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Ngoc-Sang Pham

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Collateral monetary equilibrium with liquidity constraints in an infinite horizon economy*

Ngoc-Sang PHAM †
University Paris I Panthéon-Sorbonne.

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Abstract

This paper considers an infinite-horizon monetary economy with collateralized assets. A Central Bank lends money to households by creating short- and long-term loans. Households can deposit or borrow money on both short- and long-term maturity loans. If households want to sell a financial asset, they are required to hold certain commodities as collateral. They face a cash-in-advance constraints when buying commodities and financial assets.
Under Uniform or Sequential Gains to Trade Hypothesis, the existence of collateral monetary equilibrium is ensured. I also provide some properties of equilibira, including the liquidity trap.

Keywords: Monetary economy, liquidity constraint, collateralized asset, infinite horizon, liquidity trap.

1 Introduction

In finite-time horizon models, Dubey and Geanakoplos ([DG03a], [DG03b]) have proved the existence equilibrium for monetary economies by using Gains to Trade hypothesis (GTH). Conversly, they have proved that monetary equilibrium for pure exchange economies with money does not exist unless there are sufficient gains to trade (Theorem 6 and 7 in [DG03a]). Dubey and Geanakoplos ([DG06b]) have constructed a two-period monetary economy with production. They suppose that firms sell all goods at hand in period 2, then there is always a strictly positive quantity of commodity which is sold in the economy for that the existence of an equilibrium holds.

This paper extends these results to an infinite horizon monetary economy with consumptions, money, incomplete market, and the possibility of default on financial assets. A Central Bank lends money to households by creating short- and long-term loans. Households can deposit or borrow money in both short- and long-term by trading short-term and long-term loans. There is a borrowing constraint in long-term loan. When selling a commodity or an asset, tradings face a cash-in-advance constraint. If agents want to sell a

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†Email: pns.pham@gmail.com, address: Maison des Sciences Economiques, 106-112 Boulevard de l'Hopital, 75013 Paris, France.

financial asset, they are required to hold certain commodities as collateral.

The first contribution of this paper concerns the existence of monetary equilibrium. Under a Uniform Gains to Trade Hypothesis (or Sequential Gains to Trade Hypothesis), I prove the existence of collateral monetary equilibrium. This paper gives two versions of equilibrium: under Uniform Gains to Trade Hypothesis, prices are uniformly bounded, but under Sequential Gains to Trade Hypothesis, prices are only bounded for the product topology.
Different from Bloise, Dreze and Polemarchakis [BDP05], interest rates in our model are endogeneous.

The second contribution is Theorem 4.1 about liquidity trap which generalizes the one in ([DG06b]. Our result says that: at some date, say t, if supply of money for short-term is very large with respect to supply of money which agents expect to be available in the future then the economy will fall into a liquidity trap at date t.

Literature

An excellent introduction to incomplete markets with infinite horizon can be found in [MQ08]. By extending Geanakoplos and Zame ([GZ02]), Araujo, Pascoa and Torres-Martinez ([APTM02]) proved the existence of equilibrium for an infinite-horizon models with collateral requirement on buying financial assets. Kubler and Schmeider ([KS03]) have constructed and proved the existence of Markov equilibrium in infinite-horizon asset pricing model with incomplete markets and collateral. Such Markov equilibrium is also proved to be a competitive equilibrium. Becker, Bosi, Le Van and Seegmuller ([BBLVS11]) proved the existence of a Ramsey equilibrium with endogenous labor supply and borrowing constraint on physical asset. Le Van and Pham ([LVP13]) proved the existence of intertemporal equilibrium in an infinite horizon model with physical capital, endogenous labor supply, financial asset with borrowing constraint in which aggregate capital and consumption may be not uniformly bounded.

Bloise, Dreze and Polemarchakis [BDP05] have proved *"the existence of equilibria at all overall price levels above a lower bound, provided that conditions on gains to trade, if needed, are satisfied"*.

More on liquidity trap, see Krugman [Kru98].

With the long loans of maturity of T, we refer to Magill and Quinzii ([MQ12]).

2 Model

I extend the model in Dubey and Geanakoplos ([DG03b]) to the case of infinite horizon and add collateral constraints to financial assets.

2.1 The underlying economy

I consider an infinite horizon model with uncertainty. Time runs from $t = 0$ to $+\infty$.
At each date, there are S possible exogenous states (or shocks)

$$\mathcal{S} := \{s_1, \dots, s_S\}.$$

A node ξ is charaterized by $\xi = (t, a_0, a_1, \dots, a_t)$ where $t = t(\xi)$ is the date of node ξ and $a_0, \dots, a_t \in \mathcal{S}$. The unique previous node of ξ is denoted by ξ^-. For each T, ξ, denote

- $\mathcal{D}$ is the set of all nodes. $\mathcal{D}(\xi)$ is the subtree with root ξ,
- $\mathcal{D}_T := \{\xi : t(\xi) = T\}$ is the family of nodes with date T.
- $D^T(\xi) := \bigcup_{t=t(\xi)}^{T} D_t(\xi)$, where $D_t(\xi) := \mathcal{D}_T \cap \mathcal{D}(\xi)$.
- $\xi^+ := \{\mu \in \mathcal{D}(\xi) : t(\mu) = t(\xi) + 1\}$.

A path of nodes is a sequence of nodes $(\xi_n)_{n=0}^T$ such that $\xi_{n+1} \in \xi_n^+$ for every $n \geq 0$. Note that, given ξ, there is a unique path from ξ_0 to ξ, which is denoted by $(\xi_0, \xi_1, \dots, \xi)$.
The set of commodities is $\mathcal{L} := \{1, \dots, L\}$.
There are H types of consumers, $h \in \mathcal{H} = \{1, \dots, H\}$. Each agent h is equipped with an initial vector endowment $e^h(\xi) \in \mathbb{R}_+^L$ of goods at each node ξ. We denote $e^h := (e^h(\xi)_{\xi \in \mathcal{D}})$.

Assumption (H1): *For each node ξ and each $h \in \mathcal{H}$, $\|e^h(\xi)\| > 0$, where $\|e^h(\xi)\| := \sum_{\ell=1}^{L} e_\ell^h(\xi)$. For each node ξ and each commodity $\ell \in \mathcal{L}$, $e_\ell(\xi) := \sum_{h=1}^{H} e_\ell^h(\xi) > 0$.*

Each agent h has the utility function $U^h(\cdot) = \sum_{t=0}^{\infty} \sum_{\xi \in \mathcal{D}_t} u_\xi^h(x_\xi^h)$, where, we assume that:

Assumption (H2): *$u^h : \mathcal{D} \to \mathbb{R}_+$ is C^1, concave, smooth and*

$$u_\xi^h(0) = 0, \quad \frac{\partial u_\xi^h}{\partial x_\ell(\xi)} > 0 \text{ for all } , \xi \in \mathcal{D}, \ell \in \mathcal{L}, \tag{1}$$

$$\sum_{t=0}^{\infty} \sum_{\xi \in \mathcal{D}_t} u_\xi^h(x^h) < +\infty, \quad \text{for each } x^h \in \mathbb{R}^L, \tag{2}$$

$$\lim_{x_\ell(\xi) \to \infty} u_\xi^h(x_1, \dots, x_{\ell-1}, x_\ell(\xi), x_{\ell+1}, \dots, x_L) = \infty, \text{ for all } \xi \in \mathcal{D}. \tag{3}$$

Note that, standard example of the utility function is given by the following

$$\sum_{t=0}^{\infty} \beta_h^t \sum_{\xi \in \mathcal{D}_t} P_h(\xi) u_h(x^h(\xi)) = \mathbb{E}_0^h \sum_{t=0}^{\infty} \beta_h^t u_h(x_t^h),$$

where $u^h : \mathbb{R}_+ \to \mathbb{R}_+$, and $P_h(\xi)$ is the probability of node ξ under agent h's belief.

