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Abstract

The paper proposes a new concept of solution for TU games, called multicoalitional solution, which makes sense in the context of production games, that is, where $v(S)$ is the production or income per unit of time. By contrast to classical solutions where elements of the solution are payoff vectors, multicoalitional solutions give in addition an allocation time to each coalition, which permits to realize the payoff vector. We give two instances of such solutions, called the d-multicoalitional core and the c-multicoalitional core, and both arise as the strong Nash equilibrium of two games, where in the first utility per active unit of time is maximized, while in the second it is the utility per total unit of time. We show that the d-core (or aspiration core) of Benett, and the c-core of Guesnerie and Oddou are strongly related to the d-multicoalitional and c-multicoalitional cores, respectively, and that the latter ones can be seen as an implementation of the former ones in a noncooperative framework.

Keywords: cooperative game, core, aspiration core, strong Nash equilibrium
JEL Classification: C71

Introduction

In neoclassical microeconomic models (for example, of labour market), agents are assumed to be rational and seeking to maximize their utility function. In these models, the utility function of the agents is determined by the choice between income and leisure. For such kind of agents, a natural way for assuming that an alternative A is better than another alternative B is: either alternative A ensures (at least) the same income with more time for leisure, or alternative A ensures more income and (at least) the same length of time devoted to leisure. In both cases, agents maximize their "hourly wage", either through an increase of leisure time (hence a reduction of labour time), or through an increase of income. On the other hand, agents want also to work enough time for earning a sufficient income. Hence, the ranking of alternatives by such agents should be based on a comparison of incomes and hourly incomes.

We deal with such a problem in this paper, and we assume that players can freely form coalitions in order to achieve some task and get an income, up to two restrictions:

(i) Each coalition produces a linear income over time and can redistribute this income among the players who are members of the coalition;

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(ii) One player cannot be in two different coalitions at the same time.

Defined as such, the problem appears to be a coalition formation problem viewed under the point of view of cooperative game theory with transferable utility. Indeed, cooperative game theory is a suitable tool for describing coalition formation and for defining the best redistribution of an amount generated by cooperation of some group of players. In the classical framework, it is supposed that the grand coalition will form, and a feasible redistribution is simply a redistribution of the worth of the grand coalition. Under this assumption, the natural concept of core (Gillies, 1953, 1959) is a key solution, since it ensures a distribution among individual players such that no player, no coalition, has a rational interest to leave the grand coalition: It is the concept of coalitional rationality (for applications of the core in economy, see, e.g., (Trockel, 2005; Shitovitz, 1997; Flam and Koutsougeras, 2010)). Unfortunately, in many cases, such distribution does not exist: The core is empty. To extend the notion of coalitional rationality to a larger set of games, it is necessary to change the classical view of what a feasible redistribution is. The k-additive core (Grabisch and Miranda, 2008; Gonzalez and Grabisch, 2012), for example, assumes that a feasible redistribution may not be limited to a redistribution among individual players, but a redistribution among coalitions up to size k: With this assumption, coalitional rationality is ensured for every game as soon as k is at least equal to two.

Alternatively, the c-core (Guesnerie and Oddou, 1979; Sun et al., 2008) supposes that the grand coalition is not the unique possibility of coalition formation but that every partition of the grand coalition can be formed: A feasible redistribution becomes, in this case, any distribution among individual players, which maximizes what a partition can generate.

An even older and more general way of extending feasible distributions comes from Bennett (1983) and Cross (1967) with the concept of aspiration core, or d-core (Albers, 1979): roughly speaking, a feasible distribution is any distribution among individual players, which maximizes what a balanced collection (i.e., a generalization of partitions) can achieve. The set of feasible distributions is, in that case, so large that it becomes possible to preserve coalitional rationality for every game.

Let us elaborate on balanced collections and their interpretation. It is common in the literature to see a balancing weight of a coalition in a balanced collection as the fraction of time this coalition is active (see, e.g., Peleg and Sudhölter (2003)). Then it is usually considered that a coalition being active during a given fraction of time receives the corresponding fraction of its worth (Aumann, 1989). With this view in mind, a feasible payoff corresponds to the maximum income players can generate if each of them devotes one unit of time among the coalitions to which they belong: This set is equal to the set of aspiration feasible payoffs. Then a simple use of the theorem of Bondareva-Shapley (Bondareva, 1963; Shapley, 1967) ensures that there exists a way to share time among coalitions, which builds payoff satisfying coalitional rationality. Thus, the aspiration core seems to be a suitable way to describe how coalitions must form if each player has one unit of time, and how long each coalition must be active.

However, a closer analysis reveals that the d-core does not fully take into account all the facets of our problem. First of all, the d-core proposes a set of payoff vectors but does not describe what are the coalitions who are able to achieve these payoffs. In his article about the d-core, Cross (1967) described informally his solution as a set of stable coalitions with their associated payoffs. The $\mathcal{B}$-core and the $\mathcal{M}$-core proposed by Cesco (2012) are closely related to or constitute a continuation of the paper of Cross, by considering the set of coalitions which lead to a payoff into the d-core. Under this view, a cooperative TU-game solution should be not only composed of the payoffs given to each player but should also comprise the time alloted to each coalition which permits to achieve these payoffs.

A second concern is the utility of players faced to the total time which is necessary for

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achieving a payoff given by the d-core: Imagine a situation with three players where reaching the payoff given by the d-core needs an allocation equals to one half unit of time for every pair of players[1]. With the reasonable assumption that a player cannot be in two coalitions at the same time, it follows that any implementation of this situation needs a minimum of 1.5 units of time. Hence, if the aspiration core (or d-core) gives a coalitional rational payoff per "active" unit of time, utility of players differ if we consider the payoff per "total" unit of time. Again, the solution provided by the d-core is not enough precise, in the sense that it does not take into account whether agents are concerned with the total duration of the process, or by their hourly income.

A third criticism is that each player who is active in a given coalition has the arbitrary imposition of spending the same proportion of his total time in this coalition: It is difficult to understand why an agent i who works half of his time with j requires that j works half of his time with i, even if it is not rational for one of these agents to work the same amount of time than the other one.

In order to overcome these drawbacks, we propose a completely different approach to the problem, having its root in noncooperative game theory: We suppose that each player proposes the formation of a coalition for a chosen amount of time and claims a payoff for his participation in each coalition which is formed[2], each of these proposals being seen as a strategy. The notion of Strong Nash equilibrium (Aumann, 1959) seems to be the adequate notion here, since it ensures stability of any coalition by preventing any coalitional deviation. Then, a solution in our framework is precisely the set of undominated strategies (in the sense of strong Nash equilibrium). We emphasize the fact that, in our framework, each element of the solution is a pair (x, α), where x is a payoff vector, and α is a time allocation for every coalition. This constitutes to our opinion an innovation since, up to our knowledge, no former work explicitly proposes a solution under this form. We call *multicoalitional solution* such kind of solution.

We propose two different types of utility functions, leading to two kinds of strong Nash equilibria. The first one is the utility per active unit of time, and leads to the maximization of the hourly wage. We call d-multicoalitional core the set of such equilibria, and we show that this set is never empty (Theorem 3), and that its elements satisfy nonnegativity, coalitional rationality and a notion of efficiency close to the one of the d-core (Theorem 4). We show the exact relation between the d-core and the d-multicoalitional core: in short, vectors of utility of strategies in the d-multicoalitional core are elements of the d-core (Proposition 6). The second type of utility function is the utility per total unit of time, and leads to the maximization of the total income. We prove that any strong Nash equilibrium of this type can be turned into a strategy which is also a strong Nash equilibrium of the first type (Proposition 8). Therefore, we define the c-multicoalitional core as the set of strategies which are strong Nash equilibria for both problems. They exist as soon as the c-core is not empty, moreover, a relation between the c-core and the c-multicoalitional core is established (Theorem 5).

The paper is organized as follows. Section 1 introduces the basic definitions and notation. Section 2 introduces time allocations for coalitions and timetables, that is, how to organize coalition formation so that no conflict occur, as well as the notion of minimal duration for timetables. Section 3 presents the model with the two types of utility functions, and studies the existence of strong Nash equilibria for each type, and introduces the d-multicoalitional and c-multicoalitional cores. Section 4 concludes the paper.

