Empirical Projected Copula Process and Conditional Independence An Extended Version

Lorenzo Frattarolo, Dominique Guegan

Documents de Travail du
Centre d'Economie de la Sorbonne

CES Working Papers

Empirical Projected Copula Process and Conditional Independence An Extended Version

Lorenzo FRATTAROLO, Dominique GUEGAN

2013.68

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://centredeconomiesorbonne.univ-paris1.fr/bandeau-haut/documents-de-travail/
ISSN : 1955-611X

Empirical Projected Copula Process and Conditional Independence An Extended Version

Lorenzo Frattarolo[a,b], Dominique Gueǵan[b]

[a] *University Ca Foscari of Venice, Department of Economics, Fondamenta San Giobbe, Cannareggio 873, 30121 Venice, Italy*

[b] *University Paris-1 Pantheón-Sorbonne, MSE-CES, 106, avenue de l' Hopital, 75013 Paris, France*

Abstract

Conditional dependence is expressed as a projection map in the trivariate copula space. The projected copula, its sample counterpart and the related process are defined. The weak convergence of the projected copula process to a tight centered Gaussian Process is obtained under weak assumptions on copula derivatives.

Keywords: Conditional Independence, Empirical Process, Weak convergence, Copula

1. Introduction

In this note, our purpose is to introduce the empirical projected copula process and explain its importance in non parametric conditional independence testing. The central role of conditional independence in statistical theory was first adressed in the paper of Dawid (Dawid (1979)) in which he rephrases sufficency, ancillarity, exogeneity, identification, causal inference and other relevant statistical concepts in terms of conditional independence. In this paper, he also introduced the usual notation for conditional independence that we will use all along this note:

$$\begin{aligned} X_1 \perp\!\!\!\perp X_2 | X_3 \quad &\Leftrightarrow \\ \mathbb{P}\left(X_1 \leq x_1, X_2 \leq x_2 \left| X_3 = x_3 \right.\right) \quad &= \quad \mathbb{P}\left(X_1 \leq x_1 \left| X_3 = x_3 \right.\right) \mathbb{P}\left(X_2 \leq x_2 \left| X_3 = x_3 \right.\right). \end{aligned} \tag{1}$$

Preprint submitted to Statistics & Probability Letters *October 11, 2013*

Conditional independence is a fundamental tool for probabilistic graphical models(PGM), Koller (2009) and a proper understanding of Granger causality White and Lu (2008). To investigate Conditional Independence both non parametric copulas estimators and empirical distributions have already been used independently in the literature: Bouezmarni et Al. (2012) consider empirical Bernstein copulas; Gyorfi and Walk (2012) develop a strong consistent test based on empirical distribution function. In this note, using conditional copulas with three variables we rephrase conditional independence for copula functions, Darsow et Al. (1992). The outcome is the definition of a projection map, the projected copula and the related empirical process. Then, considering recent advances in the study of empirical copula process Segers (2012), Bucher (2011) we obtain the weak convergence of the projected empirical copula process to a tight centered Gaussian process, under weak assumptions for second derivatives in the conditioning argument.
The paper is organised as follows. We inroduce some notations and assumptions in Section 2, then, in Section 3, we develop the relationship (1) using copulas and introduce the projected copula, showing that it is the proper representation of conditional independence in the copula space, and we introduce and prove the weak convergence of the projected empirical copula process. Section 4 is devoted to the conclusion and possible extensions. The technical lemma 1 and application to finite difference derivatives are postponed to the appendix.

2. Notation and Assumptions

In this section we define some notations and introduce the assumptions needed for our main theorem. Thorough the paper, arguments in boldface are collection of single arguments, for example $\mathbf{x} \equiv \{x_i\}_{i=1}^{k} \equiv x_1, \ldots, x_k$, the maximum between a and b is $a \vee b$ and the minimum is $a \wedge b$. The n-th partial derivative in the arguments $\{x_{i_k}\}_{k=1}^{n}$ is written $\partial^{(n)}_{i_1 \ldots i_n} \equiv \frac{\partial^n}{\partial x_{i_1} \ldots \partial x_{i_n}}$ and when the arguments are all equal i abbreviated in $\partial^{(n)}_{i} \equiv \frac{\partial^n}{\partial x_{i} \ldots \partial x_{i}}$. For the first derivative the one is omitted. $\partial^{(1)}_{i} \equiv \partial_i$. The indicator function on the set A is $\mathbb{I}(A)$, the space of all bounded functions defined on A is $\ell^{\infty}(A)$ and the space of k-times differentiable functions on A is $C^k(A)$. Weak convergence and convergence in outer probability in Hoffman-Jorgesen sense van der Vaart (1996) are denoted respectively by $\rightsquigarrow$ and $\xrightarrow{\mathbb{P}^*}$.We use o and O Landau symbols and their stochastic counterparts as defined in van der Vaart (1998). Given 3 ran-

dom variables $\{X_i\}_{i=1}^3$, with marginals $\mathbb{P}(X_i \le x_i) = F_i(x_i)$, $i = 1, 2, 3$ and joint cumulative distribution $\mathbb{P}(X_1 \le x_1, X_2 \le x_2, X_3 \le x_3) = F(x_1, x_2, x_3)$ by Sklar' s theorem in three dimensions Nelsen (2006)
we know it exists a copula function $C : [0,1]^3 \mapsto [0,1]$ such that $F(x_1, x_2, x_3) = C(F_1(x_1), F_2(x_2), F_3(x_3))$. Any 3-variate copula is a 3-variate distribution function with uniform marginals. We denote the space of all such functions by $\mathfrak{C}_3$. Following Darsow et Al. (1992) we define the conditional copulas of the first 2 variables given the third one $C_{U_1,U_2|U_3}(u_1, u_2 | u_3) = \partial_3 C(u_1, u_2, u_3)$. Let $(X_{11}, X_{21}, X_{31}) \ldots (X_{1N}, X_{2N}, X_{3N})$ be a random sample, distributed according to F, the empirical distribution function and its margins are $\hat{F}_N(x_1, x_2, x_3) = \frac{1}{N} \sum_{j=1}^{N} \prod_{i=1}^{3} \mathbb{I}(X_{ij} \le x_i)$ and $\hat{F}_{Ni}(x_i) = \frac{1}{N} \sum_{j=1}^{N} \mathbb{I}(X_{ij} \le x_i)$.
The empirical copula is $\hat{C}_N(u_1, u_2, u_3) = \frac{1}{N} \sum_{j=1}^{N} \prod_{i=1}^{3} \mathbb{I}\left(\hat{U}_{Nij} \le u_i\right)$ with the pseudo-observations given by $\hat{U}_{Nij} = \hat{F}_{Ni}(X_{ij})$. Several authors (Segers (2012),Bucher (2011) and references therein) have studied - both in the iid and strongly mixing case - the weak convergence of the associated empirical process $\hat{\mathbb{C}}_N(u_1, u_2, u_3) = \sqrt{N}\left(\hat{C}_N(u_1, u_2, u_3) - C(u_1, u_2, u_3)\right)$, and under the assumption:

A 1. *Segers (2012) For each $j \in \{1, 2, 3\}$, the jth first-order partial derivative $\partial_j C$ exists and is continuous on the set $V_{3,j} := \left\{u \in [0,1]^3 : 0 < u_j < 1\right\}$.*

they prove:

$$\hat{\mathbb{C}}_N(u_1, u_2, u_3) \rightsquigarrow \mathbb{C}(u_1, u_2, u_3) = \alpha_C(u_1, u_2, u_3) + \sum_{i=1}^{3} \beta_{iC}(u_i) \partial_i C(u_1, u_2, u_3) \quad (2)$$

where $\alpha_C(u_1, u_2, u_3)$ is a C-Brownian Bridge on $[0,1]^3$ and $\beta_{iC}(u_i)$ are its margins.
To obtain our result, we need an additional assumption on the second derivative in the conditioning argument.

A 2. *The function $u_3 \mapsto \partial_3^{(2)} C(u_1, u_2, u_3) \in C^0([0,1])$, $\forall u_1, u_2 \in [0,1]^2$*

In addition be able to define the sample version of the projection, that we are going to introduce in the next section, we need a functional map $\tilde{\partial}_i^{N(n)}$:

$\ell^\infty\left([0,1]^3\right) \mapsto \ell^\infty\left([0,1]^3\right)$, satisfying several hypothesis in order to properly approximate derivatives and obtain weak convergence. All the limits are for $N \to \infty$. The first assumption guaranties that when $\tilde{\partial}_i^{N(n)}$ is applied to the subspace of functions for which the partial derivative exists: it is an uniform approximation of this derivative.

A 3. *For all $G \in \ell^\infty\left([0,1]^3\right)$ s.t. $\exists \partial_i^{(n)} G$ for $i = 1,2,3$:*

$$\sup_{\mathbf{u}\in[0,1]^3} \left|\tilde{\partial}_i^{N(n)} G(\mathbf{u}) - \partial_i^{(n)} G(\mathbf{u})\right| \leq R^N_{\tilde{\partial}^{(n)}}, \ \lim_{N\to\infty} R^N_{\tilde{\partial}^{(n)}} = 0$$

The next hypothesis guaranties that $\tilde{\partial}_i^{N(n)}$ when applied to the empirical copula is a consistent estimator of copula derivatives.

A 4. *For any copula:* $\sup_{\mathbf{u}\in[0,1]^3} \left|\tilde{\partial}_i^{N(n)} \hat{C}_N(\mathbf{u}) - \partial_i^{(n)} C(\mathbf{u})\right| \overset{\mathbb{P}^*}{\to} 0$ *for $i = 1,2,3$*

The following one allows the asymptotic integration by part in the conditioning argument. We need this assumption in order to avoid the derivation of a Gaussian process.

A 5. *Given $f : u_3 \mapsto f(\mathbf{u}') = f(u_1', u_2', u_3)$, s.t. $f \in C^1([0,1])$, $u_1', u_2' \in [0,1]^2$*

$$\sup_{\mathbf{u}\in[0,1]^3} \left|J\left(f, \hat{\mathbb{C}}_N\right)\right| \overset{\mathbb{P}^*}{\to} 0, \forall a \in [0,1]$$

$$J\left(f, \hat{\mathbb{C}}_N\right) = \int_0^a f(\mathbf{u}')\, \tilde{\partial}_3^{N(1)} \hat{\mathbb{C}}_N(\mathbf{u}) + \tilde{\partial}_3^{N(1)} f(\mathbf{u}')\, \hat{\mathbb{C}}_N(\mathbf{u})\, du_3 - f(\mathbf{u}')\, \hat{\mathbb{C}}_N(\mathbf{u})\Big|_{u_3=0}^{u_3=a}$$

The last one is a technical assumption on the rate of convergence of integrated difference between true derivative and its approximation, when we apply it to the true copula.

A 6. *When $N \to \infty$:* $\sqrt{N}\int_0^{u_i}\left(\tilde{\partial}_i^{N(n)} C(\mathbf{u}) - \partial_i^{(n)} C(\mathbf{u})\right) du_i \to 0$

3. Projection and Weak Convergence

In this section we introduce the projection map in $\mathfrak{C}_3$, show that conditional independence is equivalent to invariance with respect to this map and obtain, in theorem 1, the weak convergence and asymptotic normality of the

empirical projected copula process.
Using the notion of conditional copulas, relationship (1) can be rewritten:

$$C\left(F_1\left(x_1\right), F_2\left(x_2\right), F_3\left(x_3\right)\right) = \int_0^{F_3(x_3)} \partial_3 C\left(F_1\left(x_1\right), 1, v_3\right) \partial_3 C\left(1, F_2\left(x_2\right), v_3\right) dv_3$$

Thus $X_1 \perp\!\!\!\perp X_2 | X_3$ is equivalent to invariance with respect to the map $\Pi_{|3} : \mathfrak{C}_3 \mapsto \mathfrak{C}_3$ where:

$$\Pi_{|3}\left(C\left(u_1, u_2, u_3\right)\right) = \int_0^{u_3} \partial_3 C\left(u_1, 1, v_3\right) \partial_3 C\left(1, u_2, v_3\right) dv_3 \tag{3}$$

The map $\Pi_{|3}$ is the map projection onto X_3 and the right hand side of (3) is the projected copula. The map is not new as it can be rephrased using the $\star$-product of Darsow et Al. (1992) from which follows the fact that the projected copula is always in $\mathfrak{C}_3$.Analogously,the empirical projected copula is defined as:

$$\hat{\Pi}_{N|3}\left(\hat{C}_N\left(u_1, u_2, u_3\right)\right) = \int_0^{u_3} \tilde{\partial}_3^{N(1)} \hat{C}_N\left(u_1, 1, v_3\right) \tilde{\partial}_3^{N(1)} \hat{C}_N\left(1, u_2, v_3\right) dv_3,$$

From wich follows the definition of the empirical projected copula process:

$$\hat{\mathbb{C}}_{N|3} = \sqrt{N}\left(\hat{\Pi}_{N|3}\left(\hat{C}_N\left(u_1, u_2, u_3\right)\right) - \Pi_{|3}\left(C\left(u_1, u_2, u_3\right)\right)\right) \tag{4}$$

We are now in the position to state our main theorem:

Theorem 1. *Under **A1-A6**,*

$$\begin{aligned} \hat{\mathbb{C}}_{N|3}\left(u_1, u_2, u_3\right) &\rightsquigarrow \mathbb{C}_{|3}\left(u_1, u_2, u_3\right) = \\ \partial_3 C\left(u_1, 1, u_3\right) \mathbb{C}\left(1, u_2, u_3\right) &- \int_0^{u_3} \partial_3^{(2)} C\left(u_1, 1, v_3\right) \mathbb{C}\left(1, u_2, v_3\right) dv_3 \\ +\partial_3 C\left(1, u_2, u_3\right) \mathbb{C}\left(u_1, 1, u_3\right) &- \int_0^{u_3} \partial_3^{(2)} C\left(1, u_2, v_3\right) \mathbb{C}\left(u_1, 1, v_3\right) dv_3 \end{aligned} \tag{5}$$

We remark that since $\mathbb{C}_{|3}$ is a linear combination of Gaussian processes it is also a Gaussian process.