2.2 Money

As in Dubey and Geanakoplos ([DG06b]), money is fiat and enters the economy in two ways. At each node ξ, each agent has endowment of money $m^h(\xi) \geq 0$. Denote $m^h := m^h(\xi)_{\xi \in \mathcal{D}}$. We call this outside money.

Assumption (H3):

$$\underline{m} := \sum_h m^h(\xi_0) > 0 \tag{4}$$

A central bank can make short loans totalling $M(\xi) > 0$ dollars for one period at node ξ, and also make long loans totalling $N(\xi) > 0$ for two periods at node ξ.

An agent can borrow money from the bank by promising to pay back the loan with interest. If the interest rate for long loans is r_ℓ then anyone can borrow $\nu/(1+r_\ell)$ dollars by promising to repay ν dollars in the next period. There is a borrowing constraint in long-term loan, see $(b)^h(\xi)$.
If the interest rate for short loans is r_s then anyone can borrow $\mu/(1+r_s)$ dollars by promising to repay μ) dollars at the end of the same period.
For each node ξ, denote

$$m(\xi) := \sum_{h=1}^{H} m^h(\xi), \quad \hat{m}(\xi) := \sum_{n=0}^{t(\xi)} m(\xi_n)$$

where $(\xi_0, \xi_1, \ldots, \xi)$ is the finite path whose terminal node is ξ.
The following assumption will be used to prove that prices are uniformly bounded.

Assumption (H4): *For each path of nodes* $(\xi_n)_{n=0}^{\infty}$*, we have*

$$\sum_{n=0}^{\infty} m(\xi_n) < \infty. \tag{5}$$

This assumption says that the quantity of outside money in the economy is uniformly bounded. Note that this paper gives two types of equilibrium. One type of equilibrium with Assumption (H4) is in which prices are uniformly bounded, an other type without Assumption (H4) is in which prices are bounded for the product topology.

2.3 Fundamental Macrovariables

The fundamental macrovariables are

$$\eta = (\eta(\xi))_{\xi \in \mathcal{D}} = (r_s(\xi), r_\ell(\xi), p(\xi))_{\xi \in \mathcal{D}}$$

where, at each node ξ

- $r_s(\xi)$ is the interest rate on short-term bank loans. One can borrow/deposit and repay/ get back (with interest) within the same period,
- $r_\ell(\xi)$ is the interest rate on long-term bank loans. One can borrow/deposit and repay/ get back (with interest) in the next note $\mu \in \xi^+$,
- $p(\xi) \in \mathbb{R}^L$: commodity prices,

Denote $\eta(0, \xi) = (\eta(s_0), \ldots, \eta(\xi^-), \eta(\xi))$.

2.4 Collateralized assets

There are K types of financial assets $\mathcal{K} = \{1, \ldots, K\}$. A collaterized security is a pair (A, c), where $A = (A(\eta(0, \xi)))_{\xi \in \mathcal{D}}$, with $A(\eta(0, \xi)) \in \mathbb{R}^K_+$, $A(\cdot)$ depends continuously on $\eta(0, \xi)$, and $c = (c^k_\ell)_{k \in \mathcal{K}, \ell \in \mathcal{L}} \in \mathbb{R}^{K \times L}_+$.
If one agent wants to sell one unit of financial asset k, she is required to hold $(c^k_\ell)_{\ell \in \mathcal{L}}$ units of goods.

Assumption (H5):

$$\sum_{\ell \in \mathcal{L}} c_\ell^k > 0, \quad \textit{for all } k. \tag{6}$$

As in Geanakoplos and Zame (2010), the collateral requirement is the only means of enforcing promises. Therefore, the delivery per share of security (A, c) at node ξ will be the minimum of the face value and the value of the collateral:

$$d_k(\xi) := \min \Big\{ A_k(\eta(0,\xi)), p(\xi) \cdot c_k \Big\}. \tag{7}$$

The delivery of a portfolio $\alpha = (\alpha_1, \cdots, \alpha_K) \in \mathbb{R}^K$ at note ξ is

$$\sum_{k \in \mathcal{K}} \alpha_k d_k(\xi). \tag{8}$$

We can now add asset prices into the fundamental macrovariable η

$$\eta := (\eta, \pi).$$

2.5 Liquidity constraints for households

We define the set $\sum_\eta^h = (\sum_\xi^h)_{\xi \in D}$ of feasible choices of $h \in H$ and the outcome of h as a function of ξ and her strategy, $\sigma^h(\xi) \in \sum_\xi^h$.
The strategy

$$\sigma^h(\xi) := (\mu^h, \tilde{\mu}^h, \nu^h, \tilde{\nu}^h, q^h, \tilde{q}^h, \alpha^h, \tilde{\alpha}^h)(\xi) \geq 0$$

is described as follows: $\mu^h(\xi)$: IOUs (or short-term Bank loans) sold by h on the loan market at node ξ, $\nu^h(\xi)$: long-term Bank loans sold by h, $\alpha_k^h(\xi)$: financial asset $k \in \mathcal{K}$ sold by h at node ξ (recall that selling a security is borrowing), $q_\ell^h(\xi)$: quantity of commodity ℓ sold by h at node ξ, $\tilde{\mu}^h(\xi) \equiv$ short-term money deposited by h, $\tilde{\nu}^h(\xi)$: long-term money deposited by h, $\tilde{\alpha}_k^h(\xi)$: money spent by h on asset k, $\tilde{q}_\ell^h(\xi)$: bid on h on commodity $\ell \in \mathcal{L}$ at node ξ.

The timing of trade is as follows: first, household h buys and sells bank loans; second, she buys and sells financial assets, commodities; third, she delivers on financial assets. Finally, she repays on loans. I describe the strategy of household h by the following liquidity constraints: at each node ξ
(i) deposited money $\leq$ money on hand:

$$\tilde{\mu}^h(\xi) + \tilde{\nu}^h(\xi) \leq m^h(\xi) + \tilde{m}^h(\xi^-) \qquad (1)^h(\xi)$$

where $\tilde{m}^h(\xi^-)$ is nonnegative and represents the cash hold by household h at the end of node ξ^-.
(ii) expenditures of financial assets and commodities $\leq$ money unspent in $(1)^h(\xi)$ plus money borrowed via short- and long-bonds:

$$\sum_{k \in \mathcal{K}} \tilde{\alpha}_k^h(\xi) + \sum_{\ell \in \mathcal{L}} \tilde{q}_\ell^h(\xi) \leq \Delta(1^h(\xi)) + \frac{\mu^h(\xi)}{1 + r_s(\xi)} + \frac{\nu^h(\xi)}{1 + r_\ell(\xi)}, \qquad (2)^h(\xi)$$

where $\Delta(a)$ is the defference between the right-hand-side and the left-hand-side of inequality (a).

