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Sensitivity of marketable payoffs with long-term assets

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Abstract

We consider a stochastic financial exchange economy with a finite date-event tree representing time and uncertainty and a financial structure with possibly long-term assets. We exhibit a sufficient condition under which the set of marketable payoffs depends continuously on the arbitrage free asset prices. This generalizes previous results of Angeloni-Cornet and Magill-Quinzii involving only short-term assets. We also show that, under the same condition, the useless portfolios do not depend on the arbitrage free asset prices. We then derive an existence result for nominal assets for all state prices with assumptions only on the fundamental datas of the economy.

Keywords: Incomplete markets, financial equilibrium, multi-period model, long-term assets.

JEL codes: D5, D4, G1

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1 Introduction

The space of marketable payoffs for a financial structure is the set of payoffs that are reachable by a suitable portfolio through the financial structure. The main purpose of this paper is to study the sensitivity of the marketable payoffs with respect to the arbitrage free asset prices in presence of long-term assets. More precisely, we aim to get a sufficient condition on the payoff matrix describing the returns of the assets so that the set of marketable payoffs depends continuously on the asset prices.

Economically, this means that the super-replication cost is continuous with respect to arbitrage free asset prices or equivalently to state prices. In mathematical terms, this means that the correspondence which associates the marketable payoffs to the asset prices is lower semi-continuous and has a closed graph.

In a two period economy, the continuity of marketable payoffs always holds true since the marketable payoff space is easily and continuously computed from the range of the payoff matrix, which is independent from asset prices. This results is easily extended to a multi-period model where all assets are short term, that is, have non zero return only at the immediate successors of their issuance nodes.

With more than two periods, this is no more true as already shown in [1, 5] with long term assets. Below we provide a simple numerical example where a payoff is marketable for a sequence of arbitrage free asset prices but not for the limit asset price. In other words, the marketable payoff space has not a closed graph.

After introducing notations and the model of a financial structure borrowed from [1, 5] in Section 2 , we provide a condition, Assumption **R**, on the payoff matrix in Section 3. We show that Assumption **R** is satisfied if all assets are short term, if there is a unique issuance date, or if there is no overlap of the nodes with non zero returns for two different assets. More generally, Assumption **R** translates the fact that the assets issued at a given node are true financial innovations in the sense that the payoffs cannot be replicated by assets issued before.

We prove that under Assumption **R**, the set of marketable payoffs varies continuously with respect to the arbitrage free asset prices. This continuous dependency is intimately linked with the dimension of the range of the full payoffs matrix, which is obtained from the payoff matrix by incorporating the opposite of asset prices at the issuance nodes. Actually, we prove under Assumption **R** that the dimensions of the range (or in other words, the rank) of both matrices are equal for all arbitrage free asset prices. As for the space of useless portfolio, that is the portfolios with zero returns at each node (in other words the kernel of the payoff matrix), we also prove that, again under Assumption **R**, it is equal to the kernel of the full payoffs matrix for all arbitrage free asset prices.

As a byproduct, we remark that, under Assumption **R**, we can characterize correctly a complete financial structure. Indeed, we provide an example where the financial structure is a priori incomplete since the payoff matrix has a too small range, but is complete for some arbitrage free asset prices and incomplete for some other prices. The above result shows that cannot happen under Assumption **R** and that the completeness can be checked on the payoff matrix. We finally remark that under Assumption **R**, the concept of useless portfolio i well defined since it does not depend on asset prices.

In Section 4, we consider a stochastic financial exchange economy with possibly long term nominal assets and restricted participation. We provide an existence result for all given state prices when Assumption **R** is satisfied by the payoff matrix. These results are based on the existence result (Theorem 3.1) of [1]. Assumption **R**, which involves only fundamental data of the economy, namely the returns of the assets, allows us to remove an abstract boundedness assumption in [1], which depends on the state price. So our contribution could be seen as the extension of the existence result of [1] to long term assets under Assumption **R** and is the first existence result in a multi-period model with long term assets based on assumptions only related to the fundamental datas of the model.

2 The T-period financial structure

In this section, we present the model and the notations, which are borrowed from Angeloni-Cornet[1] and are essentially the same as those of Magill-Quinzii[5].