Proof of Theorem 1. Consider the map $\Psi_N : \ell^\infty [0,1]^3 \mapsto \ell^\infty [0,1]^3$

$$\Psi_N (f) = \partial_3 C (u_1, u_2, u_3) f (u_1, u_2, u_3) - \int_0^{u_3} dv_3 \tilde{\partial}_3^{N(1)} \partial_3 C (u_1, u_2, v_3) f (u_1, u_2, v_3)$$

and the map $\Psi : \ell^\infty [0,1]^3 \mapsto \ell^\infty [0,1]^3$

$$\Psi (f) = \partial_3 C (u_1, u_2, u_3) f (u_1, u_2, u_3) - \int_0^{u_3} dv_3 \partial_3^{(2)} C (u_1, u_2, v_3) f (u_1, u_2, v_3) .$$

Under **A3-A6**,using lemma 1, we have:

$$\hat{\mathbb{C}}_{N|3} = \Psi_N \left(\hat{\mathbb{C}}_N (u_1, 1, v_3) \right) + \Psi_N \left(\hat{\mathbb{C}}_N (1, u_2, v_3) \right) + o_{\mathbb{P}^*} (1)$$

Because $\Psi_N (f_N) \to \Psi (f)$ whenever $f_N \to f$, under **A1**, we have (2) and under **A2**, $\Psi (f)$ is continous, the hypotesis of the extended continous mapping theorem (1.11.1) in van der Vaart (1996) pg. 67 are satisfied and the result follows by the application of the theorem to $\Psi_N \left(\hat{\mathbb{C}}_N (u_1, 1, v_3) \right) + \Psi_N \left(\hat{\mathbb{C}}_N (1, u_2, v_3) \right)$.

4. Conclusion

The objective of this short note is to lay the theoretical foundation for a new non parametric test of conditional independence based on invariance principle with respect to a projection map. The novelty of our approach is in defining a sample estimator, for the projection, using empirical copula processes. This makes our results valid under very general assumption and widens the range of applications for our findings. For what concern our hypothesis, a closer look to the example section of Segers (2012) reveals that the discontinuity at the boundaries of the most common copula first derivatives occurs only when two or more arguments are involved in the limit so our hypothesis of continuity of the second partial derivative only in the conditioning argument are verified for most of the examples. We are only more restictive in considering twice continously differentiable Archimedean copula generators and dependence function of extreme value copula twice continously differentiable in each argument. For what concerns derivative approximations, in the appendix is shown that finite difference approximation as in Genest et al. (2011) are copula consistent approximation. With a second order copula consistent derivative approximation, using the multiplier central limit theorem van der Vaart (1996) as in Bucher and Dette (2010) it is possible to evaluate through simulation the limit process distribution. This would be done in a different paper.

Appendix A.

Appendix A.1. Technical Lemma

In this section we state and prove the technical Lemma 1 needed for our main theorem.

Lemma 1. *let $\Psi_N : \ell^\infty [0,1]^3 \mapsto \ell^\infty [0,1]^3$ be the map:*

$$\Psi_N(f) = \partial_3 C(u_1,u_2,u_3) f(u_1,u_2,u_3) - \int_0^{u_3} dv_3 \tilde{\partial}_3^{N(1)} \partial_3 C(u_1,u_2,v_3) f(u_1,u_2,v_3)$$

*then, under **A3-A6**, we have:*

$$\hat{\mathbb{C}}_{N|3} = \Psi_N\left(\hat{\mathbb{C}}_N(u_1,1,v_3)\right) + \Psi_N\left(\hat{\mathbb{C}}_N(1,u_2,v_3)\right) + o_{\mathbb{P}^*}(1)$$

Proof of Lemma 1. The empirical projected copula process (4) can be rewritten as difference between the empirical projection map applied to the empirical copula and the empirical projection map applied to the true copula, plus the difference between the empirical and asymptotic projection map applied to the true copula:

$$\begin{aligned}\hat{\mathbb{C}}_{N|3} &= \sqrt{N}\left(\hat{\Pi}_{N|3}\left(\hat{C}_N(u_1,u_2,u_3)\right) - \hat{\Pi}_{N|3}\left(C(u_1,u_2,u_3)\right)\right) \\ &+ \sqrt{N}\left(\hat{\Pi}_{N|3}\left(C(u_1,u_2,u_3)\right) - \Pi_{|3}\left(C(u_1,u_2,u_3)\right)\right) \qquad \text{(A.1)}\end{aligned}$$

We develop first the second term of the right hand of (A.1) representing the difference between the empirical and asymptotic projection map applied to the true copula:

$$\begin{aligned}&\sqrt{N}\left(\hat{\Pi}_{N|3} - \Pi_{|3}\right) \circ \left(C(u_1,u_2,u_3)\right) = \qquad \text{(A.2)} \\ &= \sqrt{N}\left(\hat{\Pi}_{N|3}\left(C(u_1,u_2,u_3)\right) - \Pi_{|3}\left(C(u_1,u_2,u_3)\right)\right) \\ &= \sqrt{N}\left(\int_0^{u_3} dv_3 \left(\tilde{\partial}_3^{N(1)} C(u_1,1,v_3)\, \tilde{\partial}_3^{N(1)} C(1,u_2,v_3) - \partial_3 C(u_1,1,v_3)\, \partial_3 C(1,u_2,v_3)\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{N}\left(\int_0^{u_3} dv_3 \tilde{\partial}_3^{N(1)} C\left(u_1,1,v_3\right)\left(\tilde{\partial}_3^{N(1)} C\left(1,u_2,v_3\right)-\partial_3 C\left(1,u_2,v_3\right)\right)\right. \\
&\qquad \left.+\partial_3 C\left(1,u_2,v_3\right)\left(\tilde{\partial}_3^{N(1)} C\left(u_1,1,v_3\right)-\partial_3 C\left(u_1,1,v_3\right)\right)\right) \\
&= \sqrt{N}\left(\int_0^{u_3} dv_3\left(\tilde{\partial}_3^{N(1)} C\left(u_1,1,v_3\right)-\partial_3 C\left(u_1,1,v_3\right)\right)\left(\tilde{\partial}_3^{N(1)} C\left(1,u_2,v_3\right)-\partial_3 C\left(1,u_2,v_3\right)\right)\right. \\
&\qquad +\partial_3 C\left(u_1,1,v_3\right)\left(\tilde{\partial}_3^{N(1)} C\left(1,u_2,v_3\right)-\partial_3 C\left(1,u_2,v_3\right)\right) \\
&\qquad \left.+\partial_3 C\left(1,u_2,v_3\right)\left(\tilde{\partial}_3^{N(1)} C\left(u_1,1,v_3\right)-\partial_3 C\left(u_1,1,v_3\right)\right)\right)
\end{aligned}
$$

