)}(x) \left[F_{\widetilde{\alpha}_2}^{[N]}(x) - F_{\widetilde{\alpha}_1}^{[N]}(x)\right] dx. \end{aligned}$$

Since q and its N first derivatives are continuous on $\mathbb{R}_+$, they are bounded on all compact subsets and all our integrals are well defined.

By definition, all the terms in the sum are equal to 0 and $F_{\widetilde{\alpha}_2}^{[N]}(x) - F_{\widetilde{\alpha}_1}^{[N]}(x) \geq 0$ on $[0, B]$. By 2., the integral is then nonnegative and $E[q(\widetilde{\alpha}_2)] \geq E[q(\widetilde{\alpha}_1)]$.

Let us now prove that 1. implies 2. Let q be an N times continuously differentiable function on $\mathbb{R}_+$ such that $E[q(\widetilde{\alpha}_2)] \geq E[q(\widetilde{\alpha}_1)]$ for all pair $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ such that $\widetilde{\alpha}_2 \succcurlyeq_N \widetilde{\alpha}_1$. Let, if it exists, α be a nonnegative real number such that $q^{(N)}(\alpha) \neq 0$ and let ε denote a positive real number such that $q^{(N)}(t) \neq 0$ for $t \in [\alpha, \alpha + \varepsilon]$. The sign of $q^{(N)}$ remains then constant on $[\alpha, \alpha + \varepsilon]$. Let $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ be two nonnegative bounded above random variables such that $\widetilde{\beta}_2 \succcurlyeq_N \widetilde{\beta}_1$. Let B be a common upper bound for $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ and let $\widetilde{\alpha}_1 = \frac{\varepsilon}{B}\widetilde{\beta}_1 + \alpha$ and $\widetilde{\alpha}_2 = \frac{\varepsilon}{B}\widetilde{\beta}_2 + \alpha$. The random variables $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ take their values in $[\alpha, \alpha + \varepsilon]$ and it is easy to check that $F_{\widetilde{\alpha}_i}^{[k]}(t) = \left(\frac{\varepsilon}{B}\right)^{k-1} F_{\widetilde{\beta}_i}^{[k]}\left(\frac{t-\alpha}{\varepsilon}B\right)$ for $k = 1, 2...$ and $i = 1, 2$. Therefore, $\widetilde{\alpha}_2 \succcurlyeq_N \widetilde{\alpha}_1$ which implies, by 1., that $E[q(\widetilde{\alpha}_2)] \geq E[q(\widetilde{\alpha}_1)]$. We have

$$\begin{aligned}
E[q(\widetilde{\alpha}_2)] - E[q(\widetilde{\alpha}_1)] &= \int_\alpha^{\alpha+\varepsilon} q(x) dF_{\widetilde{\alpha}_2}(x) - \int_\alpha^{\alpha+\varepsilon} q(x) dF_{\widetilde{\alpha}_1}(x) \\
&= \sum_{k=1}^{N} (-1)^{k-1} q^{(k-1)}(\alpha+\varepsilon) \left[F_{\widetilde{\alpha}_2}^{[k]}(\alpha+\varepsilon) - F_{\widetilde{\alpha}_1}^{[k]}(\alpha+\varepsilon)\right] \\
&\quad + \int_\alpha^{\alpha+\varepsilon} (-1)^N q^{(N)}(t) \left[F_{\widetilde{\alpha}_2}^{[N]}(t) - F_{\widetilde{\alpha}_1}^{[N]}(t)\right] dt.
\end{aligned}$$

By definition, we have $F_{\widetilde{\alpha}_2}^{[k]}(\alpha+\varepsilon) = F_{\widetilde{\alpha}_1}^{[k]}(\alpha+\varepsilon)$ for $k = 1, ..., N$ and $F_{\widetilde{\alpha}_2}^{[N]}(t) - F_{\widetilde{\alpha}_1}^{[N]}(t) \geq 0$ and the inequality is strict for some t in $[\alpha, \alpha + \varepsilon]$, and even on a neighborhood of t by continuity of $F_{\widetilde{\alpha}_1}^{[N]}$ and $F_{\widetilde{\alpha}_2}^{[N]}$. Since the sign of $q^{(N)}$ remains constant on $[\alpha, \alpha + \varepsilon]$, this gives that $(-1)^N q^{(N)}(t) > 0$ on $[\alpha, \alpha + \varepsilon]$ and $(-1)^N q^{(N)}(x) > 0$ for all x such that $q^{(N)}(x) \neq 0$ which completes the proof.

Proof of Proposition 1

Let us prove that 2. implies 1. Let $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ be such that $\widetilde{\alpha}_2 \succcurlyeq_N \widetilde{\alpha}_1$ and let $q(\alpha) = g(x_1^*, \alpha, \mu)$. We have $\frac{\partial^N g}{\partial \alpha^N}(x_1^*, \alpha, \mu) = -pU^{(1,N)}(K - x_1^* p, x_1^* \mu + \alpha) + \mu U^{(0,N+1)}(K - x_1^* p, x_1^* \mu + \alpha)$. By 2., we have $(-1)^N \frac{\partial^N g}{\partial \alpha^N}(x_1^*, \alpha, \mu) \geq 0$ and $(-1)^N q^{(N)}(\alpha) \geq 0$. Since $\widetilde{\alpha}_i$ is bounded above $(i = 1, 2)$, $x_1^* \mu + \widetilde{\alpha}_i$ is bounded and bounded away from zero and U and its $N + 1$ first derivatives are bounded on the convex hull of the set of values taken by $(K - x_1^* p, x_1^* \mu + \widetilde{\alpha}_i)$. The same applies to q on the convex hull of the set of values taken by $\widetilde{\alpha}_i$, $i = 1, 2$. By Lemma 1, this leads to $E[q(\widetilde{\alpha}_1)] \leq E[q(\widetilde{\alpha}_2)]$. By definition, we have $E[q(\widetilde{\alpha}_1)] = 0$, which gives $E[q(\widetilde{\alpha}_2)] \geq 0$ or $E[g(x_1^*, \widetilde{\alpha}_2, \mu)] \geq 0$. By concavity of U, it is easy to check that $E[g(x, \widetilde{\alpha}_2, \mu)]$ is a decreasing function of x. Since x_2^* is characterized by $E[g(x_2^*, \widetilde{\alpha}_2, \mu)] = 0$, we obtain that $x_2^* \geq x_1^*$.