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[1]Such a situation is described with the island desert story in the introduction of Garratt and Qin (2000)

[2]A similar game is proposed by Bejan and Gómez (2012a). In their paper, agents have to share one unit of time among the set of coalitions to which they belong. The game leads to a strong Nash implementation of the d-core (see Proposition 6).

1 Notation and basic concepts

Let N denote a fixed finite nonempty set with n members, who will be called agents or players. *Coalitions of players* are nonempty subsets of N, denoted by capital letters S, T, and so on. Whenever possible, we will omit braces for singletons and pairs, denoting $\{i\}, \{i,j\}$ by i, ij respectively, in order to avoid a heavy notation. A *transferable utility* (TU) *game* on N is a pair (N, v) where v is a mapping $v : 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. We will denote by $\mathcal{G}(N)$ the set of mappings over N such that (N, v) is a TU game. For any coalition S, $v(S)$ represents the *worth* of S, i.e., what coalition S could earn regardless of other players. *A payoff vector* is a vector $x \in \mathbb{R}^n$ that assigns to agent i the payoff x_i. For any coalition $T \subset N$, we denote by v_T the restriction of v to 2^T. Given $x \in \mathbb{R}^n$, and $S \subseteq N$, denote by $x(S)$ the sum $\sum_{i \in S} x_i$ with the convention that $x(\emptyset) = 0$. A nonempty collection $\mathcal{B} \subseteq 2^N$ is called *balanced* (over N) if positive numbers $\delta_S, S \in \mathcal{B}$, exist such that :

$$\sum_{S \in \mathcal{B}} \delta_S \chi^S = \chi^N,$$

where χ^S is the characteristic vector of S given by $\chi_i^S = 1$ if $i \in S$ and 0 otherwise. The collection $(\delta_S)_{S \in \mathcal{B}}$ is called a system of balancing weights. We say that $(\mathcal{B}, (\delta_S)_{S \in \mathcal{B}})$ is a *balanced system* if $\mathcal{B}$ is balanced and $(\delta_S)_{S \in \mathcal{B}}$ is a corresponding system of balancing weights. A balanced collection is *minimal* if no subcollection of it is balanced. It is well known that a balanced collection is minimal if and only if there is a unique system of balancing weights.

Definition 1. For a game $v \in \mathcal{G}(N)$ we define

  (i) The set of *preimputations of v* as:

$$PI(v) := \{x \in \mathbb{R}^n \mid x(N) = v(N)\}.$$

  (ii) The set of *c-preimputations of v* as:

$$\text{c-}PI(v) := \{x \in \mathbb{R}^n \mid x(N) = \max_{\pi \in \Pi(N)} \sum_{S \in \pi} v(S)\}$$

  where $\Pi(N)$ is the set of partitions of N.

  (iii) The set of *d-preimputations of v* as:

$$\text{d-}PI(v) := \{x \in \mathbb{R}^n \mid x(N) = \max_{(\mathcal{B}, (\delta_S)_{S \in \mathcal{B}}) \text{ balanced}} \sum_{S \in \mathcal{B}} \delta_S v(S)\}.$$

In the classical view of preimputations, it is supposed that the grand coalition will form, and its worth is shared among the players. With a c-preimputation, the best partition of the players is sought, in order to maximize the total worth. Now, d-preimputations generalize c-preimputations since partitions are particular balanced collections. More importantly, d-preimputations have an interesting interpretation if one considers δ_S for $S \in \mathcal{B}$ as an amount of time[3] allocated to S. Assuming that $v(S)$ represents the worth or the production per unit of time, $\delta_S v(S)$ is the total worth/production achieved by S. Under this viewpoint, a d-preimputation is a sharing among the players of the total worth which can be achieved by the best arrangement of the players in time.

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[3]Although usually it is interpreted as an amount of resources (Kannai, 1992).

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Given a game (N, v) and $t \in \mathbb{R}$, define its *t-expansion* v^t by $v^t(S) = v(S)$ for all $S \subset N$, and $v^t(N) = v(N) + t$. Now, to any game (N, v) we assign

$$\bar{t}(v) := \sup_{(\mathcal{B}, \lambda) \text{ balanced system}} \sum_{S \in \mathcal{B}} (\lambda_S v(S)) - v(N).$$

It is quite easy to prove that $\bar{t}(v) = \min\{t \geq 0 \mid \mathsf{C}(v) \neq \emptyset\}$, that is, $\bar{t}(v)$ is the minimum amount to be given to the grand coalition in order to ensure balancedness

A game v is *totally balanced* if, for any $\emptyset \neq S \subseteq N$, v_S is balanced. The *totally balanced cover* of v denoted by v^{tb} is given by:

$$v^{tb}(S) = \max\{ \sum_{(\mathcal{B}, (\delta_T)_{T \in \mathcal{B}})} \delta_T v(T) \mid (\mathcal{B}, (\delta_T)_{T \in \mathcal{B}}) \text{ balanced system of S}\}$$

Definition 2. For any game $v \in \mathcal{G}(N)$ we define

(i) The *core (Gillies, 1953) of* v as:

$$\mathsf{C}(v) := \{x \in PI(v) \mid x(S) \geq v(S), \forall S \in 2^N\}.$$

(ii) The *c-core (Guesnerie and Oddou, 1979) of* v as:

$$\text{c-}\mathsf{C}(v) := \{x \in \text{c-}PI(v) \mid x(S) \geq v(S), \forall S \in 2^N\}.$$

(iii) The *d-core[4] (Albers, 1979) of* v or *aspiration core (Bennett, 1983) of* v as:

$$\text{d-}\mathsf{C}(v) := \{x \in \text{d-}PI(v) \mid x(S) \geq v(S), \forall S \in 2^N\}.$$

We say that a game v is *balanced* if $\mathsf{C}(v)$ is nonempty, and *c-balanced* if c-$\mathsf{C}(v)$ is nonempty. It is known and easy to prove that d-$\mathsf{C}(v) = \mathsf{C}(v^{tb}) = \mathsf{C}(v^{\bar{t}(v)})$ is never empty, furthermore, for any game v in $\mathcal{G}(N)$, we have:

$$\mathsf{C}(v) \subseteq \text{c-}\mathsf{C}(v) \subseteq \text{d-}\mathsf{C}(v),$$

and equality holds everywhere if and only if v is balanced. [5]

2 Time allocation and optimal timetable

Let N be a set of players, that we suppose to be finite. A *time allocation for coalitions* in N is a function $\alpha : 2^N \to [0, \infty[$. The collection $\mathcal{A} = \{S \in 2^N \mid \alpha(S) > 0\}$ is the collection of *active coalitions* under α. The time allocation is *in standard form* if $(\mathcal{A}, (\alpha(S))_{S \in \mathcal{A}})$ is a balanced system. $\alpha(S)$ can be interpreted as the initial endowment of unit of time for coalition $S \in \mathcal{A}$. We denote by $\mathcal{T}(N)$ the set of time allocations in N.

Definition 3. A *multicoalitional payoff vector* is a pair (x, α), where $x \in \mathbb{R}^N$ is a payoff vector, and $\alpha \in \mathcal{T}(N)$ is a time allocation. A *multicoalitional solution* is a mapping which assigns to every game (N, v) a set of multicoalitional payoff vectors.

Given a time allocation α, it remains to specify how each active coalition under α will spend its allocated time, by means of a "timetable": at each time t, the timetable indicates which coalitions are active under α. This timetable should satisfy two consistency requirements:

[4]See also Albers (1974) and Turbay (1977).
[5]The inclusions where shown by, e.g., Bejan and Gómez (2012b), and the last equality by Cross (1967) and Bennett (1983).

(i) *locational consistency*: A player cannot be in two different coalitions at the same time.

(ii) *time consistency*: the total amount of time spent by coalition S should be equal to its time allocation $\alpha(S)$.

We also add two requirements in order to avoid bizarre or obviously inefficient timetables. The first is that for each coalition, its active period should be a union of intervals of positive duration (isolated instants are not allowed). The second is that the timetable should not contain "holes", that is, periods during which no coalition is active. These considerations lead to the following formal definition.