. Using the last expression,$|\partial_3 C| \leq 1$ and assumption **A3** ,we can bound the absolute value of (A.2):

$$
\begin{aligned}
& \left|\sqrt{N}\left(\hat{\Pi}_{N|3}-\Pi_{|3}\right) \circ\left(C\left(u_1,u_2,u_3\right)\right)\right| \\
\leq\ & \sqrt{N}\left(1+R_{\tilde{\partial}^{(1)}}^N\right) \int_0^{u_3}\left|\tilde{\partial}_3^{N(1)} C\left(u_1,1,v_3\right)-\partial_3 C\left(u_1,1,v_3\right)\right| dv_3 \\
+\ & \sqrt{N} \int_0^{u_3} dv_3\left|\tilde{\partial}_3^{N(1)} C\left(u_1,1,v_3\right)-\partial_3 C\left(u_1,1,v_3\right)\right| dv_3
\end{aligned}
$$

By the dominated convergence theorem, the use assumption **A6** on this bound, implies that the limit of the absolute value of (A.2) is zero, i.e. that (A.2) is $o(1)$.
We consider now the first term of the right hand of the relationship (A.1) that represents the difference beetween the application of the empirical projection map to the empirical copula and the application of the empirical projection

map to the true copula :

$$
\begin{aligned}
& \sqrt{N}\left(\hat{\Pi}_{N|3}\left(\hat{C}_N\left(u_1,u_2,u_3\right)\right)-\hat{\Pi}_{N|3}\left(C\left(u_1,u_2,u_3\right)\right)\right) && \text{(A.3)}\\
= & \sqrt{N}\int_0^{u_3}\tilde{\partial}_3^{N(1)}\hat{C}_N\left(u_1,1,v_3\right)\tilde{\partial}_3^{N(1)}\hat{C}_N\left(1,u_2,v_3\right)dv_3\\
- & \sqrt{N}\int_0^{u_3}\tilde{\partial}_3^{N(1)}C\left(u_1,1,v_3\right)\tilde{\partial}_3^{N(1)}C\left(1,u_2,v_3\right)dv_3=\\
= & \sqrt{N}\int_0^{u_3}\tilde{\partial}_3^{N(1)}\hat{C}_N\left(u_1,1,v_3\right)\left(\tilde{\partial}_3^{N(1)}\hat{C}_N\left(1,u_2,v_3\right)-\tilde{\partial}_3^{N(1)}C\left(1,u_2,v_3\right)\right)dv_3\\
+ & \sqrt{N}\int_0^{u_3}\left(\tilde{\partial}_3^{N(1)}\hat{C}_N\left(u_1,1,v_3\right)-\tilde{\partial}_3^{N(1)}C\left(u_1,1,v_3\right)\right)\tilde{\partial}_3^{N(1)}C\left(1,u_2,v_3\right)dv_3\\
= & \sqrt{N}\int_0^{u_3}\tilde{\partial}_3^{N(1)}\hat{C}_N\left(u_1,1,v_3\right)\frac{1}{\sqrt{N}}\tilde{\partial}_3^{N(1)}\hat{\mathbb{C}}_N\left(1,u_2,v_3\right)dv_3\\
+ & \sqrt{N}\int_0^{u_3}\frac{1}{\sqrt{N}}\tilde{\partial}_3^{N(1)}\hat{\mathbb{C}}_N\left(u_1,1,v_3\right)\tilde{\partial}_3^{N(1)}C\left(1,u_2,v_3\right)dv_3\\
= & \int_0^{u_3}\partial_3C\left(u_1,1,v_3\right)\tilde{\partial}_3^{N(1)}\hat{\mathbb{C}}_N\left(1,u_2,v_3\right)dv_3+o_{\mathbb{P}^*}\left(1\right)\int_0^{u_3}\tilde{\partial}_3^{N(1)}\hat{\mathbb{C}}_N\left(1,u_2,v_3\right)dv_3\\
+ & \int_0^{u_3}\tilde{\partial}_3^{N(1)}\hat{\mathbb{C}}_N\left(u_1,1,v_3\right)\partial_3C\left(1,u_2,v_3\right)dv_3+o\left(R^N_{\tilde{\partial}^{(1)}}\right)\int_0^{u_3}\tilde{\partial}_3^{N(1)}\hat{\mathbb{C}}_N\left(u_1,1,v_3\right)dv_3
\end{aligned}
$$

Where the last equality follows from **A3** and **A4**.
Now, under **A5** and **A3**, we have for any $\mathbf{u}\in[0,1]^3$

$$
\begin{aligned}
\int_0^{u_3}\tilde{\partial}_3^{N(1)}\hat{\mathbb{C}}_N\left(u_1,u_2,v_3\right)dv_3 &= \hat{\mathbb{C}}_N\left(u_1,u_2,u_3\right)-\int_0^{u_3}\left(\tilde{\partial}_3^{N(1)}1\right)\hat{\mathbb{C}}_N\left(u_1,u_2,v_3\right)dv_3\\
&= \hat{\mathbb{C}}_N\left(u_1,u_2,u_3\right)+o\left(R^N_{\tilde{\partial}^{(1)}}\right)\int_0^{u_3}\hat{\mathbb{C}}_N\left(u_1,u_2,v_3\right)dv_3\\
&= \hat{\mathbb{C}}_N\left(u_1,u_2,u_3\right)+o_{\mathbb{P}^*}\left(1\right)
\end{aligned}
$$

The last expression implies that the second and the fourth term in the last inequality of (A.3) are $o_{\mathbb{P}^*}(1)$.