Let us prove 1. implies 2. As in the proof of Lemma 1, we consider $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ two nonnegative random variables with a common upper bound B such that $\widetilde{\beta}_2 \succcurlyeq_N \widetilde{\beta}_1$. As above, we introduce the random variables $\widetilde{\alpha}_{1,\varepsilon} = \frac{\varepsilon}{B}\widetilde{\beta}_1 + \alpha^*$ and $\widetilde{\alpha}_{2,\varepsilon} = \frac{\varepsilon}{B}\widetilde{\beta}_2 + \alpha^*$ for some $\alpha^* > 0$ and some $\varepsilon > 0$. The random variables $\widetilde{\alpha}_{1,\varepsilon}$ and $\widetilde{\alpha}_{2,\varepsilon}$ take their values in $[\alpha^*, \alpha^* + \varepsilon]$ and $\widetilde{\alpha}_{2,\varepsilon} \succcurlyeq_N \widetilde{\alpha}_{1,\varepsilon}$. Let us consider a given real number γ and let us define the random variables $\widetilde{\gamma}_{i,\varepsilon}$, $i = 1, 2$, as lotteries giving $\widetilde{\alpha}_{i,\varepsilon}$ with probability ε and γ with probability $1-\varepsilon$. The constant γ provides an additional degree of freedom that will prove useful in order to control the pair $(K - x_1^* p, x_1^*\mu + \alpha)$. We have $\widetilde{\gamma}_{2,\varepsilon} \succcurlyeq_N \widetilde{\gamma}_{1,\varepsilon}$ as an immediate consequence of the stability of this order relation under probability mixtures. Let $x^*_{1,\varepsilon}$ and $x^*_{2,\varepsilon}$ be respectively the solutions of $P_{U,K,p}(\widetilde{\gamma}_{1,\varepsilon}, \mu)$ and $P_{U,K,p}(\widetilde{\gamma}_{2,\varepsilon}, \mu)$. By 1., we have $x^*_{2,\varepsilon} \geq x^*_{1,\varepsilon}$. By definition, we have $E\left[g(x^*_{1,\varepsilon}, \widetilde{\gamma}_{1,\varepsilon}, \mu)\right] = 0$ and $E\left[g(x^*_{2,\varepsilon}, \widetilde{\gamma}_{2,\varepsilon}, \mu)\right] = 0$. By concavity of U, g is a decreasing function of x and we have $E\left[g(x^*_{1,\varepsilon}, \widetilde{\gamma}_{2,\varepsilon}, \mu)\right] \geq 0$. Let q be defined by $q(\alpha) = g(x^*_{1,\varepsilon}, \alpha, \mu)$. We have

$$\begin{aligned}
E\left[q(\widetilde{\gamma}_{2,\varepsilon})\right] - E\left[q(\widetilde{\gamma}_{1,\varepsilon})\right] &= \left(E\left[q(\widetilde{\alpha}_{2,\varepsilon})\right] - E\left[q(\widetilde{\alpha}_{1,\varepsilon})\right]\right)\varepsilon \\
&= \sum_{k=1}^{N} (-1)^{k-1} q^{(k-1)}(\alpha^* + \varepsilon)\left[F^{[k]}_{\widetilde{\alpha}_2,P}(\alpha^* + \varepsilon) - F^{[k]}_{\widetilde{\alpha}_1,P}(\alpha^* + \varepsilon)\right]\varepsilon \\
&\quad + \varepsilon \int_{\alpha^*}^{\alpha^*+\varepsilon} (-1)^N q^{(N)}(t)\left[F^{[N]}_{\widetilde{\alpha}_2,P}(t) - F^{[N]}_{\widetilde{\alpha}_1,P}(t)\right] dt.
\end{aligned}$$

By construction, the left side of the equality is nonnegative, all the terms in the sum are equal to zero and $F^{[N]}_{\widetilde{\alpha}_2,P}(t) - F^{[N]}_{\widetilde{\alpha}_1,P}(t)$ is nonnegative and nonzero. Note that $\widetilde{\gamma}_{i,\varepsilon}$ is bounded and bounded away from zero for $i = 1, 2$. Therefore q and its first N derivatives are bounded on the convex hull of the set of values taken by $\widetilde{\gamma}_{i,\varepsilon}$, $i = 1, 2$. Therefore, $(-1)^N q^{(N)}(t)$ is nonnegative at least on a given subinterval of $[\alpha^*, \alpha^* + \varepsilon]$. Let α^*_ε be in $[\alpha^*, \alpha^* + \varepsilon]$ such that $(-1)^N q^{(N)}(\alpha^*_\varepsilon) \geq 0$. We have then

$$(-1)^{N+1} p U^{(1,N)}\left(K - x^*_{1,\varepsilon}p, x^*_{1,\varepsilon}\mu + \alpha^*_\varepsilon\right) + (-1)^N \mu U^{(0,N+1)}\left(K - x^*_{1,\varepsilon}p, x^*_{1,\varepsilon}\mu + \alpha^*_\varepsilon\right) \geq 0 \tag{6}$$

where $x^*_{1,\varepsilon}$ satisfies

$$E\left[-pU^{(1,0)}\left(K - x^*_{1,\varepsilon}p, x^*_{1,\varepsilon}\mu + \widetilde{\gamma}_{1,\varepsilon}\right) + \mu U^{(0,1)}\left(K - x^*_{1,\varepsilon}p, x^*_{1,\varepsilon}\mu + \widetilde{\gamma}_{1,\varepsilon}\right)\right] = 0$$

or

$$\varepsilon E\left[g(x^*_{1,\varepsilon}, \frac{\varepsilon}{B}\widetilde{\beta}_1 + \alpha^*, \mu)\right] + (1-\varepsilon) g(x^*_{1,\varepsilon}, \gamma, \mu) = 0. \tag{7}$$

Remark that, until now, α^*, K and γ have been arbitrarily chosen. Let us now choose them carefully in order to derive our result. Let (Y, Z) be arbitrary in $\left(\mathbb{R}^*_+\right)^2$. By our Inada condition,