Definition 4. Let α be a time allocation, and $\mathcal{A}$ its collection of active coalitions. We call *coalition timetable* a function $f_\alpha : [0, +\infty[\to 2^{\mathcal{A}}$ which satisfies the following properties:

(i) $\forall t \in [0, +\infty[$, $\forall S, T \in f_\alpha(t)$ such that $S \neq T$, we have $S \cap T = \emptyset$;

(ii) For any coalition S in $\mathcal{A}$, $\tau(S) = \{t \in [0, +\infty[\mid f_\alpha(t) \ni S\}$ is a union of disjoint intervals of positive Lebesgue measure, and in addition, $l(\tau(S)) = \alpha(S)$, where l is the Lebesgue measure on $\mathbb{R}$;

(iii) If $f_\alpha(t) = \emptyset$ for some $t \in [0, +\infty[$, then $f_\alpha(t') = \emptyset$ for all $t' > t$.

Note that because of (iii), every timetable starts at $t = 0$. We denote by $\mathcal{F}(\alpha)$ the set of all timetables for time allocation α. Note that this set is in general not denumerable. Indeed, take $n = 3$ and coalitions $12, 23$ with alloted time 1 to each of them. Then $f(t) = 12$ for $t \in [a, a+1]$ and $f(t) = 23$ for $t \in [0, a[\cup]a+1, 2]$ is a possible timetable for any $a \in]0, 1[$.

The *duration* of a timetable f_α is defined as

$$d_{\max}(f_\alpha) = \sup\{t \in [0, +\infty[\mid f_\alpha(t) \neq \emptyset\}.$$

Due to conditions (ii) and (iii) in Definition 4, $d_{\max}(f_\alpha)$ is bounded above by $\sum_{S \in \mathcal{A}} \alpha(S) < \infty$, therefore $d_{\max}(f_\alpha)$ is finite. For a time allocation α, the *minimal duration of α* is

$$d(\alpha) = \inf\{d_{\max}(f_\alpha), f_\alpha \in \mathcal{F}(\alpha)\}.$$

Remark 1. It is left to the reader to prove the homogeneity of d, that is, $\forall r \geq 0, d(r\alpha) = rd(\alpha)$. Hence, results does not change when we change the time measurement units.

Note that $d(\alpha)$ is finite since each duration is a positive finite number,

Theorem 1. There always exists a timetable with duration $d(\alpha)$.

Proof. We denote by $\mathcal{F}(\alpha)$ the set of coalition timetables associated with time allocation α. Let $\mathcal{T}$ be the product topology on $\mathcal{F}(\alpha)$. We know by Tychonoff theorem that $\mathcal{F}(\alpha)$ is compact with the topology $\mathcal{T}$. Let $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{F}(\alpha)^{\mathbb{N}}$ be a sequence which converges to some $f \in \mathcal{F}(\alpha)$. Since $\mathcal{F}(\alpha)$ is equipped with the product topology, we have:

$$\forall t \in [0, +\infty[, \lim_{n \to +\infty} f_n(t) = f(t).$$

Since $\mathcal{F}(\alpha)$ is a space of functions valued in a finite set, we have:

$$\forall t \in [0, +\infty[, \exists N_t \in \mathbb{N}, \forall n \geq N_t, f_n(t) = f(t).$$

Hence,

$$\exists N_{d_{max}(f)} \in \mathbb{N}, \forall n \geq N_{d_{max}(f)}, f_n(d_{max}(f)) = f(d_{max}(f)) \neq \emptyset,$$

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and we deduce that:

$$\inf_{k \geq N_{d_{max}(f)}} d_{max}(f_k) \geq d_{max}(f).$$

Therefore,

$$d_{max}(f) \leq \liminf_{n \to +\infty} d_{max}(f_n). \tag{1}$$

On the other hand, the definition of inf implies that for all $j \in \mathbb{N}$, there exists $f_j \in \mathcal{F}(\alpha)$ such that $d_{max}(f_j) - d(\alpha) \leq \frac{1}{j}$. Hence, $\lim_{j \to +\infty} d_{max}(f_j) \to d(\alpha) \in [0, +\infty[$. Since $\mathcal{F}(\alpha)$ is compact, there exists a subsequence $(f_{j_k})_{k \in \mathbb{N}}$ of $(f_j)_{j \in \mathbb{N}}$ and some $f \in \mathcal{F}(\alpha)$ so that $\lim_{k \to +\infty} f_{j_k} = f$. Since the limit $\lim_{j \to +\infty} d_{max}(f_j)$ exists and is equal to $d(\alpha)$, it follows that $\lim_{k \to +\infty} d_{max}(f_{j_k})$ exists and is equal to $d(\alpha)$

Therefore, by (1),

$$d_{max}(f) \leq \liminf_{k \to +\infty} d_{max}(f_{j_k}) = \lim_{k \to +\infty} d_{max}(f_{j_k}) = d(\alpha) = \inf\{d_{\max}(f_\alpha), f_\alpha \in \mathcal{F}(\alpha)\}$$

Hence

$$d_{\max}(f) = d(\alpha)$$

$\square$

Any timetable with duration $d(\alpha)$ is called an *optimal timetable* of α.

Example 1. Let $N = \{1,2,3,4,5\}$ and consider the following collection $\mathcal{A}$ with its time allocation:

S	12	23	34	45	15
$\alpha(S)$	1/2	1/2	1/2	1/2	1/2

Note that this is a minimal balanced system. We give two examples of timetable in Figure 1.
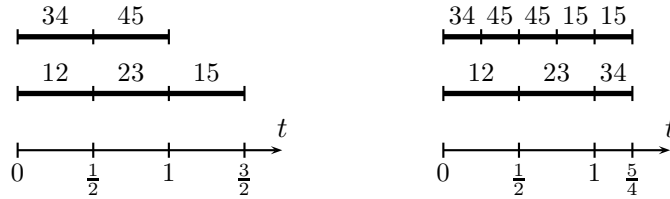

Figure 1: Time allocation for a 5-player game: Two examples of timetable

The problem of computing the minimal duration in general seems to be extremely complex. We give hereafter a lower and an upper bound of it. We recall that in a graph $G = (V, E)$ with set of vertices V and set of edges E, a *clique* is a subset of vertices which are pairwise connected (by an edge) in G. An *independent set* is a subset of vertices I such that no pair of vertices is connected in G. A partition $\mathcal{K} = \{I_1, \ldots, I_m\}$ of V in independent sets is called a *coloring* of G. Given a time allocation α in N, we construct the graph $G = (\mathcal{A}, E)$, where the set of vertices is the set of active coalitions under α, and there is an edge (S, T) if and only if $S \cap T \neq \emptyset$.

Theorem 2. For any time allocation α in N, the minimal duration satisfies

$$\max_{\mathcal{C}:\text{clique of } G} \alpha(\mathcal{C}) \leq d(\alpha) \leq \min_{\mathcal{K}:\text{coloring of } G} \alpha(\mathcal{K}),$$

with $\alpha(\mathcal{C}) = \sum_{S \in \mathcal{C}} \alpha(S)$, and $\alpha(\mathcal{K}) = \sum_{I \in \mathcal{K}} \max_{S \in I} \alpha(S)$.

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Proof. By condition (i) in Definition 4, the subsets in a clique $\mathcal{C}$ must have disjoint time slots in a timetable. This being true for any clique leads to the lower bound.

Now, observe that if no splitting of time slots is allowed, (i.e., if $\tau(S)$ is a single interval for all $S \in \mathcal{C}$), a feasible timetable amounts to find a coloring of the graph, and its duration is the sum of maximal (in term of $\alpha(S)$) coalitions in each independent set. Taking a coloring minimizing this duration gives the upper bound, since splitting of time slots can only improve this solution. $\square$

The next result concerns time allocations in standard form.

Proposition 1. Let α be a time allocation in standard form. Then $d(\alpha) \geq 1$, and a timetable f_α satisfies $d_{\max}(f_\alpha) = 1$ if and only if for each $t \in [0,1]$, $f_\alpha(t)$ is a partition of N, and is $\emptyset$ otherwise.