Summarizing, we have shown that:

$$
\begin{aligned}
\hat{\mathbb{C}}_{N|3} &= \sqrt{N}\left(\hat{\Pi}_{N|3}\left(\hat{C}_N\left(u_1,u_2,u_3\right)\right)-\hat{\Pi}_{N|3}\left(C\left(u_1,u_2,u_3\right)\right)\right)\\
&+ \sqrt{N}\left(\hat{\Pi}_{N|3}\left(C\left(u_1,u_2,u_3\right)\right)-\Pi_{|3}\left(C\left(u_1,u_2,u_3\right)\right)\right)\\
&= \sqrt{N}\left(\hat{\Pi}_{N|3}\left(\hat{C}_N\left(u_1,u_2,u_3\right)\right)-\hat{\Pi}_{N|3}\left(C\left(u_1,u_2,u_3\right)\right)\right)+o\left(1\right)\\
&= \int_0^{u_3}\partial_3 C\left(u_1,1,v_3\right)\tilde{\partial}_3^{N(1)}\hat{\mathbb{C}}_N\left(1,u_2,v_3\right)dv_3\\
&+ \int_0^{u_3}\tilde{\partial}_3^{N(1)}\hat{\mathbb{C}}_N\left(u_1,1,v_3\right)\partial_3 C\left(1,u_2,v_3\right)dv_3+o_{\mathbb{P}^*}\left(1\right) \qquad \text{(A.4)}
\end{aligned}
$$

Under **A5**,to obtain the result we can again "integrate by part". In particular, we can rewrite the last equality of (A.4) in the following way:

$$
\begin{aligned}
\hat{\mathbb{C}}_{N|3} &= \int_0^{u_3}\partial_3 C\left(u_1,1,v_3\right)\tilde{\partial}_3^{N(1)}\hat{\mathbb{C}}_N\left(1,u_2,v_3\right)dv_3\\
&+ \int_0^{u_3}\tilde{\partial}_3^{N(1)}\hat{\mathbb{C}}_N\left(u_1,1,v_3\right)\partial_3 C\left(1,u_2,v_3\right)dv_3+o_{\mathbb{P}^*}\left(1\right)\\
&= \Psi_N\left(\hat{\mathbb{C}}_N\left(1,u_2,v_3\right)\right)+J\left(\partial_3 C\left(u_1,1,v_3\right),\hat{\mathbb{C}}_N\left(1,u_2,v_3\right)\right)\\
&+ \Psi_N\left(\hat{\mathbb{C}}_N\left(u_1,1,v_3\right)\right)+J\left(\partial_3 C\left(1,u_2,v_3\right),\hat{\mathbb{C}}_N\left(u_1,1,v_3\right)\right)+o_{\mathbb{P}^*}\left(1\right)\\
&= \Psi_N\left(\hat{\mathbb{C}}_N\left(1,u_2,v_3\right)\right)+\Psi_N\left(\hat{\mathbb{C}}_N\left(u_1,1,v_3\right)\right)+o_{\mathbb{P}^*}\left(1\right)
\end{aligned}
$$

Appendix A.2. Derivatives

In the following, we introduce our definition of finite difference derivatives and then show they satisfy the requirements to be copula consistent derivatives. This is done to show that the simplest possible estimator of derivatives can be used for constructing the projected empirical copula process, if the true copula satisfies **A1** and **A2**.

Slightly adapting our definitions from Segers (2012) we have

$$\begin{aligned}
\partial_i C(\mathbf{u}) &= \limsup_{h\downarrow 0}\left\{\frac{C(\mathbf{u}+\mathbf{e}_i 2h)}{2h}\mathbb{I}(u_i=0)\right.\\
&+ \frac{C(\mathbf{u}+\mathbf{e}_i h)-C(\mathbf{u}-\mathbf{e}_i h)}{2h}\mathbb{I}(0<u_i<1)\\
&+ \left.\frac{C(\mathbf{u})-C(\mathbf{u}-\mathbf{e}_i 2h)}{2h}\mathbb{I}(u_i=1)\right\} \qquad \text{(A.5)}
\end{aligned}$$

where $\mathbf{u}=(u_1,\ldots,u_D)$ and $\mathbf{e}_i$ is the i-th D-dimensional basis vector.
In this way, it is easy to obtain the finite difference approximation used in Genest et al. (2011):

$$\begin{aligned}
\tilde{\partial}_i^h C(\mathbf{u}) &= \left\{\frac{C(\mathbf{u}+\mathbf{e}_i(2h-u_i))}{2h}\mathbb{I}(u_i\le h)\right.\\
&+ \frac{C(\mathbf{u}+\mathbf{e}_i h)-C(\mathbf{u}-\mathbf{e}_i h)}{2h}\mathbb{I}(h<u_i<1-h)\\
&+ \left.\frac{C(\mathbf{u}-\mathbf{e}_i(u_i-1))-C(\mathbf{u}-\mathbf{e}_i(u_i-(1-2h)))}{2h}\mathbb{I}(u_i\ge 1-h)\right\}
\end{aligned}$$

If we define forward, backward and central differences as

$$\Delta_i^{2h} f(\mathbf{u}) = \frac{f(\mathbf{u}+\mathbf{e}_i 2h)-f(\mathbf{u})}{2h} \qquad \text{(A.6)}$$

$$\nabla_i^{2h} f(\mathbf{u}) = \frac{f(\mathbf{u})-f(\mathbf{u}-\mathbf{e}_i 2h)}{2h} \qquad \text{(A.7)}$$

$$\delta_i^{2h} f(\mathbf{u}) = \frac{f(\mathbf{u}+\mathbf{e}_i h)-f(\mathbf{u}-\mathbf{e}_i h)}{2h} \qquad \text{(A.8)}$$

(A.5) could be rewritten

$$\begin{aligned}
\partial_i C(\mathbf{u}) &= \limsup_{h\downarrow 0}\left\{\frac{\Delta_i^{2h}C(\mathbf{u})}{2h}\mathbb{I}(u_i=0)+\frac{\delta_i^{2h}C(\mathbf{u})}{2h}\mathbb{I}(0<u_i<1)\right.\\
&+ \left.\frac{\nabla_i^{2h}C(\mathbf{u})}{2h}\mathbb{I}(u_i=1)\right\} \qquad \text{(A.9)}
\end{aligned}$$

(A.9) can be generalized to:

$$\begin{aligned}
\partial_i^{(n)} C(\mathbf{u}) &= \limsup_{h\downarrow 0}\left\{\frac{\left(\Delta_i^{2h}\right)^n C(\mathbf{u})}{(2h)^n}\mathbb{I}(u_i=0)+\frac{\left(\delta_i^{2h}\right)^n C(\mathbf{u})}{(2h)^n}\mathbb{I}(0<u_i<1)\right.\\
&+ \left.\frac{\left(\nabla_i^{2h}\right)^n C(\mathbf{u})}{(2h)^n}\mathbb{I}(u_i=1)\right\} \qquad \text{(A.10)}
\end{aligned}$$