(iii) Delivery on assets of the previous period $\leq$ money left in $(2)^h(\xi)$ plus money obtained from sales of commodities, and assets

$$d(\xi)\cdot\alpha^h(\xi^-) \quad \leq \quad \Delta(2^h(\xi)) + d(\xi)\cdot\frac{\tilde{\alpha}^h(\xi^-)}{\pi(\xi^-)} + q^h(\xi)\cdot p(\xi) + \alpha^h(\xi)\cdot\pi(\xi). \qquad (3)^h(\xi),$$

(iv) Repayments on loans

$$\mu^h(\xi) + \nu^h(\xi^-) \quad \leq \quad \Delta(3^h(\xi)) + (1+r_s(\xi))\tilde{\mu}^h(\xi) + (1+r_\ell(\xi^-))\tilde{\nu}^h(\xi^-) \quad (4)^h(\xi),$$

Moreover, we require physical constraint and borrowing constraint.
Physical constraints

$$e^h_\ell(\xi) - q^h_\ell(\xi) + \frac{\tilde{q}^h_\ell(\xi)}{p_\ell(\xi)} - \sum_{k\in\mathcal{K}} c^k_\ell \alpha^h_k(\xi) + \sum_{k\in\mathcal{K}} c^k_\ell \alpha^h_k(\xi^-) \geq 0 \qquad (pc)^h(\xi).$$

Borrowing constraints on bank loans[1]

$$\nu^h(\xi^-) \leq \sum_\ell e^h_\ell(\xi)p_\ell(\xi) \qquad (b)^h(\xi)$$

The consumption of household h is given by

$$x^h_\ell(\xi) := e^h_\ell(\xi) - q^h_\ell(\xi) + \frac{\tilde{q}^h_\ell(\xi)}{p_\ell(\xi)} - \sum_{k\in\mathcal{K}} c^k_\ell \alpha^h_k(\xi) + \sum_{k\in\mathcal{K}} c^k_\ell \alpha^h_k(\xi^-). \qquad (9)$$

And her money at the end of this node is

$$\tilde{m}^h(\xi) := \Delta(4)^h(\xi). \qquad (10)$$

2.6 Monetary equilibrium

Definition 1. *The collection $(\eta, (\sigma^a)_{a\in\mathcal{H}})$ is a collateral monetary equilibrium (CME) for the monetary economy $\mathcal{E} = \Big((u^h, e^h, m^h)_{h\in\mathcal{H}}, (A, c), (M, N)\Big)$, where $M = (M(\xi))_{\xi\in\mathcal{D}}, N = (N(\xi))_{\xi\in\mathcal{D}}$, are stocks of money from the central bank in short- and long-term loans, respectively, if*

(i) All agents maximize their utility

$$\sigma^t \in \arg\max_{\sigma^h\in\Sigma^h} U^h(x^h(\eta,\sigma^h)), \forall h\in\mathcal{H} \qquad (11)$$

(ii) All markets clear : loans, derivatives and commodities

$$\frac{1}{1+r_s(\xi)}\sum_{h\in\mathcal{H}}\mu^h(\xi) \quad = \quad M(\xi) + \sum_h \tilde{\mu}^h(\xi) \qquad (12)$$

$$\frac{1}{1+r_\ell(\xi)}\sum_h \nu^h(\xi) \quad = \quad N(\xi) + \sum_h \tilde{\nu}^h(\xi), \qquad (13)$$

$$\pi_k(\xi)\sum_h \alpha^h_k(\xi) \quad = \quad \sum_h \tilde{\alpha}^h_k(\xi) \qquad (14)$$

$$p_l(\xi)\sum_{h\in\mathcal{H}} q^h_\ell(\xi) \quad = \quad \sum_{h\in\mathcal{H}} \tilde{q}^h_\ell(\xi). \qquad (15)$$

[1] If we collateralize long-term bond as we did with financial assets then the existence of equilibrium is still ensured.

3 The existence of equilibrium

We prove the existence of collateral monetary equilibrium by taking the limit, as $T \to \infty$ of the sequence of equilibria of $T-$truncated economy $\mathcal{E}^T$. The difficulty is to bound all prices. Asset prices are bounded because of collateral constraints. Thank to the next Uniform Gains to Trade Hypothesis (UGTH), or Sequential Gains to Trade Hypothesis (SGTH) commodity prices will be bounded as well.

3.1 Gains to Trade Hypotheses

First, by following Dubey and Geanakoplos ([DG03b]) we define non $\gamma-$ Pareto optimal allocation. Denote $\mathbb{R}^{L\times\mathcal{D}} := \{x = (x_\ell(\xi))_{\ell\in\in\mathcal{L},\xi\in\mathcal{D}} : x_\ell(\xi) \in \mathbb{R}\}$.

Definition 2. $(x^1,\dots,x^h) \in \left(\mathbb{R}^{L\times\mathcal{D}}\right)^H$ *is called non γ-Pareto optimal at node $\xi \in \mathcal{D}$ if there exists $\tau^1(\xi),\cdots,\tau^H(\xi) \in \mathbb{R}^L$ such that*

$$\begin{aligned}
&\sum_{h=1}^{H} \tau^h(\xi) = 0,\\
&\tau^h(\xi) \neq 0, \quad , x^h(\xi) + \tau^h(\xi) \in \mathbb{R}^L_+, \ \textit{for all } h \in \mathcal{H}\\
&U^h(\bar{x}^h(\gamma,\tau^h(\xi))) > U^h(x^h), \quad \forall h,
\end{aligned}$$

$$\textit{where } \bar{x}^h(\gamma,\tau^h(\xi))_\ell(\mu) = \begin{cases} x^h_\ell(\mu) \ \textit{if } \mu \neq \xi,\\ x^h_\ell(\xi) + \min\{\tau^h_\ell(\xi), \dfrac{\tau^h_\ell(\xi)}{1+\gamma}\} \ \textit{if } \mu = \xi. \end{cases}$$

Definition 3. *For $x = (x^1,\dots,x^h) \in \left(\mathbb{R}^{L\times\mathcal{D}}\right)^H$, we define $\gamma(x) := \sup\{\gamma : x$ is not $\gamma-$ Pareto optimal at node ξ $\}$*

For each $a \geq 0$, $\xi \in \mathcal{D}$, we define the set $X^a(\xi)$ of allocations such that a level of trade is less than a:

$$X^a(\xi) := \Big\{ \quad \begin{aligned}
&(x^1,\dots,x^h) \in \left(\mathbb{R}^{L\times\mathcal{D}}_+\right)^H \text{ such that there exists } (p,r) \in \mathbb{R}^{(L+2)\times\mathcal{D}}_+ \text{ and}\\
&(\alpha^1,\dots,\alpha^H) \in \left(\mathbb{R}^{K\times\mathcal{D}}_+\right)^H \text{ such that}\\
\forall \mu,\ell \quad &\sum_{h=1}^{H} x^h_\ell(\mu) + \sum_{h=1}^{H}\sum_{k=1}^{K} c^k_\ell \alpha^h_k(\mu) = \sum_{h=1}^{H} e^h_\ell(\mu) + \sum_{h=1}^{H}\sum_{k=1}^{K} c^k_\ell \alpha^h_k(\mu^-)\\
\forall \ell \quad &|x^h_\ell(\xi) - e^h_\ell(\xi) - \sum_{h=1}^{H}\sum_{k=1}^{K} c^k_\ell \alpha^h_k(\xi^-) \leq a \quad \Big\}.
\end{aligned}$$

We also define, for each path $(\xi_0,\xi_1,\dots,\xi)$

$$\bar{m}(\xi) := \max_{\mu\in(\xi^-)^+} \Big(\hat{m}(\xi) - \frac{\hat{m}(\mu)}{N(\xi^-)+M(\mu)} N(\xi^-)\Big), \tag{16}$$

$$\mu_\xi(m,M) := \frac{\bar{m}(\xi)}{M(\xi)}. \tag{17}$$

Assumption (H6): (Uniform Gains to Trade Hypothesis)
There exists $a > 0$ such that at each node $\xi \in \mathcal{D}$, $\gamma_\xi(x) > \mu_\xi(m,M)$ for all $x \in X^a(\xi)$.

This hypothesis requires that there exists a trade level $a > 0$ such that there be gains to trade with this level at every node.

Denote

$$\nu_\xi(m, M) := \frac{\hat{m}(\xi) + N(\xi^-) - M(\xi)}{M(\xi)}$$

.

Assumption (H7): (Sequential Gains to Trade Hypothesis)
At each node ξ, there exists $a(\xi) > 0$ such that at each node $\xi \in \mathcal{D}$, $\gamma_\xi(x) > \nu_\xi(m, M)$ for all $x \in X^{a(\xi)}(\xi)$.

This hypothesis requires that at each node ξ, there exists a trade level $a(\xi) > 0$ such that there be gains to trade with this level.

3.2 Existence of equilibrium in the economy $\mathcal{E}^T$

We first prove the existence of $T-$ truncated economy $\mathcal{E}^T$.