Proof. For each $i \in N$, the collection $\mathcal{A}(i) = \{S \in \mathcal{A} \mid S \ni i\}$ is a clique of G, and $\alpha(\mathcal{A}(i)) = 1$ since $\mathcal{A}$ is a balanced collection. It follows from Theorem 2 that $d(\alpha) \geq 1$.

Take f_α such that $d_{\max}(f_\alpha) = 1$, and suppose that there exists a nonnull (in the sense of Lebesgue measure) time interval in $[0,1]$ on which f_α is not a partition of N. This would mean that there exists some $i \in N$ such that on a nonnull time interval $\Delta \subseteq [0,1]$, $\bigcup_{S \in f_\alpha(t)} S$ does not contain i for all $t \in \Delta$. However, since α is in standard form, we have $\alpha(\mathcal{A}(i)) = 1$, which would imply that for some nonnull interval $\Delta' \subseteq]1, \infty[$, $f_\alpha(t) \ni S \ni i$ for all $t \in \Delta'$, a contradiction with $d_{\max}(\alpha) = 1$. The converse is obvious. $\square$

Note that if f_α has the above property, it does not necessarily imply that the set of active coalitions under α is a collection of partitions, as the next example shows: Consider $N = \{1,2,3\}$, and a timetable f_α defined on $[0,1]$ as follows: $f_\alpha(t) = \{12,3\}$ for $t \in [0,\frac{1}{3}]$, $f_\alpha(t) = \{1,2,3\}$ for $t \in \left]\frac{1}{3}, \frac{2}{3}\right]$, and $f_\alpha(t) = \{13,2\}$ for $t \in \left]\frac{2}{3}, 1\right]$. Then $\mathcal{A} = \{1,2,3,12,13\}$, which is not a collection of partitions.

We say that a timetable f_α has m *full rows of identical length* if for every t such that $f_\alpha(t) \neq \emptyset$, $|f_\alpha(t)| = m$. A simple sufficient condition for optimality is the following.

Proposition 2. Let α be a time allocation, $\mathcal{A}$ its collection of active coalitions, and m be the size of a greatest independent set (i.e., one can find at most m disjoint subsets in $\mathcal{A}$). A timetable f_α is optimal if it has m full rows of identical length. Consequently, a lower bound for $d(\alpha)$ is

$$d(\alpha) \geq \frac{1}{m} \sum_{S \in \mathcal{A}} \alpha(S).$$

Proof. If f_α were not optimal, it would be possible to shorten its duration by taking the sets being active on some interval $[t_0, t_1]$ and dispatching them on other time intervals. However, for every t in the domain of f, it is not possible to add a set to $f_\alpha(t)$ because there is no independent set of size greater than m. The statement on the lower bound directly follows. $\square$

Coming back to Example 1, the corresponding graph is a cycle of length 5, therefore, maximal cliques are for example $\{12,23\}$, which gives a lower bound of 1. On Figure 1, the left timetable implements the upper bound solution (no splitting), while the right figure is the optimal solution. Indeed, it has two full rows of identical length, and there is at most two disjoint sets in $\mathcal{A}$. Therefore the optimal duration $d(\alpha)$ is $\frac{5}{4}$. This shows that time allocation in standard form does not have necessarily an optimal duration equal to 1 (see Proposition 1).

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3   Strategic coalition formation

3.1   The model

We denote by v a function that associates with each subset S of N, a real number, $v(S)$ which represents the payoff that the coalition S could create alone during one unit of time. We assume that a coalition S which is active during $\alpha(S)$ unit of time produces a payoff equals to $\alpha(S)v(S)$. Then to each time allocation α, we can associate the *total payoff* $G(v, \alpha)$ produced during at least $d(\alpha)$ units of time and defined by:

$$G(v, \alpha) := \sum_{S \subseteq N} \alpha(S)v(S).$$

We say that a payoff vector z is (v, α)-*feasible* if it satisfies $z(N) \leq G(v, \alpha)$. We assume that each player $i \in N$ chooses a *strategic time allocation* $\alpha_i \in \mathbb{R}^{2^N}$ that associates with each subset S of N a positive real number $\alpha_i(S)$ which represents the amount of unit of time that the player i wants to spend in coalition S. We suppose that $\alpha_i(S) = 0$ if $S \not\ni i$, that is, i cannot devote any time in a coalition to which he does not belong. We denote by $\mathcal{T}_i(N) \subseteq \mathcal{T}(N)$ the set of strategic time allocations of the player $i \in N$. We assume that the length of activity period $\bar{\alpha}(S)$ of a coalition S is equal to the minimum amount of time that each member wants to spend inside: The first who wants to leave a coalition decides the lifetime of this coalition. Hence, for a vector $(\alpha_i)_{i \in N} \in \prod_{i \in N} \mathcal{T}_i$ of strategic time allocations, we can associate a time allocation called *lifetime* and denoted by $\bar{\alpha}$ defined by:

$$\bar{\alpha}(S) = \min_{i \in S} \alpha_i(S).$$

We say that a coalition is active if it is active under $\bar{\alpha}$.

A *strategy* of a player $i \in N$ is a pair $(x_i, \alpha_i) \in \mathbb{R} \times \mathcal{T}_i(N)$, where x_i represents the payoff that the player i wants to have for his participation, and α_i is a strategic time allocation of the player i. We consider two kinds of utilities: The first one puts to the fore the desire for each agent to maximize his payoff with regard to the length of his active participation, that is, the desire to have the best "hourly wage", while the second utility gives emphasis to the desire of agent for maximising his payoff at the end of the timetable or equivalently to maximize his "compensation per hour lived". Moreover, we suppose that players who propose an allocation which is not "feasible" receive a punishment equals to -1. We denote by u_i and we call *utility per active unit of time of the player i* the function defined over $\mathbb{R}^N \times \prod_{j \in N} \mathcal{T}_j$ by:

$$u_i(x, (\alpha_j)_{j \in N}) := \begin{cases} 0 & \text{if } \sum_{S \ni i} \alpha_i(S) = 0 \\ \dfrac{x_i}{\sum_{S \ni i} \alpha_i(S)} & \text{if } x \text{ is } (v, \bar{\alpha})\text{-feasible and } \sum_{S \ni i} \alpha_i(S) \neq 0 \\ -1 & \text{otherwise.} \end{cases}$$

On the other hand, we denote by $\tilde{u}_i$ and we call *utility per total unit of time of the player i* the function defined over $\mathbb{R}^N \times \prod_{j \in N} \mathcal{T}_j$ by:

$$\tilde{u}_i(x, (\alpha_j)_{j \in N}) := \begin{cases} 0 & \text{if } d(\bar{\alpha}) = 0 \\ \dfrac{x_i}{d(\bar{\alpha})} & \text{if } x \text{ is } (v, \bar{\alpha})\text{-feasible and } d(\bar{\alpha}) \neq 0 \\ -1 & \text{otherwise.} \end{cases}$$

Let $G = (N, \Sigma^1, \cdots, \Sigma^n; g_1, \cdots, g_n)$ be a n-player strategic game, where Σ^i is the set of strategies of the player i, and the mapping $g_i : \prod_{j \in N} \Sigma^j \mapsto \mathbb{R}$ represents the utility of i. For

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$S \subseteq N$, we denote by Σ^S the set $\Sigma^S := \prod\limits_{i \in S} \Sigma^i$, and for $\sigma \in \Sigma^N$, we denote by σ_S the projection of σ on Σ^S.

We say that $\sigma^* \in \Sigma^N$ is a *strong Nash equilibrium* of G, (Aumann, 1959) or strong equilibrium of G if one has :

$$\forall S \subseteq N, \forall \sigma_S \in \Sigma^S, \exists i \in S, g_i(\sigma^*) \geq g_i(\sigma_S, \sigma^*_{N \setminus S}).$$

In the following two subsections, we define and study the strong Nash equilibria of $(N, \mathbb{R}^N \times \prod\limits_{j \in N} \mathcal{T}_j, (u_i)_{i \in N})$ and $(N, \mathbb{R}^N \times \prod\limits_{j \in N} \mathcal{T}_j, (\tilde{u}_i)_{i \in N})$

3.2 Strong Nash equilibria of $(N, \mathbb{R}^N \times \prod\limits_{j \in N} \mathcal{T}_j, (u_i)_{i \in N})$

Let v be a TU-game, we denote by d-$MC(v)$ and we call *d-multicoalitional core* of v the set of strong Nash equilibria of $(N, \mathbb{R}^N \times \prod\limits_{j \in N} \mathcal{T}_j, (u_i)_{i \in N})$.