From this expression the approximation is:

$$\tilde{\partial}_i^{h(n)} C(\mathbf{u}-\mathbf{e}_i u_i) = \left\{ \frac{\left(\Delta_i^{2h}\right)^n C(\mathbf{u}-\mathbf{e}_i u_i)}{(2h)^n} \mathbb{I}(u_i \le nh) + \frac{\left(\delta_i^{2h}\right)^n C(\mathbf{u})}{(2h)^n} \mathbb{I}(nh < u_i < 1-nh) + \frac{\left(\nabla_i^{2h}\right)^n C(\mathbf{u}-\mathbf{e}_i(u_i-1))}{(2h)^n} \mathbb{I}(u_i \ge 1-nh) \right\} \tag{A.11}$$

*Appendix A.2.1. Finite difference Approximation satisfies **A3** and **A4***

In this paragraph, we will show that finite difference approximations are uniform approximations of the derivatives for derivable functions (i.e. assumption **A3**) and that are a consistent estimators for copula derivatives (i.e. assumption **A4**)
Let us start with **A3**, it is well known that for n times differentiable functions G

$$\left(\frac{\left(\Delta_i^{2h}\right)^n G(\mathbf{u})}{(2h)^n} - \partial_i^{(n)} G(\mathbf{u}) \right) = o(1)$$

$$\left(\frac{\left(\nabla_i^{2h}\right)^n G(\mathbf{u})}{(2h)^n} - \partial_i^{(n)} G(\mathbf{u}) \right) = o(1)$$

$$\left(\frac{\left(\delta_i^{2h}\right)^n G(\mathbf{u})}{(2h)^n} - \partial_i^{(n)} G(\mathbf{u}) \right) = o(1)$$

thus

$$\begin{aligned} \left(\tilde{\partial}_i^{h(n)} C(\mathbf{u}) - \partial_i^{(n)} C(\mathbf{u})\right) &= \qquad\qquad (\text{A.12}) \\ &= \left(o(1) + \partial_i^{(n)} C(\mathbf{u}-\mathbf{e}_i u_i) - \partial_i^{(n)} C(\mathbf{u})\right) \mathbb{I}(u_3 \le nh) \\ &+ o(1)\, \mathbb{I}(nh < u_3 < 1-nh) \\ &+ o(1) \left(\partial_i^{(n)} C(\mathbf{u}-\mathbf{e}_i(u_i-1)) - \partial_i^{(n)} C(\mathbf{u})\right) \mathbb{I}(u_3 > 1-nh) \end{aligned}$$

When $u_i \in \{0,1\}$ the derivative difference (A.12) is 0 otherwise is $o(1)$ by the continuity in u_i, so in the end the difference between the approximation and the derivative is $o(1)$. Then, using the previous result, **A4** follows:

$$\begin{aligned}
&\left(\tilde{\partial}_i^{h(n)}\hat{C}_N(\mathbf{u})-\partial_i^{(n)}C(\mathbf{u})\right)\\
=\;&\left(\tilde{\partial}_i^{h(n)}\hat{C}_N(\mathbf{u})-\tilde{\partial}_i^{h(n)}C(\mathbf{u})\right)+o(1)\\
=\;&\frac{1}{\sqrt{N}}\tilde{\partial}_i^{h(n)}\hat{\mathbb{C}}_N(\mathbf{u})+o(1)\\
=\;&\left\{\frac{\left(\Delta_i^{2h}\right)^n\hat{\mathbb{C}}_N(\mathbf{u}-\mathbf{e}_iu_i)}{\sqrt{N}(2h)^n}\mathbb{I}(u_i\leq nh)\right.\\
+\;&\frac{\left(\delta_i^{2h}\right)^n\hat{\mathbb{C}}_N(\mathbf{u})}{\sqrt{N}(2h)^n}\mathbb{I}(nh<u_i<1-nh)\\
+\;&\left.\frac{\left(\nabla_i^{2h}\right)^n\hat{\mathbb{C}}_N(\mathbf{u}-\mathbf{e}_i(u_i-1))}{\sqrt{N}(2h)^n}\mathbb{I}(u_i\geq 1-nh)\right\}+o(1)\\
=\;&\\
=\;&\left\{\frac{\sum_{j=1}^n(-1)^j\begin{pmatrix}n\\j\end{pmatrix}\hat{\mathbb{C}}_N(\mathbf{u}-\mathbf{e}_i(u_i+(n-i)2h))}{\sqrt{N}(2h)^n}\mathbb{I}(u_i\leq nh)\right.\\
+\;&\frac{\sum_{j=1}^n(-1)^j\begin{pmatrix}n\\j\end{pmatrix}\hat{\mathbb{C}}_N\left(\mathbf{u}-\mathbf{e}_i\left(\frac{n}{2}-j\right)2h\right)}{\sqrt{N}(2h)^n}\mathbb{I}(nh<u_i<1-nh)\\
+\;&\left.\frac{\sum_{j=1}^n(-1)^j\begin{pmatrix}n\\j\end{pmatrix}\hat{\mathbb{C}}_N(\mathbf{u}-\mathbf{e}_i(u_i-1-j2h))}{\sqrt{N}(2h)^n}\mathbb{I}(u_i\geq 1-nh)\right\}+o(1)
\end{aligned}$$

Since $\sum_{j=1}^n(-1)^j\begin{pmatrix}n\\j\end{pmatrix}=0$,if $h^n=O\left(\frac{1}{\sqrt{N}}\right)$, it goes to zero in probability by the continuity of the paths of $\mathbb{C}$

Appendix A.2.2. Finite Difference Approximations satisfies ***A6***

Finite difference approximation satisfies also the integrated difference rate of convergence condition **A6**.
Under hypothesis **A1** and **A2** on copula derivatives needed for theorem 1,

by Taylor expansion we can show:

$$\left(\frac{\left(\Delta_i^{2h}\right)C\left(\mathbf{u}\right)}{(2h)}-\partial_i C\left(\mathbf{u}\right)\right)=O\left(h\right)$$
$$\left(\frac{\left(\nabla_i^{2h}\right)C\left(\mathbf{u}\right)}{(2h)}-\partial_i C\left(\mathbf{u}\right)\right)=O\left(h\right)$$
$$\left(\frac{\left(\delta_i^{2h}\right)C\left(\mathbf{u}\right)}{(2h)}-\partial_i C\left(\mathbf{u}\right)\right)=o\left(h\right)$$

Then we have

$$\begin{aligned}
& \sqrt{N}\int_0^{u_3} dv_3\left(\tilde{\partial}_3^{N(1)}C\left(u_1,u_2,v_3\right)-\partial_3 C\left(u_1,u_2,v_3\right)\right) \\
= & \sqrt{N}\int_0^{u_3\wedge h} dv_3 O\left(h\right)+\mathbb{I}\left(u_3\geq 1-h\right)\sqrt{N}\int_{1-h}^{u_3} dv_3 O\left(h\right)+\sqrt{N}o\left(h\right) \\
= & \sqrt{N}u_3\wedge hO\left(h\right)+\mathbb{I}\left(u_3\geq 1-h\right)\sqrt{N}\left(\left(1-h\right)-u_3\right)O\left(h\right)+\sqrt{N}o\left(h\right) \\
= & \sqrt{N}o\left(h\right)=o\left(1\right)
\end{aligned}$$

Appendix A.2.3. Finite Difference Approximations Allows ***A5***

The most challenging requirement is asymptotic integration by part given in the assumption **A5**. We, now, show that finite difference approximation are asymptotically integrable by part.