$\lim_{z\to 0}\frac{U^{(0,1)}(Y,z)}{U^{(1,0)}(Y,z)}=\infty$ and $\lim_{z\to\infty}\frac{U^{(0,1)}(Y,z)}{U^{(1,0)}(Y,z)}=0$ which gives that there exists some $z^*>0$ such that $\frac{U^{(0,1)}(Y,z^*)}{U^{(1,0)}(Y,z^*)}=\frac{p}{\mu}$ or $-pU^{(1,0)}(Y,z^*)+\mu U^{(0,1)}(Y,z^*)=0$. We choose $x^*>0$ such that $\mu x^*<\inf(Z,z^*)$ and α^* and K are taken such that $\alpha^*=Z-\mu x^*>0$ and $K=Y+px^*$. Let γ be given by $\gamma=z^*-\mu x^*>0$. We have

$$\begin{aligned} g(x^*,\widetilde{\gamma},\mu) &= -pU^{(1,0)}(K-x^*p,x^*\mu+\gamma)+\mu U^{(0,1)}(K-x^*p,x^*\mu+\gamma)\\ &= -pU^{(1,0)}(Y,z^*)+\mu U^{(0,1)}(Y,z^*)=0. \end{aligned}$$

The solution of Equation (7) for $\varepsilon=0$ is then given by x^*. In a well chosen neighborhood of x^*, $x\longmapsto K-xp$ and $x\longmapsto x\mu+\frac{\varepsilon}{B}\widetilde{\beta}_1+\alpha^*$ are bounded and bounded away from 0. The functions $U^{(1,0)}$ and $U^{(0,1)}$ being continuously differentiable, the function $(x,\varepsilon)\longmapsto \varepsilon E\left[g(x,\frac{\varepsilon}{B}\widetilde{\beta}_1+\alpha^*,\mu)\right]+(1-\varepsilon)g(x,\widetilde{\gamma},\mu)$ is then differentiable with respect to x at $(x^*,0)$. Furthermore, the derivative of this last function with respect to x at $(x^*,0)$ is nonzero (concavity of U). The solution $x^*_{1,\varepsilon}$ of Equation (7) is then continuous with respect to ε in a neighborhood of 0 which gives $\lim_{\varepsilon\to 0}x^*_{1,\varepsilon}=x^*$. Furthermore, we clearly have $\lim_{\varepsilon\to 0}\alpha^*_\varepsilon=\alpha^*$. Taking the limit in Equation (6) when ε tends to 0, we obtain

$$(-1)^{N+1}pU^{(1,N)}(K-x^*p,x^*\mu+\alpha^*)+(-1)^N\mu U^{(0,N+1)}(K-x^*p,x^*\mu+\alpha^*)\geq 0$$

or, by construction

$$(-1)^{N+1}pU^{(1,N)}(Y,Z)+(-1)^N\mu U^{(0,N+1)}(Y,Z)\geq 0$$

Proof of Proposition 2

Let us prove that 2. implies 1. Let $(\widetilde{\mu}_1,\widetilde{\mu}_2)$ be such that $\widetilde{\mu}_2\succcurlyeq_N\widetilde{\mu}_1$ and let $q(\mu)=g(x_1^*,\alpha,\mu)$. We have $q^{(N)}(\mu)=-p\,(x_1^*)^N\,U^{(1,N)}(K-x_1^*p,x_1^*\mu+\alpha)+\mu\,(x_1^*)^N\,U^{(0,N+1)}(K-x_1^*p,x_1^*\mu+\alpha)+N\,(x_1^*)^{N-1}\,U^{(0,N)}(K-x_1^*p,x_1^*\mu+\alpha)$. Since we assumed that $x_1^*\geq 0$, by 2, we have

$(-1)^N\,p\,(x_1^*)^N\,U^{(1,N)}(K-x_1^*p,x_1^*\mu+\alpha)\leq 0$ and

$$\begin{aligned} & (-1)^N\left(x_1^*\mu U^{(0,N+1)}(K-x_1^*p,x_1^*\mu+\alpha)+NU^{(0,N)}(K-x_1^*p,x_1^*\mu+\alpha)\right)\\ = & (-1)^N\frac{x_1^*\mu}{x_1^*\mu+\alpha}\left((x_1^*\mu+\alpha)\,U^{(0,N+1)}(K-x_1^*p,x_1^*\mu+\alpha)+NU^{(0,N)}(K-x_1^*p,x_1^*\mu+\alpha)\right)\\ & +(-1)^N\frac{\alpha N}{x_1^*\mu+\alpha}U^{(0,N)}(K-x_1^*p,x_1^*\mu+\alpha)\\ \geq & \ 0 \end{aligned}$$

which gives $(-1)^N q^{(N)}(\mu) \geq 0$. Since $\widetilde{\mu}_i$ is bounded above $(i=1,2)$, $x_1^*\widetilde{\mu}_i+\alpha$ is bounded and bounded away from zero and U and its $N+1$ first derivatives are bounded on the convex hull of the set of values taken by $(K-x_1^*p, x_1^*\widetilde{\mu}_i+\alpha)$. The same applies to q on the convex hull of the set of values taken by $\widetilde{\mu}_i$, $i=1,2$. By Lemma 1, this leads to $E\left[q(\widetilde{\mu}_1)\right] \leq E\left[q(\widetilde{\mu}_2)\right]$. By definition, we have $E\left[q(\widetilde{\mu}_1)\right]=0$ which gives $E\left[q(\widetilde{\mu}_2)\right]=E\left[g(x_1^*,\alpha,\widetilde{\mu}_2)\right]\geq 0$. By concavity of U, it is easy to check that $g(x,\alpha,\mu)$ is a decreasing function of x. From there we derive that $x_2^* \geq x_1^*$.