Definition 4. (T-truncated economy $\mathcal{E}^T$) *We define $\mathcal{E}^T$ as $\mathcal{E}$ but for all $t > T$, $\eta_t = \sigma_t = 0$ and, at period T, there are neither trade in loans nor trades in financial assets, i.e. $\nu^h(\xi) = \tilde{\nu}^h(\xi) = \tilde{\alpha}^h_k(\xi) = \tilde{\alpha}^h_k(\xi) = 0$ for every h, k, and $\xi \in \mathcal{D}_T$.*

Theorem 3.1. *Under Assumptions (H1), (H2), (H3), (H4), (H5) and (H6), there exists a collateral monetary equilibrium for $\mathcal{E}^T$. Moreover, at equilibrium, all prices are uniformly bounded.*

Proof. See Appendix A.1. □

Theorem 3.2. *Under Assumptions (H1), (H2), (H3), (H5) and (H7), there exists a collateral monetary equilibrium for $\mathcal{E}^T$.*

Proof. See Appendix A.2. □

Note that Assumption (H4) ensures that quantity of money at every node is uniformaly bounded, hence prices are so. But without Assumption (H4), prices are only bounded for the product topology.

3.3 The existence of equilibrium: infinite horizon

Theorem 3.3. *Under Assumptions in Theorem 3.1 (or Theorem 3.2), there exists a collateral monetary equilibrium for the infinite horizon economy.*

Proof. By observing the proof of Theorem 3.1 (or Theorem 3.2), we can assume that the sequence of equilibra $(\eta^{*,T}, (\sigma^{*h,T})_{h\in\mathcal{H}})$ tends to $(\eta^*, (\sigma^{*h})_{h\in\mathcal{H}})$ when T tends to ∞.
We will prove that $(\eta^*, (\sigma^{*h})_{h\in\mathcal{H}})$ is an equilibrium of $\mathcal{E}$. It is clear that condition (ii) in Definition 1 hold. We only need to prove the optimality of plan $(\sigma^{*h})_{h\in\mathcal{H}}$.
Denote

$$\Sigma^{h,T} = \{\sigma^h : \quad \sigma^h(\xi) = 0 \quad \forall \xi \notin \mathcal{D}^T(\xi_0), \\ \text{and} \quad \nu^h(\xi) = \tilde{\nu}^h(\xi) = \tilde{\alpha}^h_k(\xi) = \tilde{\alpha}^h_k(\xi) = 0 \quad \forall \xi \notin \mathcal{D}_T(\xi_0)\}$$

For a plan $\sigma^h \in \Sigma^h_{\eta^*}$, denote $U^{h,T}(\sigma^h) := \sum_{\xi \in \mathcal{D}^T} u^h_\xi(\sigma^h(\xi), \sigma^h(\xi^-))$.
Suppose that there exists a plan $\bar{\sigma}^h \in \Sigma^h_{\eta^*}$, $\epsilon > 0$, and $T_1 \in \mathbb{N}$ such that for all $T \geq T_1$

$$U^{h,T}(\bar{\sigma}^h) - U^{h,T}(\sigma^{*h}) > 3\epsilon, \forall T \geq T_1.$$

Note that there exists $T_2 > T_1, \hat{\sigma}^h \in \Sigma^h_{\eta^*} \cap \Sigma^{h,T_2}$ such that $U^{h,T}(\hat{\sigma}^h) - U^{h,T}(\bar{\sigma}^h) > -\epsilon$ and $U^{h,T}(\sigma^{*h}) - U^h(\sigma^{*h}) > -\epsilon$ for all $T \geq T_2$. Hence, there exists $\hat{\sigma}^h \in \Sigma^h_{\eta^*} \cap \Sigma^{h,T_2}$ such that

$$U^{h,T}(\hat{\sigma}^h) - U^h(\sigma^{*h}) > \epsilon, \forall T \geq T_2.$$

We define $\psi^h : \Sigma^h_\eta \to \Sigma^{h,T_2}_\eta$ by $\psi(\sigma^h) := \{\hat{\sigma}^h \in \Sigma^{h,T_2} : U^{h,T_2}(\hat{\sigma}^h) - U^h(\sigma^h) > \epsilon\}$.
Denote Θ is the space of prices that is compact in Theorem 3.1. Define F^h is a correspondence from $\Theta \times \Sigma^h$ to Σ^{h,T_2} by

$$F^h(\eta, \sigma^h) = \Sigma^{h,T_2}_\eta \cap \psi^h(\sigma^h).$$

So F^h is lower semi-continuous[2] with respect to the product topology.
By definition of $\hat{\sigma}^h$, we see that $\hat{\sigma}^h \in F^h(\eta^*, \sigma^{*h})$. Combining with $\lim_{T\to\infty}(\eta^{*T}, (\sigma^{*h,T})_h) = (\eta^*, (\sigma^{*h})_h)$, there exists a sequence $(\hat{\sigma}^h_T)_{T \geq T_0} \subset \Sigma^{h,T_2}$ such that $\lim_{T \in \infty} \hat{\sigma}^h_T = \hat{\sigma}^h$ and $\hat{\sigma}^h_T \in F^h(\eta^{*T}, \sigma^{*h,T})$ for all $T \geq T_0$.
Without the generality, we can assume that $T_0 \geq T_2$. Therefore, $\hat{\sigma}^h_{T_0} \in \Sigma^{h,T_0}_{\eta^{*T_0}}$ and

$$U^{h,T_2}(\hat{\sigma}^h_{T_0}) - U^h(\sigma^{h,T_0}) > \epsilon.$$

Hence,

$$U^{h,T_0}(\hat{\sigma}^h_{T_0}) \geq U^{h,T_2}(\hat{\sigma}^h_{T_0}) > U^h(\sigma^{h,T_0}) + \epsilon > U^{h,T_0}(\sigma^{h,T_0}).$$

This is a contradiction with the optimality of the truncated economy $\mathcal{E}^{T_0}$. □

4 Some properties of equilibria

Lemma 4.1. *We have*

$$\begin{aligned} & \sum_{h \in \mathcal{H}, \ell \in \mathcal{L}} \tilde{q}^h_\ell(\xi) + \sum_{h \in \mathcal{H}, k \in \mathcal{K}} \tilde{\alpha}^h_k(\xi) \\ \leq & \sum_h \Big(m^h(\xi) + \tilde{m}^h(\xi^-)\Big) + M(\xi) + N(\xi), \end{aligned} \tag{18}$$

and

$$\begin{aligned} & (1 + r_\ell(\xi^-))N(\xi^-) + (1 + r_s(\xi))M(\xi) \\ \leq & \sum_h \Big(m^h(\xi) + \tilde{m}^h(\xi^-)\Big) + M(\xi) + N(\xi). \end{aligned} \tag{19}$$

[2] See Pascoa and Seghir ([PS09])

Proof. To prove (18), we use (2^h_ξ) and the fact that markets are clear. (19) is proved by using (4^h_ξ) and the fact that markets are clear. □

Proposition 4.1. *At any equilibrium*

(i) $r_s(\xi), r_\ell(\xi) \geq 0$ *for all* t.

(ii) $1 + r_\ell(\xi) \geq \min_{\xi^+}\{(1 + r_s(\xi))(1 + r_s(\xi^+))\}$. *A directly consequence of this result* [3] *is that* $r_\ell(\xi) \geq r_s(\xi)$.

(iii) $r_s(\xi) \leq \dfrac{\sum_h \Big(m^h(\xi) + \tilde{m}^h(\xi^-)\Big) + N(\xi) - (1 + r_\ell(\xi^-))N(\xi^-)}{M(\xi)}$,

and $r_\ell(\xi^-) \leq \dfrac{\sum_h \Big(m^h(\xi) + \tilde{m}^h(\xi^-)\Big) + N(\xi) - r_s(\xi)M(\xi) - N(\xi^-)}{N(\xi^-)}$.