We show that the d-multicoalitional core is never empty, and that it has properties close to the ones of the classical core, which justify its name.

Theorem 3. For any game v in $\mathcal{G}(N)$, the set d-$MC(v)$ is nonempty, and, for any $(x, (\alpha_i)_{i \in N}) \in$ d-$MC(v)$, we have:

(i) $\forall i \in N, u_i(x, (\alpha_i)_{i \in N}) \geq 0$

(ii) $\forall S \subseteq N, \sum\limits_{i \in S} u_i(x, (\alpha_i)_{i \in N}) \geq v(S)$ (coalitional rationality)

(iii) $x(N) = G(v, \bar{\alpha})$ (($v, \bar{\alpha}$)-efficiency)

Proof. Let v be a game in $\mathcal{G}(N)$. First, we prove the nonemptiness property. Let I be the subset of N defined by $I := \{i \in N, v(i) < 0\}$. We define the game $\tilde{v}$ by $\tilde{v}(i) = 0$ if $i \in I$ and $\tilde{v}(S) = v(S)$ otherwise. Let $x \in$ d-$\mathsf{C}(\tilde{v})$. There exists a balanced system $(\mathcal{A}, (\lambda_S)_{S \subseteq N})$ such that

$$x(N) = \sum_{S \subseteq N} \lambda_S \tilde{v}(S).$$

We put $\alpha_i(i) = 0$ if $i \in I$ and $\alpha_i(S) = \lambda_S$ otherwise. Hence:

$$\sum_{S \subseteq N} \lambda_S \tilde{v}(S) = \sum_{S \subseteq N} \bar{\alpha}(S) v(S) = x(N).$$

We want to prove that the strategy $(x, (\alpha_i)_{i \in N})$ belongs to the d-multicoalitional core. Suppose there exists for a coalition $S \subseteq N$ a coalitional deviation $((x'_i)_{i \in S}, (\alpha'_i)_{i \in S})$ such that $\forall i \in S$, $u_i(x, (\alpha_i)_{i \in N}) < u_i((x_i)_{i \in N \setminus S}, (x'_i)_{i \in S}, (\alpha_i)_{i \in N \setminus S}, (\alpha'_i)_{i \in S})$. Since $\forall i \in N, 0 \leq u_i(x, (\alpha_i)_{i \in N})$, the strict inequality $\forall i \in S, u_i(x, (\alpha_i)_{i \in N}) < u_i((x_i)_{i \in N \setminus S}, (x'_i)_{i \in S}, (\alpha_i)_{i \in N \setminus S}, (\alpha'_i)_{i \in S})$ and the definition of u_i ensure that the $(v, ((\alpha_i)_{i \in N \setminus S}, (\alpha'_i)_{i \in S}))$-efficiency is satisfied. We denote by $\bar{\alpha}'$ the lifetime which corresponds to $((\alpha_i)_{i \in N \setminus S}, (\alpha'_i)_{i \in S})$ and we adopt the convention $\frac{0}{0} = 0$. We have:

$$u_i((x_i)_{i \in N \setminus S}, (x'_i)_{i \in S}, (\alpha_i)_{i \in N \setminus S}, (\alpha'_i)_{i \in S}) = \frac{x'_i}{\sum\limits_{S \ni i} \alpha'_i(S)}$$

if $i \in S$ and

$$u_i((x_i)_{i \in N \setminus S}, (x'_i)_{i \in S}, (\alpha_i)_{i \in N \setminus S}, (\alpha'_i)_{i \in S}) = \frac{x_i}{\sum\limits_{S \ni i} \alpha_i(S)} = u_i(x, (\alpha_i)_{i \in N})$$

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if $i \in N \setminus S$. The $(v, ((\alpha_i)_{i \in N \setminus S}, (\alpha'_i)_{i \in S}))$-efficiency ensures that

$$x'(S) + x(N \setminus S) \leq \sum_{T \subseteq N} \bar{\alpha}'(T) v(T). \tag{2}$$

Furthermore, $(\mathcal{A}, (\lambda_S)_{S \subseteq N})$ is a balanced system, then, $\forall i \in N$, $\sum_{S \ni i} \alpha_i(S) \leq \sum_{S \ni i} \lambda_S = 1$, therefore, $\forall i \in N$, $u_i(x, (\alpha_i)_{i \in N}) = \frac{x_i}{\sum_{S \ni i} \alpha_i(S)} \geq x_i$. Since $x \in \mathsf{d\text{-}C}(\tilde{v})$, we deduce that:

$$\sum_{T \subseteq N} \bar{\alpha}'(T) v(T) \leq \sum_{T \subseteq N} \alpha'(T) x(T) \leq \sum_{T \subseteq N} \bar{\alpha}'(T) \sum_{i \in T} u_i(x, (\alpha_i)_{i \in N}). \tag{3}$$

Then, we know that, $\forall i \in N$, $u_i(x, (\alpha_i)_{i \in N}) \leq u_i((x_i)_{i \in N \setminus S}, (x'_i)_{i \in S}, (\alpha_i)_{i \in N \setminus S}, (\alpha'_i)_{i \in S})$ with equality for $i \in N \setminus S$ and strict inequality for $i \in S$. Hence,

$$\sum_{T \subseteq N} \bar{\alpha}'(T) \sum_{i \in T} u_i(x, (\alpha_i)_{i \in N}) < \sum_{T \subseteq N} \bar{\alpha}'(T) \sum_{i \in T} u_i((x_i)_{i \in N \setminus T}, (x'_i)_{i \in T}, (\alpha_i)_{i \in N \setminus T}, (\alpha'_i)_{i \in T}). \tag{4}$$

A simple rewriting gives:

$$\sum_{T \subseteq N} \bar{\alpha}'(T) \sum_{i \in T} u_i((x_i)_{i \in N \setminus T}, (x'_i)_{i \in T}, (\alpha_i)_{i \in N \setminus T}, (\alpha'_i)_{i \in T}) =$$
$$\sum_{i \in N} \sum_{T \ni i} \bar{\alpha}'(T) u_i((x_i)_{i \in N \setminus T}, (x'_i)_{i \in T}, (\alpha_i)_{i \in N \setminus T}, (\alpha'_i)_{i \in T}). \tag{5}$$

The definition of $\bar{\alpha}'$ ensures that $\forall T \subseteq N$, and $\forall i \in N$, $\bar{\alpha}'(T) \leq \alpha_i(T)$. Hence,

$$\sum_{i \in N} \sum_{T \ni i} \bar{\alpha}'(T) u_i((x_i)_{i \in N \setminus T}, (x'_i)_{i \in T}, (\alpha_i)_{i \in N \setminus T}, (\alpha'_i)_{i \in T}) \leq$$
$$\sum_{i \in S} \left(\frac{x'_i}{\sum_{T \ni i} \bar{\alpha}'(T)} \sum_{T \ni i} \bar{\alpha}'(T) \right) + \sum_{i \in N \setminus S} \left(\frac{x_i}{\sum_{T \ni i} \bar{\alpha}'(T)} \sum_{T \ni i} \bar{\alpha}'(T) \right), \tag{6}$$

and, finally,

$$\sum_{i \in S} \left(\frac{x'_i}{\sum_{T \ni i} \bar{\alpha}'(T)} \sum_{T \ni i} \bar{\alpha}'(T) \right) + \sum_{i \in N \setminus S} \left(\frac{x_i}{\sum_{T \ni i} \bar{\alpha}'(T)} \sum_{T \ni i} \bar{\alpha}'(T) \right) = x'(S) + x(N \setminus S). \tag{7}$$

The inequalities (2) to (7) leads to the contradiction $x'(S) + x(N \setminus S) < x'(S) + x(N \setminus S)$.