Let us prove that 1. implies 2. As in the proof of Lemma 1 and Proposition 1, we consider $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ two nonnegative random variables with a common upper bound B such that $\widetilde{\beta}_2 \succcurlyeq_N \widetilde{\beta}_1$. As above, we introduce the random variables $\widetilde{\mu}_{1,\varepsilon}=\frac{\varepsilon}{B}\widetilde{\beta}_1+\mu^*$ and $\widetilde{\mu}_{2,\varepsilon}=\frac{\varepsilon}{B}\widetilde{\beta}_2+\mu^*$ for some $\mu^*>0$ and some $\varepsilon>0$. The random variables $\widetilde{\mu}_{1,\varepsilon}$ and $\widetilde{\mu}_{2,\varepsilon}$ take their values in $[\mu^*,\mu^*+\varepsilon]$ and $\widetilde{\mu}_{2,\varepsilon} \succcurlyeq_N \widetilde{\mu}_{1,\varepsilon}$. Let us consider a given γ and let us define the random variables $\widetilde{\gamma}_{i,\varepsilon}$, $i=1,2$, as a lottery that takes the value $\widetilde{\mu}_{i,\varepsilon}(\omega)$ with probability ε and the value γ with probability $1-\varepsilon$. As previously, we have $\widetilde{\gamma}_{2,\varepsilon} \succcurlyeq_N \widetilde{\gamma}_{1,\varepsilon}$. Let $x_{1,\varepsilon}^*$ and $x_{2,\varepsilon}^*$ be respectively the solutions of $P_{U,K,p}(\alpha,\widetilde{\gamma}_{1,\varepsilon})$ and $P_{U,K,p}(\alpha,\widetilde{\gamma}_{2,\varepsilon})$. If $x_{1,\varepsilon}^*$ is nonnegative then, by 1., we have $x_{2,\varepsilon}^* \geq x_{1,\varepsilon}^*$. By definition, we have $E\left[g(x_{1,\varepsilon}^*,\alpha,\widetilde{\gamma}_{1,\varepsilon})\right]=0$ and $E\left[g(x_{2,\varepsilon}^*,\alpha,\widetilde{\gamma}_{2,\varepsilon})\right]=0$. By concavity of U, g is a decreasing function of x and we have $E\left[g(x_{1,\varepsilon}^*,\alpha,\widetilde{\gamma}_{2,\varepsilon})\right]\geq 0$. Let q be defined by $q(\mu)=g(x_{1,\varepsilon}^*,\alpha,\mu)$. We have

$$
\begin{aligned}
E\left[q(\widetilde{\gamma}_{2,\varepsilon})\right]-E\left[q(\widetilde{\gamma}_{1,\varepsilon})\right] &= \left(E\left[q(\widetilde{\mu}_{2,\varepsilon})\right]-E\left[q(\widetilde{\mu}_{1,\varepsilon})\right]\right)\varepsilon \\
&= \sum_{k=1}^{N}(-1)^{k-1}q^{(k-1)}(\mu^*+\varepsilon)\left[F_{\widetilde{\mu}_2,P}^{[k]}(\mu^*+\varepsilon)-F_{\widetilde{\mu}_1,P}^{[k]}(\mu^*+\varepsilon)\right]\varepsilon \\
&\quad +\varepsilon\int_{\mu^*}^{\mu^*+\varepsilon}(-1)^N q^{(N)}(t)\left[F_{\widetilde{\mu}_2,P}^{[N]}(t)-F_{\widetilde{\mu}_1,P}^{[N]}(t)\right]dt.
\end{aligned}
$$

By construction, the left side of the equality is nonnegative, all the terms in the sum are equal to zero and $F_{\widetilde{\mu}_2,P}^{[N]}(t)-F_{\widetilde{\mu}_1,P}^{[N]}(t)$ is nonnegative and nonzero. Note that $\widetilde{\gamma}_{i,\varepsilon}$ is bounded and bounded away from zero for $i=1,2$. Therefore q and its first N derivatives are bounded on the convex hull of the set of values taken by $\widetilde{\gamma}_{i,\varepsilon}$, $i=1,2$. Therefore, $(-1)^N q^{(N)}$ is nonnegative at least on a given subinterval of $[\mu^*,\mu^*+\varepsilon]$. Let μ_ε^* be in $[\mu^*,\mu^*+\varepsilon]$ such that $(-1)^N q^{(N)}(\mu_\varepsilon^*)\geq 0$. We have then

$$
\begin{aligned}
&(-1)^{N+1}p\left(x_{1,\varepsilon}^*\right)^N U^{(1,N)}\left(K-x_{1,\varepsilon}^*p, x_{1,\varepsilon}^*\mu_\varepsilon^*+\alpha\right) \\
&+(-1)^N\mu_\varepsilon^*\left(x_{1,\varepsilon}^*\right)^N U^{(0,N+1)}\left(K-x_{1,\varepsilon}^*p, x_{1,\varepsilon}^*\mu_\varepsilon^*+\alpha\right) \\
&+(-1)^N N\left(x_{1,\varepsilon}^*\right)^{N-1} U^{(0,N)}\left(K-x_{1,\varepsilon}^*p, x_{1,\varepsilon}^*\mu_\varepsilon^*+\alpha\right) \\
\geq\ & 0
\end{aligned} \tag{8}
$$

where $x^*_{1,\varepsilon}$ satisfies

$$E\left[-pU^{(1,0)}\left(K - x^*_{1,\varepsilon}p, x^*_{1,\varepsilon}\widetilde{\gamma}_{1,\varepsilon} + \alpha\right) + \widetilde{\gamma}_{1,\varepsilon}U^{(0,1)}\left(K - x^*_{1,\varepsilon}p, x^*_{1,\varepsilon}\widetilde{\gamma}_{1,\varepsilon} + \alpha\right)\right] = 0$$

or

$$\varepsilon E\left[g(x^*_{1,\varepsilon}, \alpha, \frac{\varepsilon}{B}\widetilde{\beta}_1 + \mu^*)\right] + (1-\varepsilon)g(x^*_{1,\varepsilon}, \alpha, \gamma) = 0. \tag{9}$$

Remark that, until now p, μ^*, K, α and γ have been arbitrarily chosen. Let us now choose them carefully in order to derive our result. We assume first that γ is equal to 1. Let $x^* > 0$ be given and let (M, Y, Z) in $(\mathbb{R}_+)^3$ such that $M < Z$. We take $p = \frac{U^{(0,1)}(Y, x^*+Z-M)}{U^{(1,0)}(Y, x^*+Z-M)}$, $K = Y + px^*$, $\alpha = Z - M$ and $\mu^* = \frac{M}{x^*}$. By construction, we have $\alpha > 0$ and we have

$$\begin{aligned}
g(x^*, \alpha, \gamma) &= -pU^{(1,0)}\left(K - x^*p, x^*\gamma + \alpha\right) + \widetilde{\gamma}U^{(0,1)}\left(K - x^*p, x^*\gamma + \alpha\right) \\
&= -pU^{(1,0)}\left(K - x^*p, x^* + \alpha\right) + U^{(0,1)}\left(K - x^*p, x^* + \alpha\right) \\
&= -pU^{(1,0)}\left(Y, x^* + Z - M\right) + U^{(0,1)}\left(Y, x^* + Z - M\right) \\
&= 0.
\end{aligned}$$