(iv) **Public debt:** *the following result can explain the fact in monetary policy: for every path* $(\xi_0, \xi_1, \ldots, \xi_n)$, *we have*

$$\sum_{k=0}^{n-1} \Big[r_\ell(\xi_k)N(\xi_k) + r_s(\xi_k)M(\xi_k)\Big] + r_s(\xi_n)M(\xi_n) \leq \hat{m}(\xi_n) + N(\xi_n), \tag{20}$$

where $\hat{m}(\xi) = \sum_{h=1}^{H} (m^h(\xi_0) + \cdots + m^h(\xi))$.

Proof. Use Lemma 4.1. □

Corollary 1. *Under Assumptions in Theroem 3.1, at equilibrium we have*

$$\lim_{t(\xi)\to\infty} r_\ell(\xi)N(\xi) + r_s(\xi)M(\xi) = 0. \tag{21}$$

This result implies that if the Central Bank can control all quantity of money in long run, and the quantity of money injected by the Central Bank is uniformly bounded then all interest rates tend to a very low level. Indeed, if if the Central Bank can control all quantity of money in long run, then Assumption 2.2 is hold. Hence for every infinite path, we have

$$\sum_{k=0}^{\infty} \Big[r_\ell(\xi_k)N(\xi_k) + r_s(\xi_k)M(\xi_k)\Big] + r_s(\xi_n)M(\xi_n) < \infty. \tag{22}$$

Consequently, $\lim_{n\to\infty} r_\ell(\xi_n)N(\xi_n) + r_s(\xi_n)M(\xi_n) = 0$.

As in [DG06a], the following result shows that if agents borrow money from the bank and do not use all the money to purchase then liquidity trap occurs.

[3] Generally, if we consider the long-term loans of t-period with $t = 1, \ldots, T$ then

$$1 + r_{L(T)}(\xi) \geq \min_{\xi^{t_1},\ldots,\xi^{t_a}:\sum_{i=1}^{a} t_i = T} \prod_{i=1}^{a} \Big(1 + r_{L(t_i)}(\xi^{t_1+\cdots+t_i})\Big).$$

Lemma 4.2. *If $\mu^h(\xi) > 0$ and $r_s(\xi) > 0$ then $\Delta(2^h(\xi)) = 0$.*

Proof. Consider at node ξ. Assume that $\mu^h(\xi) > 0$ and $r_s(\xi) > 0$. Let h borrow ϵ less on short-term loan $r_s(\xi)$. This action well leave agent h with $\epsilon r_s(\xi)$ more money after trade in node ξ. Agent h can spend this amount to buy more consumption good in next node, a contradiction. □

For each monetary economy $\mathcal{E} = \Big((u^h, e^h, m^h)_{h\in\mathcal{H}}, A, c, M, N\Big)$ we denote $M^b(\xi) := m(\xi_0) + \cdots + m(\xi) + N(\xi^-) + N(\xi) + M(\xi)$. Note that the total money at node ξ is smaller that $M^b(\xi)$, and that $M^b(\xi)$ does not depend on $M(\xi^-)$.

The following result say that if the Central Bank injects a quantity of money which is larger than the expected quantity of money in the future then the interest rate of short-term loan is zero. This result is consistent with those in Dubey and Geanakoplos [DG06b].

Theorem 4.1. Liquidity Trap Theorem
Consider a monetary economy $\mathcal{E} = \Big((u^h, e^h, m^h)_{h\in\mathcal{H}}, A, c, M, N\Big)$. Consider node ξ. If there exists a finite constant B and $T > t(\xi)$ such that: if $\dfrac{M(\xi)}{M^b(\xi')} > B$ for each $\xi' \in D_T \cap D(\xi)$ then $r_s(\xi) = 0$.

Proof. This is a consequence of Lemma 4.2 and the proofs of Theorem 3.1, and Theorem 3.2.
Since $M(\xi) \leq \dfrac{1}{1 + r_s(\xi)} \sum_{h=1}^{H} \mu^h(\xi)$ then there exists h such that $\dfrac{\mu^h(\xi)}{1 + r_s(\xi)} \geq M(\xi)/H$. We will prove that $\Delta(2^h(\xi)) > 0$ when $M(\xi)$ is large enough.
We have

$$\sum_{\ell\in\mathcal{L}} \tilde{q}^h_\ell(\xi) \;\leq\; \sum_{h\in\mathcal{H}}\sum_{\ell\in\mathcal{L}} \tilde{q}^h_\ell(\xi) = \sum_{h\in\mathcal{H}}\sum_{\ell\in\mathcal{L}} p_\ell(\xi) q^h_\ell(\xi) \leq \sum_{\ell\in\mathcal{L}}\sum_{h\in\mathcal{H}} p_\ell(\xi) e_\ell(\xi).$$

Lemma A.8 implies that $p_\ell(\xi)$ is not greater than a constant that does not depend on $M(\xi)$, but this constant depends on $M^b(\xi')$. Consequently $\sum_{\ell\in\mathcal{L}} \tilde{q}^h_\ell(\xi)$ is so.
By using the same argument and note that

$$\sum_{k\in\mathcal{K}} \tilde{\alpha}^h_k(\xi) \;\leq\; \sum_{h\in\mathcal{H}}\sum_{k\in\mathcal{K}} \tilde{\alpha}^h_k(\xi) \leq \sum_{k\in\mathcal{K}}\sum_{h\in\mathcal{H}} \pi_k(\xi) \alpha^h_k(\xi),$$

we see that $\sum_{k\in\mathcal{K}} \tilde{\alpha}^h_k(\xi)$ is not greater than a constant that does not depend on $M(\xi)$. Therefore, if $M(\xi)$ is large enough then $\Delta(2^h(\xi)) > 0$, hence $r_s(\xi) = 0$. □

Definition 5. Constant of Liquidity Trap
$A_1 := \inf\{A :$ *if* $M(\xi) > AM^b(\xi') \quad \forall \xi' \in \xi^+$ *then* $r_s(\xi) = 0\}$ *is called constant of liquidity trap.*

Theorem 4.1 say that this constant exists and finite. However, an other question appears: how can we estimate this contant? in pratice, we face some challenges when we want to estimate this constant, for example we don't know exactly $M^b(\xi_0^+)$.

5 Conclusion and future agenda

We have constructed an infinite-time horizon monetary economy model and given a positive answer to the question on the existence of monetary equilibirum in infinite horizon model with incomplete financial market and liquidity constraints. Gains to Trade Hypotheses ensure the existence of equilibrium. We have studied liquidity trap at equilibrium.
It would be interesting in future work to introduce productive sector into model and study the impacts of liquidity constraints on strategy of firm as well as on aggregate economic activities.
An other avenue for future research is to construct a model to analyse financial fragility and stability. This model would include households, commercial banks, firms. We refer to Goodhart, Sunirand, Tsomocos ([GST06]), Goodhart ([Goo06]) and Goodhart, Tsomocos ([GT10]).