We prove now the other statements of the theorem.

(i) If $\exists i \in N$, such that $u_i(x, (\alpha_j)_{j \in N}) < 0$ then the strategy (x_i, α_i) is strictly dominated by the strategy which consists in claiming a payoff $x'_i = 0$ for a strategic time allocation equals to 0 for every coalition.

(ii) We want to prove that $\forall S \subseteq N$, $\sum_{i \in S} u_i(x, \alpha) \geq v(S)$. Suppose there exists $S \subseteq N$ such that $\sum_{i \in S} u_i(x, (\alpha_i)_{i \in N}) < v(S)$.

  (a) If there exists $i \in N$ such that $u_i(x, (\alpha_i)_{i \in N}) = -1$ then it is clear that $(x, (\alpha_i)_{i \in N})$ is not a strong Nash equilibrium.

  (b) If $\forall i \in S$, $u_i(x, (\alpha_i)_{i \in N}) \neq -1$, the strategy of the players belonging to S are strictly dominated by $\alpha'_i(S) = \alpha_i(S) + 1$, $\alpha'_i(T) = \alpha_i(T)$ if $T \neq S$, $\forall i \in S$ and $x'_i = x_i + \frac{v(S) - \sum_{i \in S} u_i(x, (\alpha_i)_{i \in N})}{|S|}$.

(iii) If $x(N) < G(v, \bar{\alpha})$ then there exists a deviation of the coalition N which consists in claiming $x'_i = x_i + \frac{G(v, \bar{\alpha}) - x(N)}{|N|}$.

$\square$

Observe that, unlike the d-core, the d-multicoalitional core has only nonnegative payoff vectors. We show now the converse result: the d-multicoalitional core is precisely the set of strategies satisfying requirements (i) to (iii) in Theorem 3.

Theorem 4. For any game $v \in \mathcal{G}(N)$ the d-multicoalitional core of v is the set of $(x, (\alpha_i)_{i \in N}) \in \mathbb{R}^N \times \mathcal{T}(N)$ such that:

(i) $u_i(x, (\alpha_i)_{i \in N}) \geq 0$.

(ii) $\forall S \subseteq N, \sum_{i \in S} u_i(x, (\alpha_i)_{i \in N}) \geq v(S)$.

(iii) $x(N) = \sum_{S \subseteq N} \bar{\alpha}(S) v(S)$.

Proof. Let $(x, (\alpha_i)_{i \in N})$ be a couple satisfying (i), (ii) and (iii). We want to prove that the strategy $(x, (\alpha_i)_{i \in N})$ belongs to the d-multicoalitional core. The proof is very similar to the proof of the nonemptiness of the d-multicoalitional core in Theorem 3. Suppose there exists for a coalition $S \subseteq N$ a coalitional deviation $((x'_i)_{i \in S}, (\alpha'_i)_{i \in S})$ such that $\forall i \in S, u_i(x, (\alpha_i)_{i \in N}) < u_i((x_i)_{i \in N \setminus S}, (x'_i)_{i \in S}, (\alpha_i)_{i \in N \setminus S}, (\alpha'_i)_{i \in S})$. Since $\forall i \in N, 0 \leq u_i(x, (\alpha_i)_{i \in N})$, the strict inequality $\forall i \in S, u_i(x, (\alpha_i)_{i \in N}) < u_i((x_i)_{i \in N \setminus S}, (x'_i)_{i \in S}, (\alpha_i)_{i \in N \setminus S}, (\alpha'_i)_{i \in S})$ and the definition of u_i ensure that the $(v, ((\alpha_i)_{i \in N \setminus S}, (\alpha'_i)_{i \in S}))$-efficiency is satisfied. We denote by $\bar{\alpha}'$ the lifetime which corresponds to $((\alpha_i)_{i \in N \setminus S}, (\alpha'_i)_{i \in S})$ and we adopt the convention $\frac{0}{0} = 0$. Since $u_i(x, (\alpha_i)_{i \in N}) \geq 0$, we have:

$$u_i((x_i)_{i \in N \setminus S}, (x'_i)_{i \in S}, (\alpha_i)_{i \in N \setminus S}, (\alpha'_i)_{i \in S}) = \frac{x'_i}{\sum_{S \ni i} \alpha'_i(S)}$$

if $i \in S$ and

$$u_i((x_i)_{i \in N \setminus S}, (x'_i)_{i \in S}, (\alpha_i)_{i \in N \setminus S}, (\alpha'_i)_{i \in S}) = \frac{x_i}{\sum_{S \ni i} \alpha_i(S)} = u_i(x, (\alpha_i)_{i \in N})$$

if $i \in N \setminus S$. The $(v, ((\alpha_i)_{i \in N \setminus S}, (\alpha'_i)_{i \in S}))$-efficiency ensures that

$$x'(S) + x(N \setminus S) \leq \sum_{T \subseteq N} \bar{\alpha}'(T) v(T).$$

Furthermore, by hypothesis,

$$\sum_{T \subseteq N} \bar{\alpha}'(T) v(T) \leq \sum_{T \subseteq N} \bar{\alpha}'(T) \sum_{i \in T} u_i(x, (\alpha_i)_{i \in N}).$$

Then, by following the proof of nonemptiness of the d-multicoalitional core in Theorem 3, we deduce

$$\sum_{T \subseteq N} \bar{\alpha}'(T) \sum_{i \in T} u_i(x, (\alpha_i)_{i \in N}) < x'(S) + x(N \setminus S),$$

a contradiction. $\square$

Proposition 3. Let $(x, (\alpha_i)_{i \in N})$ be an element of the d-multicoalitional core, and $i \in N$ such that $u_i(x, (\alpha_i)_{i \in N}) \neq 0$. Then for all $S \ni i$, we have $\alpha_i(S) = \bar{\alpha}(S)$.

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Proof. By definition of u_i, we have $u_i = \frac{x_i}{\sum_{S \ni i} \alpha_i(S)}$. Since $\sum_{S \ni i} \alpha_i(S) \geq \sum_{S \ni i} \bar{\alpha}(S)$, choosing $\alpha_i \neq \bar{\alpha}$ would make the strategy $(x, (\alpha_i)_{i \in N})$ dominated. $\square$

Essentially, the proposition says that in an equilibrium, all players ask for the same time allocation $\bar{\alpha}$. The important consequence of this is, if all players can reach a positive income, then *the d-multicoalitional core is a multicoalitional solution* in the sense of Definition 3.

Proposition 4. Let $(x, (\alpha_i)_{i \in N})$ an element of the d-multicoalitional core of a game v. Let S be a coalition such that $\bar{\alpha}(S) > 0$. Then $\sum_{i \in S} u_i(x, (\alpha_i)_{i \in N}) = v(S)$.

Proof. Let S be a coalition such that $\bar{\alpha}(S) > 0$. Then if $\sum_{i \in S} \frac{x(i)}{\sum_{S \ni i} \alpha_i(S)} > v(S)$, we have by Theorem 3

$$x(N) = \sum_{T \subseteq N} \bar{\alpha}(T) v(T) < \sum_{T \subseteq N} \bar{\alpha}(T) \sum_{i \in T} u_i(x, (\alpha_i)_{i \in N}) = \sum_{i \in N} \sum_{T \ni i} \bar{\alpha}(T) \frac{x_i}{\sum_{T \ni i} \bar{\alpha}(T)} = x(N),$$

a contradiction.

Then $\alpha(S) > 0$ implies $\sum_{i \in S} u_i(x, (\alpha_i)_{i \in S}) = v(S)$. $\square$

Proposition 5. If there exists $(x, (\alpha_i)_{i \in N})$ such that $(u_i(x, (\alpha_i)_{i \in N}))_{i \in N} \in$ d-$\mathsf{C}(v)$ and $u_i(x, (\alpha_i)_{i \in N}) \geq 0 \; \forall i \in N$, then $(x, (\alpha_i)_{i \in N}) \in$ d-$MC(v)$.

Proof. The proof is similar to that of Theorem 3. $\square$

Remark 2. As we can see from the following example, unlike the d-core or the $\mathcal{B}$-core (Cesco, 2012) it is not necessary to require that each player is active during exactly one unit of time. Moreover, $(x, (\alpha_i)_{i \in N})$ in the d-multicoalitional core does not necessarily imply that $(u_i(x, (\alpha_i)_{i \in N}))$ is in the d-core.