Let $F(u_1, u_2, u_3) = \int_0^{u_3} f(u_1, u_2, v_3)\, dv_3 + k(u_1, u_2)$, then:

$$\begin{aligned}
& \int_0^{u_3} dv_3 f(u_1', u_2', v_3)\, \tilde{\partial}_3^h \hat{\mathbb{C}}_N(u_1, u_2, v_3) \\
+ & \int_0^{u_3} dv_3 \tilde{\partial}_3^h f(u_1', u_2', v_3)\, \hat{\mathbb{C}}_N(u_1, u_2, v_3) \\
- & f(u_1', u_2', u_3)\, \hat{\mathbb{C}}_N(u_1, u_2, u_3) \\
= & \int_0^{u_3 \wedge h} dv_3 f(u_1', u_2', v_3)\, \frac{\hat{\mathbb{C}}_N(u_1, u_2, 2h)}{2h} \\
+ & \int_h^{u_3 \wedge (1-h)} dv_3 f(u_1', u_2', v_3)\, \frac{\hat{\mathbb{C}}_N(u_1, u_2, v_3 + h) - \hat{\mathbb{C}}_N(u_1, u_2, v_3 - h)}{2h} \\
+ & \mathbb{I}(u_3 \geq (1-h)) \int_{(1-h)}^{u_3} dv_3 f(u_1', u_2', v_3)\, \frac{\hat{\mathbb{C}}_N(u_1, u_2, 1) - \hat{\mathbb{C}}_N(u_1, u_2, 1-h)}{2h} \\
+ & \int_0^{u_3 \wedge h} dv_3 \frac{f(u_1', u_2', 2h)}{2h} \hat{\mathbb{C}}_N(u_1, u_2, v_3) \\
+ & \int_h^{u_3 \wedge (1-h)} dv_3 \frac{f(u_1', u_2', v_3 + h) - f(u_1', u_2', v_3 - h)}{2h} \hat{\mathbb{C}}_N(u_1, u_2, v_3) \\
+ & \mathbb{I}(u_3 \geq (1-h)) \int_{(1-h)}^{u_3} dv_3 \frac{f(u_1', u_2', 1) - f(u_1', u_2', 1-h)}{2h} \hat{\mathbb{C}}_N(u_1, u_2, v_3) \\
- & f(u_1', u_2', u_3)\, \hat{\mathbb{C}}_N(u_1, u_2, u_3)
\end{aligned}$$

$$
\begin{aligned}
= &\quad \hat{\mathbb{C}}_N(u_1,u_2,2h)\,\frac{F(u_1',u_2',u_3\wedge h)-F(u_1',u_2',0)}{u_3\wedge h}\,\frac{u_3\wedge h}{2h} && \text{(A.13)}\\
+ &\quad \int_0^{u_3\wedge(1-h)-h} dv_3 f(u_1',u_2',v_3-h)\,\frac{\hat{\mathbb{C}}_N(u_1,u_2,v_3)}{2h} && \text{(A.14)}\\
- &\quad \int_{2h}^{u_3\wedge(1-h)+h} dv_3 f(u_1',u_2',v_3+h)\,\frac{\hat{\mathbb{C}}_N(u_1,u_2,v_3)}{2h} && \text{(A.15)}\\
+ &\quad \left(\hat{\mathbb{C}}_N(u_1,u_2,1)-\hat{\mathbb{C}}_N(u_1,u_2,1-h)\right)\mathbb{I}(u_3\geq(1-h))\times \\
\times &\quad \frac{f(u_1',u_2',(1-h))-f(u_1',u_2',u_3)}{(1-h)-u_3}\,\frac{(1-h)-u_3}{2h} && \text{(A.16)}\\
+ &\quad \int_0^{u_3\wedge h} dv_3\frac{f(u_1',u_2',2h)}{2h}\hat{\mathbb{C}}_N(u_1,u_2,v_3) && \text{(A.17)}\\
+ &\quad \int_h^{u_3\wedge(1-h)} dv_3\frac{f(u_1',u_2',v_3+h)-f(u_1',u_2',v_3-h)}{2h}\hat{\mathbb{C}}_N(u_1,u_2,v_3) && \text{(A.18)}\\
+ &\quad \mathbb{I}(u_3\geq(1-h))\int_{(1-h)}^{u_3} dv_3\frac{f(u_1',u_2',1)-f(u_1',u_2',1-h)}{2h}\hat{\mathbb{C}}_N(u_1,u_2,v_3) && \text{(A.19)}\\
- &\quad f(u_1',u_2',u_3)\,\hat{\mathbb{C}}_N(u_1,u_2,u_3) && \text{(A.20)}
\end{aligned}
$$

When $N\to\infty$ we have that

$$
\begin{aligned}
\frac{u_3\wedge h}{2h} &\to \frac{1}{2}\\
\frac{F(u_1',u_2',u_3\wedge h)-F(u_1',u_2',0)}{u_3\wedge h} &\to f(u_1',u_2',0)\\
\hat{\mathbb{C}}_N(1,u_2,2h) &\rightsquigarrow \mathbb{C}(1,u_2,0)=0
\end{aligned}
$$

where the last term follows from the continuity of the sample paths of $\mathbb{C}$. So (A.13) goes to zero.
Analogously (A.16) goes to zero since:

$$
\begin{aligned}
\mathbb{I}(u_3\geq 1-h) &\to \mathbb{I}(u_3=1)\\
\frac{u_3-1+h}{2h}\mathbb{I}(u_3\geq 1-h) &\to \frac{1}{2}\mathbb{I}(u_3=1)\\
\frac{F(u_1',u_2',u_3)-F(u_1',u_2',1-h)}{u_3-1+h}\mathbb{I}(u_3\geq 1-h) &\to f(u_1',u_2',1)\\
\hat{\mathbb{C}}_N(u_1,u_2,1) &\rightsquigarrow \mathbb{C}(1,u_2,1)\\
\hat{\mathbb{C}}_N(u_1,u_2,1-2h) &\rightsquigarrow \mathbb{C}(u_1,u_2,1)
\end{aligned}
$$