A Existence of equilibrium

A.1 Proof of Theorem 3.1

We follows Dubey and Geanakoplos [DG03b].
For $\epsilon > 0, h \in \mathcal{H}$, define the ambient strategy space of an agent of type h:

$$\Sigma^h_\epsilon = \Big\{(\sigma^h(\xi))_{\xi\in\mathcal{D}} : 0 \leq \sigma^h(\xi) \leq \frac{1}{\epsilon}\Big\}.$$

These spaces are clearly convex and compact.
Given choices $\sigma \in \prod_{h\in\mathcal{H}} \Sigma^h_\epsilon$, define macrovariables $\eta_\epsilon(\sigma) = (r, \pi, p)(\sigma)$ as follow

$$\begin{aligned}
\frac{1}{1+r^\epsilon_s(\xi)} &= \frac{\epsilon + M(\xi) + \sum_h \tilde{\mu}^h(\xi)}{\epsilon + \sum_h \mu^h(\xi)}, \quad \frac{1}{1+r^\epsilon_\ell(\xi)} = \frac{\epsilon + N(\xi) + \sum_h \tilde{\nu}^h(\xi)}{\epsilon + \sum_h \nu^h(\xi)},\\
\pi^\epsilon_k(\xi) &= \frac{\epsilon + \sum_h \tilde{\alpha}^h_k(\xi)}{\epsilon + \sum_h \alpha^h_k(\xi)}, \quad p^\epsilon_\ell(\xi) = \frac{\epsilon + \sum_{h\in\mathcal{H}} \tilde{q}^h_\ell(\xi)}{\epsilon + \sum_{h\in\mathcal{H}} q^h_\ell(\xi)}.
\end{aligned}$$

And the delivery is defined

$$d^{\epsilon,\sigma}_k(\xi) = \frac{\epsilon^2}{\epsilon^2 + \sum_h \alpha^h_k(\xi)} \min\Big(d_k(\xi), \frac{1}{\epsilon}\Big) + \frac{\sum_h \alpha^h_k(\xi)}{\epsilon^2 + \sum_h \alpha^h_k(\xi)} d_k(\xi). \tag{23}$$

The payoff to any players of type $h \in \mathcal{H}$

$$\Pi^h(\sigma, \sigma^h) = u^h(x^h). \tag{24}$$

Denote $\tilde{\Sigma}^h_{\eta_\epsilon(\sigma)}$ is defined in exactly the same manner as $\Sigma^h_{\eta_\epsilon(\sigma)}$, but replacing d^k for $d^{\epsilon,\sigma}_k$

$$\tilde{\Sigma}^h_{\eta_\epsilon(\sigma)} = \Sigma^h_{\eta_\epsilon(\sigma)}(d^{\epsilon,\sigma}_k).$$

Define $\psi : \Sigma_\epsilon \to \Sigma_\epsilon$, where $\Sigma_\epsilon := \prod_{h\in\mathcal{H}} \Sigma^h_\epsilon$, by the following

$$\psi^h_\epsilon(\sigma) = \underset{\bar{\sigma}^h \in \tilde{\Sigma}^h_{\eta_\epsilon(\sigma)} \cap \Sigma^h_\epsilon}{\arg\max}\ \Pi^h(\sigma, \bar{\sigma}^h). \tag{25}$$

We see that all the standard assumptions are satisfied, hence there exists an ϵ - collateral monetary equilibrium (i.e. type-symmetric Nash equilibrium for Γ^ϵ) for every $\epsilon > 0$.
We will prove that $\lim_{\epsilon \searrow 0}(\eta_\epsilon(\sigma^\epsilon), \sigma^\epsilon) = (\eta, \sigma)$ is an equilibrium of $\mathcal{E}^T$.

Lemma A.1. *at every node ξ, the total commodity is uniformly bounded by a constant C which does not depend on the quantity of money for sufficiently small ϵ.*

Proof. Clear. □

Lemma A.2. *for every $h \in \mathcal{H}, \xi \in \mathcal{D}^T$; $\mu^{t,\epsilon}(\xi), \nu^{t,\epsilon}(\xi), r_s^\epsilon(\xi), r_\ell^\epsilon(\xi)$ are bounded for sufficiently small ϵ.*

Proof. Notice that the total amount of money at a node is bounded by some constant D, so $\mu^{h,\epsilon}(\xi), \nu^{h,\epsilon}(\xi) \leq D$ for all small enough ϵ.
The fact that $r_s^\epsilon(\xi), r_\ell^\epsilon(\xi)$ are bounded from above, is easily proved by the boundedness of $\mu^{t,\epsilon}(\xi), \nu^{t,\epsilon}(\xi)$.. □

Remark A.1. *The constant D does not depend on T, it only depends on node ξ.*

Lemma A.3. *$r_s^\epsilon(\xi), r_\ell^\epsilon(\xi)$ are non negative for sufficiently small ϵ.*

Proof. Clearly. □

Lemma A.4. *There exists $\underline{p}$ such that $p_\ell^\epsilon(\xi) \geq \underline{p}$ for sufficiently small ϵ.*

Proof. We choose H^* such that $u_\xi^h(0, \ldots, 0, H^*, 0, \ldots, 0) > U^h(C)$ for all h. By assumption on outside money, there exists $h \in \mathcal{H}$ such that $m_0^{h,\epsilon} \geq \underline{\text{m}}/H$, so we have

$$p_\ell^\epsilon(\xi) \geq \frac{\underline{\text{m}}}{H.H^*}. \tag{26}$$

Indeed, if $p_\ell^\epsilon(\xi) < \dfrac{\underline{\text{m}}}{H.H^*}$ then let h spend $\underline{\text{m}}/H$ to buy H^* units of commodity l at node ξ, so h obtain a final utility $\geq u^h(0, \ldots, 0, H^*, 0, \ldots, 0) > u^h(C)$. Contradiction! □

Lemma A.5. *We have $1 + r_\ell^\epsilon(\xi) \geq \min_{\mu \in \xi^+}(1 + r_s^\epsilon(\xi))1 + r_s^\epsilon(\mu)$ at each node ξ.*

Proof. Clear. □

Lemma A.6. *At each final node $\xi \in D_T(\xi_0)$, for path $(\xi_0, \xi_1, \ldots, \xi)$ we have*

$$r_s^\epsilon(\xi_0)M(\xi_0) + r_\ell^\epsilon(\xi_0)N(\xi_0) + \cdots + r_s^\epsilon(\xi^-)M(\xi^-) + r_\ell^\epsilon(\xi^-)N(\xi^-) + r_s^\epsilon(\xi)M(\xi) \leq \hat{m}(\xi) + B\epsilon,$$

where $\hat{m}(\xi) = \sum_{h=1}^{H}(m^h(\xi_0) + \cdots + m^h(\xi))$, and B is a constant which depends on ξ.

Proof. □

Denote $\mu_\xi(m, M) := \dfrac{1}{M(\xi)}\big(\hat{m}(\xi) - \min_{\mu \in (\xi^-)^+} \dfrac{\hat{m}(\mu)}{M(\mu) + N(\xi^-)} N(\xi^-)\big)$.

Lemma A.7. *Given $\xi^- \in D_{T-1}(\xi_0)$, then there exists a final node $\xi \in D_T(\xi_0)$ such that*

$$r_s(\xi) \leq \frac{\hat{m}(\xi)}{M(\xi) + N(\xi^-)} \leq \mu_\xi(m, M). \tag{27}$$

Proof. Given $\xi^- \in D_{T-1}(\xi_0)$. Since Lemma A.5, there exists $\xi \in (\xi^-)^+$ such that $1+r_l^\epsilon(\xi) \geq (1+r_s^\epsilon(\xi^-))(1+r_s^\epsilon(\xi))$. Hence $r_l^\epsilon(\xi) \geq r_s^\epsilon(\xi^-) + r_s^\epsilon(\xi)$. Lemma A.6 implies that

$$r_s^\epsilon(\xi^-)(N(\xi^-)+N(\xi)) + r_s^\epsilon(\xi)(N(\xi^-)+M(\xi)) \leq \hat{m}(\xi) + B\epsilon.$$

Consequently, $r_s^\epsilon(\xi) \leq \dfrac{\hat{m}(\xi)+B\epsilon}{N(\xi^-)+M(\xi)}$. Let $\epsilon \to$, we obtain $r_s^\epsilon(\xi) \leq \dfrac{\hat{m}(\xi)}{N(\xi^-)+M(\xi)}$.
On the other hand, definition of $\mu_\xi(m,M)$ implies that

$$\mu_\xi(m,M) = \frac{1}{M(\xi)}\big(\hat{m}(\xi) - \min_{\mu\in(\xi^-)^+} \frac{\hat{m}(\mu)}{M(\mu)+N(\xi^-)} N(\xi^-)\big) \tag{28}$$

$$\geq \frac{1}{M(\xi)}\big(\hat{m}(\xi) - \frac{\hat{m}(\xi)}{M(\xi)+N(\xi^-)} N(\xi^-)\big) = \frac{\hat{m}(\xi)}{M(\xi)+N(\xi^-)}. \tag{29}$$