The solution of Equation (9) for $\varepsilon = 0$ is then given by x^*. In a well chosen neighborhood of x^*, $x \longmapsto K - xp$ and $x \longmapsto x\left(\mu^* + \frac{\varepsilon}{B}\widetilde{\beta}_1\right) + \alpha$ are bounded and bounded away from 0. The functions $U^{(1,0)}$ and $U^{(0,1)}$ being continuously differentiable, the function $(x, \varepsilon) \longmapsto \varepsilon E\left[g(x, \alpha, \frac{\varepsilon}{B}\widetilde{\beta}_1 + \mu^*)\right] + (1-\varepsilon)g(x, \alpha, \gamma)$ is then differentiable with respect to x at $(x^*, 0)$. Furthermore, the derivative with respect to x at $(x^*, 0)$ is nonzero. The solution $x^*_{1,\varepsilon}$ of Equation (9) is then continuous with respect to ε in a neighborhood of 0 which gives $\lim_{\varepsilon\to 0} x^*_{1,\varepsilon} = x^*$ and guarantees that $x^*_{1,\varepsilon} > 0$ for ε small enough. Since we clearly have $\lim_{\varepsilon\to 0} \mu^*_\varepsilon = \mu^*$, taking the limit in Equation (8) when ε tends to 0, we obtain

$$(-1)^N (x^*)^{N-1}\left(-px^*U^{(1,N)}(Y, Z) + \mu^*x^*U^{(0,N+1)}(Y, Z) + NU^{(0,N)}(Y, Z)\right) \geq 0$$

or

$$(-1)^N\left(-px^*U^{(1,N)}(Y, Z) + MU^{(0,N+1)}(Y, Z) + NU^{(0,N)}(Y, Z)\right) \geq 0 \tag{10}$$

This result being true for all (Y, Z) in $\left(\mathbb{R}^*_+\right)^2$, all $M \in (0, Z)$ all $x^* > 0$ and for $p = \frac{U^{(0,1)}(Y, x^*+Z-M)}{U^{(1,0)}(Y, x^*+Z-M)}$. When x^* goes to 0, Y, M and Z being fixed, $x^*\frac{U^{(0,1)}(Y, x^*+Z-M)}{U^{(1,0)}(Y, x^*+Z-M)}$ goes to 0 and we obtain that

$$(-1)^N\left(MU^{(0,N+1)}(Y, Z) + NU^{(0,N)}(Y, Z)\right) \geq 0$$

for all (Y,Z) in $\left(\mathbb{R}_+^*\right)^2$ and all $M \in (0,Z)$ or equivalently

$(-1)^N \left(ZU^{(0,N+1)}(Y,Z) + NU^{(0,N)}(Y,Z)\right) \geq 0$ and $(-1)^N U^{(0,N)}(Y,Z) \geq 0$ for all (Y,Z) in $\left(\mathbb{R}_+^*\right)^2$.

We assumed that $z\frac{U^{(0,1)}(Y,z)}{U^{(1,0)}(Y,z)}$ is unbounded. We then have either $\lim_{z\to 0} z\frac{U^{(0,1)}(Y,z)}{U^{(1,0)}(Y,z)} = \infty$ or $\lim_{z\to\infty} z\frac{U^{(0,1)}(Y,z)}{U^{(1,0)}(Y,z)} = \infty$. If $\lim_{z\to 0} z\frac{U^{(0,1)}(Y,z)}{U^{(1,0)}(Y,z)} = \infty$, it suffices to take $x^* = \frac{\zeta}{2}$ and $M = Z - \frac{\zeta}{2}$ for ζ arbitrarily small to make the quantity $px^* = \frac{U^{(0,1)}(Y,x^*+Z-M)}{U^{(1,0)}(Y,x^*+Z-M)}x^* = \frac{1}{2}\zeta\frac{U^{(0,1)}(Y,\zeta)}{U^{(1,0)}(Y,\zeta)}$ arbitrarily large. Since M, Y and Z are bounded, Equation (10) gives then $(-1)^N U^{(1,N)}(Y,Z) \leq 0$. If $\lim_{z\to\infty} z\frac{U^{(0,1)}(Y,z)}{U^{(1,0)}(Y,z)} = \infty$, it suffices to take x^* sufficiently large to make the quantity x^*+Z-M sufficiently large and $\frac{U^{(0,1)}(Y,x^*+Z-M)}{U^{(1,0)}(Y,x^*+Z-M)}(x^*+Z-M)$ arbitrarily large. Since Z is kept fixed, $\frac{x^*}{x^*+Z-M}$ is arbitrarily close to 1 and px^* arbitrarily large. Since M, Y and Z are bounded, Equation (10) gives then $(-1)^N U^{(1,N)}(Y,Z) \leq 0$.

The Proof of Corollary 1 follows directly by reversing the inequalities above.