For (A.17) and (A.19) we recall a rough bound

$$\left|\hat{\mathbb{C}}_N\right| \leq \sqrt{N}\left(\left|\hat{C}_N\right| + |C|\right) \leq 2\sqrt{N}$$

so that

$$\begin{aligned}
&\left|\frac{f(u_1', u_2', 2h) - f(u_1', u_2', 0)}{2h} \int_0^{u_3 \wedge h} dv_3 \hat{\mathbb{C}}_N(u_1, u_2, v_3)\right| \\
\leq\ & 2\left|f(u_1', u_2', 2h) - f(u_1', u_2', 0)\right| \frac{\sqrt{N}}{2h} u_3 \wedge h \\
&\left|\mathbb{I}(u_3 \geq (1-h)) \frac{f(u_1', u_2', 1) - f(u_1', u_2', 1-h)}{2h} \int_{(1-h)}^{u_3} \hat{\mathbb{C}}_N(u_1, u_2, v_3)\, dv_3\right| \\
\leq\ & \left|f(u_1', u_2', 1) - f(u_1', u_2', 1-h)\right| \frac{\sqrt{N}}{2h} 2\left(\left((1-h) - u_3\right) \mathbb{I}(u_3 \geq (1-h))\right)
\end{aligned}$$

By the continuity of f the limit is zero in both cases.
If we sum (A.14),(A.15) and (A.18) we obtain

$$\begin{aligned}
& \int_{u_3 \wedge (1-h) - h}^{u_3 \wedge (1-h)} dv_3 f(u_1', u_2', v_3 - h) \frac{\hat{\mathbb{C}}_N(u_1, u_2, v_3)}{2h} \\
- & \int_h^{2h} dv_3 f(u_1', u_2', v_3 - h) \frac{\hat{\mathbb{C}}_N(u_1, u_2, v_3)}{2h} \\
+ & \int_{u_3 \wedge (1-h)}^{u_3 \wedge (1-h) + h} dv_3 f(u_1', u_2', v_3 + h) \frac{\hat{\mathbb{C}}_N(u_1, u_2, v_3)}{2h} \\
- & \int_0^h dv_3 f(u_1', u_2', v_3 + h) \frac{\hat{\mathbb{C}}_N(u_1, u_2, v_3)}{2h}
\end{aligned} \tag{A.21}$$

All those terms can be written as particular instances of the following integral

$$\begin{aligned}
& \frac{1}{k} \int_a^{a+k} dv_3 f(u_1', u_2', v_3)\, \mathbb{C}_N(u_1, u_2, v_3) \\
= & \frac{1}{k}\left[\sqrt{N} \int_a^{a+k} dv_3 f(u_1', u_2', v_3)\, C(u_1, u_2, v_3)\right. \\
- & \left.\sqrt{N} \int_a^{a+k} dv_3 f(u_1', u_2', v_3)\, \hat{C}_N(u_1, u_2, v_3)\right]
\end{aligned}$$

Since both C and $\hat{C}_N$ are bounded monotonic non decreasing non negative function, and f is a bounded integrable function, we can use the second mean value theorem Gradshteyn (2000). For some $\eta, \eta' \in [0,k]$ we have:

$$\begin{aligned}
&= \frac{1}{k}\left[\sqrt{N}C\left(u_1,u_2,a+k\right)\int_{a+\eta}^{a+k} dv_3 f\left(u_1',u_2',v_3\right)\right.\\
&- \left.\sqrt{N}\hat{C}_N\left(u_1,u_2,a+k\right)\int_{a+\eta'}^{a+k} dv_3 f\left(u_1',u_2',v_3\right)\right]\\
&= \frac{1}{k}\left[\hat{\mathbb{C}}_N\left(u_1,u_2,a+k\right)\int_{a}^{a+k} dv_3 f\left(u_1',u_2',v_3-k\right)\right.\\
&- \sqrt{N}C\left(u_1,u_2,a+k\right)\int_{a}^{a+\eta} dv_3 f\left(u_1',u_2',v_3\right)\\
&+ \left.\sqrt{N}\hat{C}_N\left(u_1,u_2,a+k\right)\int_{a}^{a+\eta'} dv_3 f\left(u_1',u_2',v_3\right)\right]\\
&= \frac{1}{k}\left[\hat{\mathbb{C}}_N\left(u_1,u_2,a+k\right)k\left[f\left(u_1',u_2',a\right)+O\left(k\right)\right]\right.\\
&- \sqrt{N}C\left(u_1,u_2,a+k\right)\eta\left[f\left(u_1',u_2',a\right)+O\left(\eta\right)\right]\\
&+ \left.\sqrt{N}\hat{C}_N\left(u_1,u_2,a+k\right)\eta'\left[f\left(u_1',u_2',a\right)+O\left(\eta'\right)\right]\right]\\
&= \hat{\mathbb{C}}_N\left(u_1,u_2,a+k\right)\left[f\left(u_1',u_2',a\right)+O\left(k\right)\right]
\end{aligned}$$

Applying this result to (A.21), subtracting (A.18) we get a zero limit by the continuity of the paths of $\mathbb{C}$, so that **A5** is verified.

References

Bucher, A., Dette, H. (2010).Anote on bootstrap approximations for the empirical copula process. Statistics and Probability Letters, 80, 1925-1932.

Bucher, A. and Volgushev, S. (2011). Empirical and sequential empirical copula processes under serial dependence. arXiv:1111.2778.

Bouezmarni T. , Rombouts J. V.K.& Taamouti A. (2012), Nonparametric Copula-Based Test for Conditional Independence with Applications to Granger Causality, Journal of Business & Economic Statistics, 30:2, 275-287

Darsow WF, Nguyen B, Olsen ET (1992) Copulas and Markov processes. Illinois J Math 36,600-642.

Dawid, A. D. (1979), Conditional Independence in Statistical Theory, Journal of the Royal Statistical Society, Series B 41, 1-31.

Frattarolo, L., and Guegan, D., (2013) Empirical Projected Copula Process and Conditional Independence, MSE working paper

Genest, C., Neslehova, J., and Quessy, J.-F. (2011). Tests of symmetry for bivariate copulas. The Annals of the Institute of Statistical Mathematics, 64:,811-834.

Gradshteyn I.S., Ryzhik I.M. (2000), Table of integrals, series, and Product, Academic Press London.

Gyorfi L. and Walk H. (2012) Strongly consistent nonparametric tests of conditional independence Statistics and Probability Letters 82 1145-1150

Koller, D. & Friedman, N. (2009). Probabilistic Graphical Models, MIT Press, Cambridge, Massachusetts.

Nelsen, R. B. (2006). An introduction to copulas (2nd ed.). Springer: New York.

Segers, J. (2012). Asymptotics of empirical copula processes under nonrestrictive smoothness assumptions. Bernoulli,18:3,764-782.

van der Vaart, A. W., Wellner, J. A. (1996). Weak convergence and empirical processes. Springer: New York.

van der Vaart, A. W. (1998). Asymptotic Statistics. Cambridge University Press, New York.

White, H., and Lu, X. (2010), Granger Causality and Dynamic Structural Systems, Journal of Financial Econometrics,8:2,193-243.