Consider the game v defined over the subset of $N = \{1, 2, 3\}$ by:

S	1	2	3	12	13	23	123
$v(S)$	1	1	1	3	3	3	4

and the strategy $(x, (\alpha_i)_{i \in N})$ defined by: $x(1) = 3, x(2) = x(3) = 2$ and $\alpha_1(1) = \alpha_1(12) = \alpha_1(13) = 1, \alpha_2(12) = \alpha_3(13) = 1$, and $\alpha_i(S) = 0$ otherwise. This yields

$$x(N) = 7 = \sum_{S \subseteq N} \bar{\alpha}(S) v(S), u_1(x, (\alpha_i)_{i \in N}) = 1, u_2(x, (\alpha_i)_{i \in N}) = u_3(x, (\alpha_i)_{i \in N}) = 2.$$

Then $(x, (\alpha_i)_{i \in N}) \in$ d-$MC(v)$ but $(u_i(x, (\alpha_i)_{i \in N}))_{i \in N} \notin$ d-$\mathsf{C}(v)$, because $\sum_{i \in N} u_i(x, (\alpha_i)_{i \in N}) = 5 > 4.5 = v(N) + \bar{t}(v)$.

The following proposition gives a result very close to the one of (Bejan and Gómez, 2012a).

Proposition 6. If $(x, (\alpha_i)_{i \in N}) \in \mathbb{R}^N \times \prod_{j \in N} \mathcal{T}_j$ is such that $\bar{\alpha}$ is in standard form, and $x_i \geq 0$ for all $i \in N$, then the following equivalence is satisfied:

$$(x, (\alpha_i)_{i \in N}) \in \text{d-}MC(v) \Leftrightarrow (u_i(x, (\alpha_i)_{i \in N}))_{i \in N} \in \text{d-}\mathsf{C}(v)$$

Proof. By Proposition 5, it suffices to prove the right implication. Let $(x, (\alpha_i)_{i \in N}) \in$ d-$MC(v)$, we have $\sum_{i \in S} u_i(x, (\alpha_i)_{i \in N}) \geq v(S)$ for every $S \subseteq N$. Hence,

$$v(N) + \bar{t}(v) \leq \sum_{i \in N} u_i(x, (\alpha_i)_{i \in N}). \tag{8}$$

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Furthermore, by Proposition 4, we have equality if $\bar{\alpha}(S) > 0$. Hence, since $\bar{\alpha}$ is in standard form, there exists $(\delta_S)_{S \in \{T \subseteq N, \bar{\alpha}(T) > 0\}}$ such that

$$\sum_{i \in N} u_i(x, (\alpha_i)_{i \in N}) = \sum_{S \in \{T \subseteq N, \bar{\alpha}(T) > 0\}} \delta_S \sum_{i \in S} u_i(x, (\alpha_i)_{i \in N}) = \sum_{S \in \{T \subseteq N, \bar{\alpha}(T) > 0\}} \delta_S v(S) \leq v(N) + \bar{t}(v).$$

By combining with (8), equality holds throughout, and $(u_i(x, (\alpha_i)_{i \in N}))_{i \in N} \in$ d-$\mathsf{C}(v)$. $\qquad \square$

3.3 The c-multicoalitional core

Let v be a TU-game, we denote by $SN\tilde{u}(v)$ the set of strong Nash equilibria of $(N, \mathbb{R}^N \times \prod_{j \in N} \mathcal{T}_j, (\tilde{u}_i)_{i \in N})$. The next proposition shows that these strong Nash equilibria have properties similar to elements of the d-multicoalitional core.

Proposition 7. For any $(x, (\alpha_i)_{i \in N}) \in SN\tilde{u}(v)$, we have:

   (i) $\forall i \in N, \tilde{u}_i(x, (\alpha_i)_{i \in N}) \geq 0$

   (ii) $\forall S \subseteq N, \sum_{i \in S} \tilde{u}_i(x, (\alpha_i)_{i \in N}) \geq v(S)$ (coalitional rationality)

   (iii) $x(N) = G(v, \bar{\alpha})$ (($v, \bar{\alpha}$)-efficiency)

Proof. The proof is similar to the corresponding points of Theorem 3. $\qquad \square$

The next proposition investigates the existence of such equilibria.

Proposition 8. $SN\tilde{u}(v) \neq \emptyset$ if and only if $SN\tilde{u}(v) \cap$ d-$MC(v) \neq \emptyset$.

Proof.   (i) Suppose $SN\tilde{u}(v) \neq \emptyset$, and let $(x, (\alpha_i)_{i \in N})$ be an element of $SN\tilde{u}(v)$. By using Theorem 2, we know that:
$$\max_{\mathcal{C}:\text{clique of } G} \alpha(\mathcal{C}) \leq d(\alpha).$$

Hence, by using the convention $\frac{0}{0} = 0$ and since $\forall i \in N, x_i \geq 0$, and $\{S \ni i, \bar{\alpha}(S) > 0\}$ is a clique, we have:
$$\frac{x_i}{d(\bar{\alpha})} \leq \frac{x_i}{\sum_{S \ni i} \bar{\alpha}(S)}.$$

We put $\bar{\alpha}_i(S) = \bar{\alpha}(S)$ for all $i \in N$ and $S \subseteq N$. We deduce that:
$$\tilde{u}_i(x, (\alpha_i)_{i \in N}) \leq u_i(x, (\bar{\alpha}_i)_{i \in N}).$$

Hence, if $x_i = 0$, we have $\tilde{u}_i(x, (\alpha_i)_{i \in N}) = u_i(x, (\bar{\alpha}_i)_{i \in N})$ and if $x_i > 0$, there exists an active coalition S under $\bar{\alpha}$ which contains i, and by using Propositions 3 and 7,
$$v(S) \leq \sum_{i \in S} \tilde{u}_i(x, (\alpha_i)_{i \in N}) \leq \sum_{i \in S} u_i(x, (\bar{\alpha}_i)_{i \in N}) = v(S)$$

from which we deduce the equality
$$\tilde{u}_i(x, (\alpha_i)_{i \in N}) = u_i(x, (\bar{\alpha}_i)_{i \in N}) \quad \forall i \in N.$$

It is clear that:
$$\tilde{u}_i(x, (\alpha_i)_{i \in N}) = \tilde{u}_i(x, (\bar{\alpha}_i)_{i \in N}) \quad \forall i \in N.$$

Therefore $(x, (\bar{\alpha}_i)_{i \in N}) \in SN\tilde{u}(v)$ and furthermore, we have by Proposition 7:

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(a) $u_i(x, (\bar{\alpha}_i)_{i \in N}) \geq 0$.

(b) $\forall S \subseteq N, \sum_{i \in S} u_i(x, (\bar{\alpha}_i)_{i \in N}) \geq v(S)$.

(c) $x(N) = \sum_{S \subseteq N} \bar{\alpha}(S) v(S)$.

Therefore $(x, (\bar{\alpha}_i)_{i \in N}) \in SN\tilde{u}(v) \cap \text{d-}MC(v)$

(ii) The converse is obvious.

$\square$

Unlike the case where the utility is defined per active unit of time, it could be the case that no strong equilibrium exists when utility per total unit of time is used instead. The foregoing proposition has shown that if a strong Nash equilibrium exists, it can be turned into a strong equilibrium of both problems, i.e., with the two types of utility. The following definition is a natural consequence of this fact.

Definition 5. Let v be a TU-game, we denote by $c\text{-}MC(v)$ and we call *c-multicoalitional core of* v the set
$$c\text{-}MC(v) := SN\tilde{u}(v) \cap \text{d-}MC(v)$$

Elements of the c-multicoalitional core have the remarkable property to be undominated strategies both for utility per active unit of time *and* total unit of time. Since the c-multicoalitional core is a subset of the d-multicoalitional core, it follows that it is also a multicoalitional solution in the sense of Definition 3. The next theorem gives a necessary and sufficient condition for the nonemptiness of the c-multicoalitional core. It turns out that this condition is equivalent to the nonemptiness of the c-core of a related game.