Proof of Proposition 3

In an expected utility framework, Nth-degree risk aversion in the direction of $\left(\rho_y, \rho_z\right)$ is equivalent to

$$\begin{aligned} & \frac{1}{2}E\left[U\left(y + x_2\rho_y, z + x_2\rho_z + \widetilde{\alpha}_2\right)\right] - \frac{1}{2}E\left[U\left(y + x_2\rho_y, z + x_2\rho_z + \widetilde{\alpha}_1\right)\right] \\ > \; & \frac{1}{2}E\left[U\left(y + x_1\rho_y, z + x_1\rho_z + \widetilde{\alpha}_2\right)\right] - \frac{1}{2}E\left[U\left(y + x_1\rho_y, z + x_1\rho_z + \widetilde{\alpha}_1\right)\right]. \end{aligned}$$

for all $(y,z,x_1,x_2) \in \mathbb{R}_+^4$ with $x_2 > x_1$. The previous inequation is satisfied for all $x_2 > x_1$ if and only if $E\left[U\left(y + t\rho_y, z + t\rho_z + \widetilde{\alpha}_2\right)\right] - E\left[U\left(y + t\rho_y, z + t\rho_z + \widetilde{\alpha}_1\right)\right]$ is increasing in t or, equivalently, if and only if $E\left[f\left(y + t\rho_y, z + t\rho_z + \widetilde{\alpha}_2\right)\right]$ is larger than $E\left[f\left(y + t\rho_y, z + t\rho_z + \widetilde{\alpha}_1\right)\right]$ for all $(y,z,t) \in \mathbb{R}_+^3$. This is satisfied if and only if $E\left[f\left(y, z + \widetilde{\alpha}_2\right)\right]$ is larger than $E\left[f\left(y, z + \widetilde{\alpha}_1\right)\right]$ for all $(y,z) \in \mathbb{R}_+^2$. Since we want this inequality to be true for all $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ such that $\widetilde{\alpha}_2$ is an increase in Nth-degree risk over $\widetilde{\alpha}_1$, this inequality is equivalent, by Lemma 1, to $(-1)^N f^{(0,N)} \geq 0$.

Proof of Proposition 4

In an expected utility model, Nth-degree multiplicative-risk attraction in the direction of $\left(\rho_y, \rho_z\right)$ is equivalent to

$$\begin{aligned} & \frac{1}{2}E\left[U\left(y + x_2\rho_y, z + x_2\rho_z\widetilde{\alpha}_2\right)\right] - \frac{1}{2}E\left[U\left(y + x_2\rho_y, z + x_2\rho_z\widetilde{\alpha}_1\right)\right] \\ > \; & \frac{1}{2}E\left[U\left(y + x_1\rho_y, z + x_1\rho_z\widetilde{\alpha}_2\right)\right] - \frac{1}{2}E\left[U\left(y + x_1\rho_y, z + x_1\rho_z\widetilde{\alpha}_1\right)\right] \end{aligned} \tag{11}$$

for all $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ such that $\widetilde{\alpha}_1 \preccurlyeq_N \widetilde{\alpha}_2$ and all (x_1, x_2) such that $0 < x_1 < x_2$ and $y + x_i\rho_y \geq 0$, $i = 1, 2$. This inequality is satisfied for all $x_2 > x_1$ if and only if $E\left[U\left(y + t\rho_y, z + t\rho_z\widetilde{\alpha}_2\right)\right] - E\left[U\left(y + t\rho_y, z + t\rho_z\widetilde{\alpha}_1\right)\right]$ is increasing on $T(y) = \left\{t \in \mathbb{R}_+ : y + t\rho_y \geq 0\right\}$ or, equivalently, if and only if $E\left[h\left(\widetilde{\alpha}_2, y, z, t\right)\right]$ is larger than $E\left[h\left(\tilde{\alpha}_1, y, z, t\right)\right]$ for all $(y, z) \in \mathbb{R}_+^2$ and for $t \in T(y)$ where $h(\alpha, y, z, t) = \rho_y U^{(1,0)}\left(y + t\rho_y, z + t\rho_z\alpha\right) + \alpha\rho_z U^{(0,1)}\left(y + t\rho_y, z + t\rho_z\alpha\right)$. By Lemma 1, this last inequality is satisfied for every pair $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ such that $\widetilde{\alpha}_2$ is an increase in Nth-degree risk over $\widetilde{\alpha}_1$ if and only if $(-1)^N \frac{\partial^N h}{\partial\alpha^N} \geq 0$.

Simple calculus gives $\frac{\partial^N h}{\partial\alpha^N} = t^N\rho_y\rho_z^N U^{(1,N)} + \alpha t^N\rho_z^{N+1}U^{(0,N+1)} + Nt^{N-1}\rho_z^N U^{(0,N)}$ where derivatives of U are taken at $\left(y + t\rho_y, z + \alpha t\rho_z\right)$. Since $t \geq 0$ and $\rho_z \geq 0$, our necessary and sufficient condition can then be rewritten as $(-1)^N\left(t\rho_y U^{(1,N)} + \alpha t\rho_z U^{(0,N+1)} + NU^{(0,N)}\right) \geq 0$ for all $(\alpha, y, z) \in \mathbb{R}_+^3$ and for $t \in T(y)$. Denoting $y + t\rho_y$ by Y and $z + \alpha t\rho_z$ by Z, our necessary and sufficient condition is equivalent to $(-1)^N\left((Y - y)U^{(1,N)} + (Z - z)U^{(0,N+1)} + NU^{(0,N)}\right) \geq 0$ for all $(Y, Z) \in \mathbb{R}_+^2$ and all (y, z) such that $y \geq Y$ and $0 \leq z \leq Z$ and where the derivatives are taken at (Y, Z). Taking successively $y = Y$ and $z = Z$, $y = Y$ and $z = 0$ we obtain that $(-1)^N U^{(0,N)} \geq 0$ and $(-1)^N\left(ZU^{(0,N+1)} + NU^{(0,N)}\right) \geq 0$. Letting y go to ∞, we obtain $(-1)^N U^{(1,N)} \leq 0$. Conversely, if these 3 conditions are satisfied and for a given $y \geq Y$, we have $(-1)^N (Y - y)U^{(1,N)} \geq 0$ and

$$\begin{aligned}
& (-1)^N\left((Z - z)U^{(0,N+1)} + NU^{(0,N)}\right) \\
= \; & \frac{Z - z}{Z}(-1)^N\left(ZU^{(0,N+1)} + NU^{(0,N)}\right) + \frac{z}{Z}(-1)^N NU^{(0,N)} \\
\geq \; & 0
\end{aligned}$$

which gives that our necessary and sufficient condition is satisfied.