□

Lemma A.8. *At final node $\xi \in D_T(\xi_0)$, if $r_s(\xi) \leq \mu_\xi(m,M)$ then for each μ in the path $(\xi_0, \xi_1, \dots, \xi)$, and $\ell \in \mathcal{L}$*

$$p_\ell(\mu) < \frac{M^b(\xi)}{a} \max(1, \frac{H^*H}{\underline{e}(\mu)}), \tag{30}$$

Proof. Consider a final node $\xi \in D_T(\xi_0)$. Suppose that $r_s(\xi) \leq \mu_\xi(m,M)$.
Assume for each $\ell \in \mathcal{L}, p_\ell(\xi) \geq \dfrac{M^b(\xi)}{a}$ then $a \geq \dfrac{M^b(\xi)}{p_\ell(\xi)} \geq \dfrac{\tilde{q}_\ell^h(\xi)}{p_\ell(\xi)} - q_\ell^h(\xi)$.
If $\dfrac{\tilde{q}_\ell^h(\xi)}{p_\ell(\xi)} - q_\ell^h(\xi) < -a$ then $q_\ell^h(\xi) > a$, so $p_\ell(\mu) < \dfrac{M^b(\xi)}{a}$, contradiction. If $\dfrac{\tilde{q}_\ell^h(\xi)}{p_\ell(\xi)} - q_\ell^h(\xi) \geq -a$ then $\dfrac{\tilde{q}_\ell^h(\xi)}{p_\ell(\xi)} - q_\ell^h(\xi) \in [-a,a]$. Consequently,

$$|x_\ell^h(\xi) - e_\ell^h(\xi) - \sum_k c_\ell^k \alpha_k^h(\xi^-)| = |\frac{\tilde{q}_\ell^h(\xi)}{p_\ell(\xi)} - q_\ell^h(\xi)| \leq a.$$

It means that $x = (x^1, \dots, x^h) \in \left(\mathbb{R}_+^{L\times D^T}\right)^H$ belongs to $X^a(\xi)$.
Uniform Gains to Trade Hypothesis implies that $\gamma_\xi(x) > \mu_\xi(m,M)$. Combining with Lemma A.7, we get $\gamma_\xi(x) > r_s(\xi)$. By definition of $\gamma_\xi(x)$, there exists $\gamma > r_s(\xi)$ such that x is not γ - Pareto optimal at node ξ. Therefore, there exists $\tau(\xi) = (\tau^h(\xi))_{h\in\mathcal{H}} \in \mathbb{R}^{L\times H}$ such that

$$\tau^h(\xi) \neq 0, \quad \text{for all } h \in \mathcal{H}, \text{ and } \sum_{h=1}^H \tau^h(\xi) = 0, \tag{31}$$

$$x^h(\xi) + \tau^h(\xi) \in \mathbb{R}_+^L, \quad \text{for all } h \in \mathcal{H}, \tag{32}$$

$$U^h(\bar{x}^h(\gamma, \tau^h(\xi))) > U^h(x^h), \text{ for all } h \in \mathcal{H}. \tag{33}$$

Since $\sum_{h=1}^H \tau^h(\xi) = 0$, there exists $i \in \mathcal{H}$ such that $p(\xi)\tau^i(\xi) \leq 0$. Without the generality, we can assume that $\tau^i(\xi) = (\tau_1^i(\xi), \dots, \tau_m^i(\xi), -\tau_{m+1}^i(\xi), \dots, -\tau_L^i(\xi))$, with $\tau_\ell^i(\xi) \geq 0$. We have $\sum_{\ell=1}^m p_\ell(\xi)\tau_\ell^i(\xi) \leq \sum_{\ell=m+1}^L p_\ell(\xi)\tau_\ell^i(\xi)$.

We construct a new strategy $(\hat{\sigma}^i(\mu))_{\mu\in\mathcal{D}^T}$ of agent i as the following: $\hat{\sigma}^i(\mu) = \sigma^i(\mu), \quad \forall \mu \neq \xi$ and at node ξ

$$\begin{aligned}
\hat{\mu}^i(\xi) &:= \mu^i_(\xi) + \sum_{\ell=1}^{m} p_\ell(\xi)\tau^i_\ell(\xi),\\
\hat{\tilde{q}}^i_\ell(\xi) &:= \tilde{q}^i_\ell(\xi) + p_\ell(\xi)\frac{\tau^i_\ell(\xi)}{1+\gamma}, \quad \forall \ell = 1,\dots,m\\
\hat{q}^i_\ell(\xi) &:= q^i_\ell(\xi) + \tau^i_\ell(\xi), \quad \forall \ell = m+1,\dots,L.
\end{aligned}$$

Since $x^i(\xi) + \tau^i(\xi) \in \mathbb{R}^L_+$, then this new trategy satisfies the physical constraint $(pc)^h(\xi)$. Thank to $\gamma^a_\xi(x) > r_s(\xi)$, then $\frac{1}{1+\gamma} < \frac{1}{1+r_s(\xi)}$, hence liquidity constraint $(2^i(\xi))$ is hold. Liquidity constraint $(4^i(\xi))$ is satisfied because of $\sum_{\ell=1}^{m} p_\ell(\xi)\tau^i_\ell(\xi) \leq \sum_{\ell=m+1}^{L} p_\ell(\xi)\tau^i_\ell(\xi)$.
On the other hand $U^i(\bar{x}^i(\gamma,\tau^i(\xi))) > U^i(x^i)$, contradiction to the optimality of σ^i.

We now consider a node μ in the path $(\xi_0,\xi_1,\dots,\xi)$. Let $k \in \mathcal{L}$. Since $\sum_{h=1}^{H} e^h_k(\mu) \geq \underline{e}(\mu)$, there exists h such that $e^h_k(\xi) \geq \underline{e}(\mu)/H$.
If $\frac{p_k(\mu)}{p_\ell(\xi)} > \frac{2H^*H}{\underline{e}(\mu)}$. Let h do nothing, just sell $\underline{e}(\mu)/(2H)$ units of commodity k, obtain $p_k(\mu)\underline{e}(\mu)/(2H)$ dollars. Hence h can buy at leastH^* units of commodity ℓ, contradiction. Therefore, $\frac{p_k(\mu)}{p_\ell(\xi)} \leq \frac{2H^*H}{\underline{e}(\mu)}$, so we obtain (30) □

Lemma A.9. *Price of financial asset k at node ξ: $\pi_k(\xi)$ is bounded from above if commodity prices $p_\ell(\xi)$ are bounded from above.*

Proof. Choose h with $m^h_0 > \underline{m}/h$. Let h uses m^h_0 to buy a vector of commodities $(\frac{m^h_0}{p_\ell(\xi)L})_{\ell\in\mathcal{L}}$, at initial node in order to use these commodities as collateral at node ξ. So at note ξ, h can sell at least $w = \min_{\ell: c^k_\ell > 0}\left\{\frac{m^h_0}{\bar{p}c^k_\ell L}\right\} > 0$ units of asset k and obtain $w\pi^\epsilon_k(\xi)$ dollars. Of course, $w\pi^\epsilon_k(\xi) \leq \bar{p}H^*$. Thus $\pi^\epsilon_k(\xi) \leq \bar{p}H^*/w$. □

Lemma A.10. *Let ξ as in Lemma A.7, we have*

$$r_s(\xi_0)M(\xi_0) + r_\ell(\xi_0)N(\xi_0) + \cdots + r_s(\xi^-)M(\xi^-) + r_\ell(\xi^-)N(\xi^-) + r_s(\xi)M(\xi) = \hat{m}(\xi). \quad (34)$$