Theorem 5. Let $v \in \mathcal{G}(N)$ be a game, and denote by $\tilde{v}$ the game defined by $\tilde{v}(i) = 0$ if $v(i) < 0$ and $\tilde{v}(S) = v(S)$ otherwise. Then $c\text{-}MC(v) \neq \emptyset$ if and only if $\tilde{v}$ is c-balanced, and moreover
$$c\text{-}\mathsf{C}(\tilde{v}) = \{(\tilde{u}_i(x, (\alpha_i)_{i \in N}))_{i \in N} \mid (x, (\alpha_i)_{i \in N}) \in c\text{-}MC(v)\}.$$

Proof. (i) Let us prove that if the c-multicoalitional core is nonempty, then so is the c-core. Let us take $(x, (\alpha_i)_{i \in N}) \in c\text{-}MC(v)$ and prove that $(\tilde{u}_i(x, (\alpha_i)_{i \in N}))_{i \in N} \in c\text{-}\mathsf{C}(\tilde{v})$. Firstly, we have
$$x(N) = \sum_{S \subseteq N} \bar{\alpha}(S) v(S) \leq \sum_{S \subseteq N} \bar{\alpha}(S) \tilde{v}(S).$$

Moreover, since $\tilde{u}_i(x, (\alpha_i)_{i \in N}) \geq 0$ and $\sum_{i \in S} \tilde{u}_i(x, (\alpha_i)_{i \in N}) \geq v(S)$, we deduce easily that $\sum_{i \in S} \tilde{u}_i(x, (\alpha_i)_{i \in N}) \geq \tilde{v}(S)$. This establishes coalitional rationality. In addition, we can deduce that
$$\sum_{S \subseteq N} \bar{\alpha}(S) \tilde{v}(S) \leq \sum_{S \subseteq N} \bar{\alpha}(S) \sum_{i \in S} \tilde{u}_i(x, (\alpha_i)_{i \in N}) \leq \sum_{i \in N} x_i \sum_{S \ni i} \frac{\bar{\alpha}(S)}{d(\bar{\alpha})}.$$

According to Theorem 2, $\sum_{S \ni i} \bar{\alpha}(S) \leq d(\bar{\alpha})$, therefore
$$\sum_{i \in N} x_i \sum_{S \ni i} \frac{\bar{\alpha}(S)}{d(\bar{\alpha})} \leq x(N).$$

Hence, equality holds throughout and for $i \in N$, if $x_i > 0$ then $\sum_{S \ni i} \bar{\alpha}(S) = d(\bar{\alpha})$, while if $x_i = 0$ then $\tilde{v}(i) = 0$ and $\sum_{S \ni i} \bar{\alpha}(S) \leq d(\bar{\alpha})$.

We put:
$$\alpha_i'(i) = \alpha_i(i) + (d(\bar{\alpha}) - \sum_{S \ni i} \bar{\alpha}(S)), \quad \text{if } x_i = 0$$

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and
$$\alpha_i'(S) = \alpha_i(S), \quad \text{otherwise.}$$

Then we have $d(\bar{\alpha}') = d(\bar{\alpha}) = \sum_{S \ni i} \bar{\alpha}_i'(S)$. Let $f_{\bar{\alpha}'}$ be an optimal timetable of $\bar{\alpha}'$. It is clear that $\forall t \in [0, d(\bar{\alpha}')], f_{\bar{\alpha}'}(t) \in \Pi(N)$.

Since $\Pi(N)$ is a finite set, we deduce that there exists $\pi_1, \ldots, \pi_k \in \Pi(N)$ and $p_1, \ldots, p_k \in [0, d(\bar{\alpha}')]$ such that

$$\forall j \in \{1, \ldots, k\}, \quad \lambda(\{t \in [0, d(\bar{\alpha}')], f_{\bar{\alpha}'}(t) = \pi_j\}) = p_j,$$

where λ is the Lebesgue measure, and $\sum_{j=1}^k p_j = d(\bar{\alpha}')$. The time consistency given by the definition of the timetable implies $\lambda(\{t \in [0, d(\bar{\alpha}')], S \in f_{\bar{\alpha}'}(t)\}) = \bar{\alpha}'(S)$ for every $S \subseteq N$, we have $\sum_{j | \pi_j \ni S} p_j = \bar{\alpha}'(S)$ for every $S \subseteq N$, hence

$$\sum_{j=1}^k p_j \Big(\sum_{S \in \pi_j} \tilde{v}(S) \Big) = \sum_{S \subseteq N} \bar{\alpha}'(S) \tilde{v}(S).$$

Furthermore

$$\sum_{j=1}^k p_j \Big(\sum_{S \in \pi_j} \tilde{v}(S) \Big) \leq d(\bar{\alpha}) (\max_{\pi \in \Pi(N)} \sum_{S \in \pi} v(S)) \leq d(\bar{\alpha})(\tilde{v}(N) + \bar{t}(\tilde{v})).$$

Therefore, we deduce that $x(N) \leq d(\bar{\alpha})(\tilde{v}(N) + \bar{t}(\tilde{v}))$. Since, $\tilde{u}_i(x, (\alpha_i)_{i \in N}) \geq 0 \ \forall i \in N$ and $\sum_{i \in S} \tilde{u}_i(x, (\alpha_i)_{i \in N}) \geq \tilde{v}(S) \ \forall S \subseteq N$, the definition of $\tilde{u}_i$ and $\bar{t}(\tilde{v})$ gives $x(N) \geq d(\bar{\alpha})(\tilde{v}(N) + \bar{t}(\tilde{v}))$.

and finally,

$$x(N) = d(\bar{\alpha})(\max_{\pi \in \Pi(N)} \sum_{S \in \pi} \tilde{v}(S)) = d(\bar{\alpha})(\tilde{v}(N) + \bar{t}(\tilde{v})).$$

Hence, $(\frac{x_i}{d(\bar{\alpha})})_{i \in N} = (\tilde{u}_i(x, (\alpha_i)_{i \in N}))_{i \in N} \in$ c-$\mathsf{C}(\tilde{v})$.

(ii) We prove the converse implication. If $\tilde{v}$ is c-balanced, then there exists a partition $\pi \in \Pi(N)$ such that $\sum_{S \in \pi} \tilde{v}(S) = \tilde{v}(N) + \bar{t}(v)$. Let $x \in$ c-$\mathsf{C}(\tilde{v})$. It is easy to verify that $(x, (\alpha_i)_{i \in N})$, with $\alpha_i(S) = 1$ if $(i \in S$ and $S \in \pi)$ and $\alpha_i(S) = 0$ otherwise, belongs to the c-multicoalitional core. $\square$

Remark 3. There exist games which are not c-balanced, but with a nonempty c-multicoalitional core. For example, the game v defined on $N = \{1, 2, 3\}$ by

S	1	2	3	12	13	23	123
$v(S)$	0	0	-100	10	0	0	0

is not c-balanced. However, it can be checked that $(x, (\alpha_i)_{i \in N})$ with $x = (5, 5, 0)$ and $\alpha_1(12) = \alpha_2(12) = 1$, $\alpha_i(S) = 0$ otherwise, is an element of c-$MC(v)$[6].

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[6]We observe, in particular, that it is more rational in our framework that any agent with a nonpositive productivity does not work, while most of classical models require that this kind of agents work during exactly one unit of time.

4 Concluding remarks

We summarize the main achievements of the paper:

(i) We have provided a new type of solution, called multicoalitional solution, in the context of production games, which is innovative in the following aspect: An element of the solution is not limited to a payoff vector x, it also explicitly gives together a time allocation α to each coalition, which permits to realize the payoff vector x;

(ii) We have proposed two examples of multicoalitional solutions, namely, the d-multicoalitional core and the c-multicoalitional core. They are strong Nash equilibria, which respectively maximize the utility per active unit of time, and the utility per total unit of time;

(iii) These solutions can be seen as an implementation in a noncooperative framework of some generalization of the d-core (aspiration core) and the c-core.

We believe that this work opens new horizons in game theory, in particular at the intersection of cooperative and noncooperative game theories.

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