A far as Nth-degree multiplicative-risk aversion is concerned, it is characterized by the fact that $E\left[U\left(y + t\rho_y, z + t\rho_z\widetilde{\alpha}_2\right)\right] - E\left[U\left(y + t\rho_y, z + t\rho_z\widetilde{\alpha}_1\right)\right]$ is decreasing on $T(y) = \left\{t \in \mathbb{R}_+ : y + t\rho_y \geq 0\right\}$ or, equivalently, by the fact that $E\left[-h\left(\widetilde{\alpha}_2, y, z, t\right)\right]$ is larger than $E\left[-h\left(\tilde{\alpha}_1, y, z, t\right)\right]$ for all $(y, z) \in \mathbb{R}_+^2$ and for $t \in T(y)$ or, finally, by $(-1)^{N+1}\frac{\partial^N h}{\partial\alpha^N} \geq 0$. The rest of the proof is then directly adapted from the multiplicative-risk attraction setting.

Appendix B. Nth-Degree Stochastic Dominance

In this appendix we generalize Lemma 1, Propositions 1 and 2, and Corollary 1. We begin by formalizing the concept of Nth-degree stochastic dominance.

Definition 4 (Jean) *Let* $\widetilde{\alpha}_1$ *and* $\widetilde{\alpha}_2$ *denote two random variables with values in* $[0, B]$. *We say that* $\widetilde{\alpha}_2$ *is dominated in the sense of Nth-degree stochastic dominance by* $\widetilde{\alpha}_1$, *and we denote it by* $\widetilde{\alpha}_2 \succcurlyeq_{NSD} \widetilde{\alpha}_1$, *if* $F^{[N]}_{\widetilde{\alpha}_2}(x) \geq F^{[N]}_{\widetilde{\alpha}_1}(x)$ *for all* $x \in [0, B]$ *where the inequality is strict for some* x *and* $F^{[k]}_{\widetilde{\alpha}_2}(B) \geq F^{[k]}_{\widetilde{\alpha}_1}(B)$ *for* $k = 1, ..., N-1$.

Jean (1980) characterizes Nth-degree stochastic dominance: He establishes that $\widetilde{\alpha}_2$ is dominated in the sense of Nth-degree stochastic dominance by $\widetilde{\alpha}_1$ if and only if $E[q(\widetilde{\alpha}_2)] > E[q(\widetilde{\alpha}_1)]$ for all N times continuously differentiable real valued function q such that $(-1)^k q^{(k)} > 0$ for $k = 1, ..., N$ where $q^{(k)} = \frac{d^k q}{d\alpha^k}$. The following Lemma characterizes the set of N times continuously differentiable functions for which $E[q(\widetilde{\alpha}_2)] \geq E[q(\widetilde{\alpha}_1)]$ for all pair $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ where $\widetilde{\alpha}_2 \succcurlyeq_{NSD} \widetilde{\alpha}_1$.

Lemma 2 *Let* q *be a given real valued function that is* N *times continuously differentiable on* $\mathbb{R}_+$. *The following are equivalent.*

1. *For all pair* $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ *such that* $\widetilde{\alpha}_2 \succcurlyeq_{NSD} \widetilde{\alpha}_1$, *we have* $E[q(\widetilde{\alpha}_2)] \geq E[q(\widetilde{\alpha}_1)]$.

2. *For all* $x \geq 0$, *we have* $(-1)^k q^{(k)} \geq 0$ *for* $k = 1, ..., N$.

Proof. The fact that 2. implies 1. results directly from Jean (1980). Let us now prove that 1. implies 2. Let q be an N times continuously differentiable function on $\mathbb{R}_+$ such that $E[q(\widetilde{\alpha}_2)] \geq E[q(\widetilde{\alpha}_1)]$ for all pair $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ such that $\widetilde{\alpha}_2 \succcurlyeq_{NSD} \widetilde{\alpha}_1$. Let k be in $\{1, ..., N\}$ and let $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ such that $\widetilde{\alpha}_2 \succcurlyeq_k \widetilde{\alpha}_1$. By definition, we have $F^{[k]}_{\widetilde{\alpha}_2}(x) \geq F^{[k]}_{\widetilde{\alpha}_1}(x)$ for all $x \in [0, B]$ where the inequality is strict for some x and $F^{[i]}_{\widetilde{\alpha}_2}(B) = F^{[i]}_{\widetilde{\alpha}_1}(B)$ for $i = 1, ..., k-1$. By integration, we obtain $F^{[j]}_{\widetilde{\alpha}_2}(x) \geq F^{[j]}_{\widetilde{\alpha}_1}(x)$ for all $j \geq k$ and all x in $[0, B]$ and, in particular, $\widetilde{\alpha}_2 \succcurlyeq_{NSD} \widetilde{\alpha}_1$ hence $E[q(\widetilde{\alpha}_2)] \geq E[q(\widetilde{\alpha}_1)]$. By Lemma 2 we have $(-1)^k q^{(k)} \geq 0$ on $[0, B]$ for $k = 1, ..., N$.■

The following Proposition establishes the effect on optimal choice of an Nth-degree stochastically dominated shift on the initial endowment α.

Proposition 5 *Let* U *be a given increasing, strictly concave and infinitely differentiable utility function satisfying Assumption A1. Let us consider* p *and* μ *as given. The following properties are equivalent:*

1. *For all initial endowment* $(K, \widetilde{\alpha}_1)$, *a switch from* $\widetilde{\alpha}_1$ *to* $\widetilde{\alpha}_2$ *such that* $\widetilde{\alpha}_2 \succcurlyeq_{NSD} \widetilde{\alpha}_1$ *increases the optimal level of the choice variable, i.e.* $x^*_2 \geq x^*_1$.

2. *For all* (y, z), *we have* $(-1)^k \left(-pU^{(1,k)}(y, z) + \mu U^{(0,k+1)}(y, z)\right) \geq 0$ *for* $k = 1, ..., N$.

Proof. Let us prove that 2. implies 1. Let $(\widetilde{\alpha}_1, \widetilde{\alpha}_2)$ be such that $\widetilde{\alpha}_2 \succcurlyeq_{NSD} \widetilde{\alpha}_1$ and let $q(\alpha) = g(x_1^*, \alpha, \mu)$. By 2., we have $(-1)^k q^{(k)}(\alpha) \geq 0$ for $k = 1, ..., N$. By Lemma 2, this leads to $E[q(\widetilde{\alpha}_1)] \leq E[q(\widetilde{\alpha}_2)]$. By definition, we have $E[q(\widetilde{\alpha}_1)] = 0$, which gives $E[q(\widetilde{\alpha}_2)] \geq 0$ or $E[g(x_1^*, \widetilde{\alpha}_2, \mu)] \geq 0$. By concavity of U, it is easy to check that $E[g(x, \widetilde{\alpha}_2, \mu)]$ is a decreasing function of x. Since x_2^* is characterized by $E[g(x_2^*, \widetilde{\alpha}_2, \mu)] = 0$, we obtain that $x_2^* \geq x_1^*$.