Proof. Similarly Claim V in Dubey and Geanakoplos [DG03b]. □

Lemma A.11. *At each final node ξ, we have $r_s(\xi) \leq \mu_s(m,M)$*

Proof. Assume there is a final node ξ at which $r_s(\xi) > \mu_s(m,M)$. Then

$$M(\xi)r_s(\xi) + \min_{\mu\in(\xi^-)^+} \frac{\hat{m}(\mu)}{M(\mu)+N(\xi^-)} N(\xi^-) > \hat{m}(\xi).$$

Lemma A.6 implies that

$$r_s(\xi_0)M(\xi_0) + r_\ell(\xi_0)N(\xi_0) + \cdots + r_s(\xi^-)M(\xi^-) + r_\ell(\xi^-)N(\xi^-) + r_s(\xi)M(\xi) \leq \hat{m}(\xi).$$

Hence

$$\min_{\mu\in(\xi^-)^+} \frac{\hat{m}(\mu)}{M(\mu)+N(\xi^-)}N(\xi^-) > r_s(\xi_0)M(\xi_0)+r_\ell(\xi_0)N(\xi_0)+\cdots+r_s(\xi^-)M(\xi^-)+r_\ell(\xi^-)N(\xi^-).$$

Let $\mu \in (\xi^-)^+$ be as in Lemma A.7, i.e, $r_s(\mu) \leq \dfrac{\hat{m}(\mu)}{M(\mu)+N(\xi^-)}$ then

$$\begin{aligned}
&\min_{\mu\in(\xi^-)^+} \frac{\hat{m}(\mu)}{M(\mu)+N(\xi^-)}N(\xi^-) + r_s(\mu)M(\mu)\\
\leq\;& \frac{\hat{m}(\mu)}{M(\mu)+N(\xi^-)}N(\xi^-) + r_s(\mu)M(\mu)\\
\leq\;& \frac{\hat{m}(\mu)}{M(\mu)+N(\xi^-)}N(\xi^-) + \frac{\hat{m}(\mu)}{M(\mu)+N(\xi^-)}M(\mu) = \hat{m}(\mu).
\end{aligned}$$

Consequently, we get

$$\hat{m}(\mu) > r_s(\xi_0)M(\xi_0)+r_\ell(\xi_0)N(\xi_0)+\cdots+r_s(\xi^-)M(\xi^-)+r_\ell(\xi^-)N(\xi^-)+r_s(\mu)M(\mu),$$

contradiction to Lemma A.10. □

Lemma A.12. *All prices are bounded from above.*

Proof. This is a direct sequence of the above results. □

Lemma A.13. $d_k^{\epsilon,\sigma}(\xi,\eta) = d_k(\xi,\eta)$ *for sufficiently small* ϵ.

Proof. By collateral constraints, $d_k(\xi)$ is bounded. Consequently we get $d_k^{\epsilon,\sigma}(\xi,\eta) = d_k(\xi,\eta)$ for sufficiently small ϵ. □

Lemma A.14. *We have* $\alpha_k^h(\xi)$ *is bounded from above.*

Proof. By collateral constraints, we get

$$\underline{c}\sum_k \alpha_k^{h,\epsilon}(\xi) \leq \sum_k \big[\sum_\ell c_\ell^k\big]\alpha_k^{h,\epsilon}(\xi) \leq \sum_\ell e_\ell^h(\xi) \leq \bar{e}.$$

□

A.2 Proof of Theorem 3.2

I use the same method in the proof of Theorem 3.1. However, in order to prove that all prices are bounded when $\epsilon \to 0$, I use Sequential Gains to Trade Hypothesis.

At each node $\xi \in D^T(\xi_0)$, consider path $(\xi_0,\xi_1,\ldots,\xi)$. Let denote $\tilde{m}(\xi) := \sum_{h=1}^{h} \tilde{m}^h(\xi)$ be the total stock of money unspent at the end of node ξ. Then we have

$$\begin{aligned}
\tilde{m}(\xi) &:= \tilde{m}(\xi^-)+m(\xi)+N(\xi)+M(\xi)-(1+r_\ell(\xi^-))N(\xi^-)-(1+r_s(\xi))M(\xi)\\
&= \cdots\\
&= \hat{m}(\xi)+N(\xi)-r_s(\xi)M(\xi)\\
&\quad -r_s(\xi^-)M(\xi^-)-r_\ell(\xi^-)N(\xi^-)-\cdots-r_s(\xi_0)M(\xi_0)-r_\ell(\xi_0)N(\xi_0)
\end{aligned}$$

Since markets clear, we have

$$\begin{aligned} r_s(\xi)M(\xi) &\leq \sum_h \mu^h(\xi) - M(\xi) \\ &\leq m(\xi) + \tilde{m}(\xi^-) - M(\xi) \\ &\leq \hat{m}(\xi) + N(\xi) - M(\xi). \end{aligned}$$

Therefore, $r_s(\xi) \leq \frac{\hat{m}(\xi) + N(\xi) - M(\xi)}{M(\xi)} < \gamma_\xi(x)$ for all $x \in X^{a(\xi)}(\xi)$. By using the same argument in Lemma A.8, we obtain that all commodity prices are bounded from above, and so financial asset prices.

References

[APTM02] A. Araujo, M. P. Pascoa, and J. P. Torres-Martinez, *Collateral avoids ponzi schemes in incomplete markets*, Econometrica **70** (2002), no. 4, 1613–1638.

[BBLVS11] Robert Becker, Stefano Bosi, Cuong Le Van, and Thomas Seegmuller, *On the existence, efficiency and bubbles of a ramsey equilibrium with endogenous labor supply and borrowing constraints.*, CES, Working paper, 2011.

[BDP05] G. Bloise, J. Dreze, and H. Polemarchakis, *Monetary equilibria over an infinite horizon*, Economic Theory **25(1)** (2005), 51–74.

[DG03a] P. Dubey and J. Geanakoplos, *Inside and outside money, gains to trade, and IS-LM*, Economic Theory **21** (2003), no. 2, 347–397.

[DG03b] ______, *Monetary equilibrium with missing markets*, Journal of Mathematical Economics **39** (2003), no. 5-6, 585–618.

[DG06a] ______, *Determinacy with nomial assets and outside money*, Economic Theory **27** (2006), no. 1, 79–106.

[DG06b] ______, *Money and production, and liquidity trap*, International Journal of Economic Theory **2** (2006), no. 3-4, 295–317.

[Goo06] C. A. E. Goodhart, *A framework for assessing financial stability?*, Journal of Banking & Finance **30** (2006), no. 12, 3415–3422.

[GST06] C. A. E. Goodhart, P. Sunirand, and D. P. Tsomocos, *A model to analyse financial fragility*, Economic Theory **27** (2006), no. 1, 107–142.

[GT10] C. A. E. Goodhart and D. P. Tsomocos, *Analysis of financial stability*, ch. 5 in Challenges in Central Banking: The current institutional environment and forces affecting monetary policy, Cambridge University Press, 2010.

[GZ02] J. Geanakoplos and W. Zame, *Collateral and the enforcement of intertemporal contracts*, working paper, 2002.

[Kru98] P. R. Krugman, *"it's baaack: Japan's slump and the return of the liquidity trap*, Brookings Papers on Economic Activity, Economic Studies Program **vol. 29(2),** (1998), 137–206.

[KS03] F. Kubler and K. Schmedders, *Stationary equilibria in asset-pricing models with incomplete markets and collateral*, Econometrica **71** (2003), no. 6, 1767–1793.

[LVP13] Cuong Le Van and Ngoc-Sang Pham, *Intertemporal equilibrium with financial asset and physical capital*, CES working paper, 2013.

[MQ08] M. J.P. Magill and M. Quinzii, *Incomplete markets, volume 2, infinite horizon economies*, Edward Elgar Publishing Company, 2008.

[MQ12] M. Magill and M. Quinzii, *Interest rate policy and expectations of inflation*, Working paper, 2012.

[PS09] M. R. Pascoa and A. Seghir, *Harsh default penalties lead to ponzi schemes*, Games and Economic Behavior **65** (2009), no. 1, 270–286.