Let us prove 1. implies 2. As in the proof of the Lemma 2, it suffices to remark that, for $k = 1, ..., N$, $\widetilde{\alpha}_2 \succcurlyeq_k \widetilde{\alpha}_1$ implies $\widetilde{\alpha}_2 \succcurlyeq_{NSD} \widetilde{\alpha}_1$.■

Finally, the following Proposition establishes the effect on optimal choice of an Nth-degree stochastically dominated shift on the multiplicative variable μ.

Proposition 6 *Let U be a given increasing, strictly concave and infinitely differentiable utility function satisfying Assumption A2. The following properties are equivalent:*

1. *For all initial endowment (K, α) and all asset cost and payoff $(p, \widetilde{\mu}_1)$ such that $x_1^* \geq 0$, and any switch from $\widetilde{\mu}_1$ to $\widetilde{\mu}_2$ with $\widetilde{\mu}_2 \succeq_{NSD} \widetilde{\mu}_1$ increases (decreases) the optimal level of the choice variable, i.e. $x_2^* \geq x_1^*$.*

2. *For all (y, z), we have $(-1)^k U^{(1,k)}(y, z) \leq 0$ (resp. ≥ 0) $R^k(y, z) \leq N$ and $(-1)^k U^{(0,k)}(y, z) \geq 0$ (resp. ≤ 0) for $k = 1, ..., N$.*

Proof. Follows the proof of Proposition 5. ■

References

[1] Baiardi, D., Menegatti, M. (2011). Pigouvian Tax, Abatement Policies and Uncertainty on the Environment, Journal of Economics 103 (3), 221-251.

[2] Block, M.K., Heineke J.M. (1973) The allocation of effort under uncertainty: the case of risk-averse behavior. Journal of Political Economy 81, 376–385.

[3] Chiu, W.H., Eeckhoudt L. (2010) The effects of stochastic wages and non-labor income on labor supply: update and extensions. Journal of Economics 100, 69-83.

[4] Chiu, W.H., Eeckhoudt, L., Rey, B. (2012) On relative and partial risk attitudes: Theory and Implications. Economic Theory 50, 151-167.

[5] Courbage, C., Rey, B. (2012) Priority Setting in Health Care and Higher Order Degree Change in Risk. Journal of Health Economics 31, 484-489.

[6] Dardanoni V. (1988) Optimal choices under uncertainty: the case of two-argument utility functions. Economic Journal 98, 429–450.

[7] Dardanoni, V., Wagstaff, A. (1990) Uncertainty and the demand for medical care. Journal of Health Economics 9, 23–38.

[8] Denuit, M., De Vylder, E., Lefevre, C. (1999) Extremal generators and extremal distributions for the continuous s-convex stochastic orderings. Insurance: Mathematics and Economics 24, 201-217.

[9] Denuit, M., Eeeckhoudt, L., Tsetlin, I., Winkler, R.L. (2010a) Multivariate concave and convex stochastic dominance. INSEAD working paper 2010/29.

[10] Denuit, M., Eeckhoudt, L., Rey, B. (2010b). Some consequences of correlation aversion in decision science. Annals of Operations Research 176, 259-269.

[11] Denuit, M., Eeckhoudt, L., Menegatti, M. (2011) Correlated risks, bivariate utility and optimal choices. Economic Theory 46, 39-54.

[12] Eeckhoudt, L., Schlesinger, H. (2006) Putting risk in its proper place. American Economic Review 96, 280-289.

[13] Eeckhoudt, L., Schlesinger, H. (2008) Changes in risk and the demand for saving. Journal of Monetary Economics 55, 1329-1336.

[14] Eeckhoudt, L., Schlesinger, H., Tselin, I. (2009). Apportioning of risks via stochastic dominance. Journal of Economic Theory 144, 994-1003.

[15] Eeckhoudt, L., Rey, B., Schlesinger, H. (2007). A good sign for multivariate risk taking. Management Science 53, 117-124.

[16] Ekern, S. (1980). Increasing Nth-degree risk. Economics Letters 6, 329–333.

[17] Gollier, C. (2001) The Economics of Risk and Time. M.I.T. Press.

[18] Hadar, J., Seo, T.K. (1990) The effects of shifts in a return distribution on optimal portfolios. International Economic Review 31, 721-736.

[19] Jean, W.H. (1980) The geometric mean and stochastic dominance. Journal of Finance 35, 151-158.

[20] Keenan, D.C., Kim, I., Warren, R.S. (2006) The Private Provision of Public Goods under Uncertainty: A Symmetric-Equilibrium Approach. Journal of Public Economic Theory 8, 863-873.

[21] Kimball, M. (1990) Precautionary saving in the small and in the large. Econometrica 58, 58-73.

[22] Leland, H. E. (1968) Saving and uncertainty: The precautionary demand for saving. Quarterly Journal of Economics 82, 465-73.

[23] Menezes, C., Geiss, C., Tressler, J. (1980). Increasing Downside Risk. American Economic Review 70, 921-932.

[24] Menezes, C. Wang, X.H. (2005) Increasing outer risk. Journal of Mathematical Economics 41, 875–886.

[25] Rothschild, M., Stiglitz, J. (1970), "Increasing risk I: A definition." Journal of Economic Theory 2, 225-243.

[26] Rothschild, M., Stiglitz, J. (1971), "Increasing risk II: Its Economic Consequences." Journal of Economic Theory 3, 66-84.

[27] Sandler, T., Sterbenz, F.P., Posnett, J. (1987) Free riding and uncertainty. European Economic Review 31, 1605–1617.

[28] Sandmo, A. (1970) The effect of uncertainty on saving decisions. Review of Economic Studies 37, 353-60.

[29] Tressler, J.H., Menezes, C.F. (1980) Labor supply and wage rate uncertainty. Journal of Economic Theory 23, 425–436.