2.1 Time and uncertainty

We[1] consider a time structure with $(T+1)$ dates, $t \in \mathcal{T} := \{0, ..., T\}$, and a finite set of agents $\mathcal{I}$. The uncertainty is described by a date-event tree $\mathbb{D}$ of length $T+1$. The set $\mathbb{D}_t$ is the set of nodes (also called date-events) that could occur at date t and the family $(\mathbb{D}_t)_{t\in\mathcal{T}}$ defines a partition of the set $\mathbb{D}$; for each $\xi \in \mathbb{D}$, we denote by $t(\xi)$ the unique date $t \in \mathcal{T}$ such that $\xi \in \mathbb{D}_t$.

At date $t = 0$, there is a unique node ξ_0, that is $\mathbb{D}_0 = \{\xi_0\}$. As $\mathbb{D}$ is a tree, each node ξ in $\mathbb{D} \setminus \{\xi_0\}$ has a unique immediate predecessor denoted $pr(\xi)$ or ξ^-. The mapping pr maps $\mathbb{D}_t$ to $\mathbb{D}_{t-1}$. Each node $\xi \in \mathbb{D} \setminus \mathbb{D}_T$ has a nonempty set of immediate successors defined by $\xi^+ = \left\{\bar{\xi} \in \mathbb{D} : \xi = \bar{\xi}^-\right\}$.

For $\tau \in \mathcal{T} \setminus \{0\}$ and $\xi \in \mathbb{D} \setminus \cup_{t=0}^{\tau-1}\mathbb{D}_t$, we define $pr^\tau(\xi)$ by the recursive formula: $pr^\tau(\xi) = pr\left(pr^{\tau-1}(\xi)\right)$. We then define the set of successors and the set of predecessors of ξ as follows:

[1]We use the following notations. A $(\mathbb{D} \times \mathcal{J})$-matrix A is an element of $\mathbb{R}^{\mathbb{D}\times\mathcal{J}}$, with entries $(a_\xi^j)_{(\xi\in\mathbb{D}, j\in\mathcal{J})}$; we denote by $A_\xi \in \mathbb{R}^{\mathcal{J}}$ the ξ-th row of A and by $A^j \in \mathbb{R}^{\mathbb{D}}$ the j-th column of A. We recall that the transpose of A is the unique $(\mathcal{J} \times \mathbb{D})$-matrix ${}^t\mathrm{A}$ satisfying $(Ax) \bullet_{\mathbb{D}} y = x \bullet_{\mathcal{J}} \left({}^t\mathrm{Ay}\right)$ for every $x \in \mathbb{R}^{\mathcal{J}}$, $y \in \mathbb{R}^{\mathbb{D}}$, where $\bullet_{\mathbb{D}}$ [resp. $\bullet_{\mathcal{J}}$] denotes the usual inner product in $\mathbb{R}^{\mathbb{D}}$ [resp. $\mathbb{R}^{\mathcal{J}}$]. We denote by rank$A$ the rank of the matrix A and by Vect(A) the range of A, that is the linear sub-space spanned by the column vectors of A. For every subset $\tilde{\mathbb{D}} \subset \mathbb{D}$ and $\tilde{\mathcal{J}} \subset \mathcal{J}$, the matrix $A_{\tilde{\mathbb{D}}}^{\tilde{\mathcal{J}}}$ is the $(\tilde{\mathbb{D}}\times\tilde{\mathcal{J}})$-sub-matrix of A with entries a_ξ^j for every $(\xi, j) \in (\tilde{\mathbb{D}}\times\tilde{\mathcal{J}})$. Let x, y be in $\mathbb{R}^n$; $x \geq y$ (resp. $x \gg y$) means $x_h \geq y_h$ (resp. $x_h > y_h$) for every $h = 1, \ldots, n$ and we let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$, $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x \gg 0\}$. We also use the notation $x > y$ if $x \geq y$ and $x \neq y$. The Euclidean norm in the Euclidean different spaces is denoted $\|.\|$ and the closed ball centered at x and of radius $r > 0$ is denoted $\bar{B}(x, r) := \{y \in \mathbb{R}^n \mid \|y - x\| \leq r\}$.

$$\mathbb{D}^{+}(\xi) = \{\xi' \in \mathbb{D} : \exists \tau \in \mathcal{T} \setminus \{0\} \mid \xi = pr^{\tau}(\xi')\}$$
$$\mathbb{D}^{-}(\xi) = \{\xi' \in \mathbb{D} : \exists \tau \in \mathcal{T} \setminus \{0\} \mid \xi' = pr^{\tau}(\xi)\}$$

If $\xi' \in \mathbb{D}^{+}(\xi)$ [resp. $\xi' \in \mathbb{D}^{+}(\xi) \cup \{\xi\}$], we shall use the notation $\xi' > \xi$ [resp. $\xi' \geq \xi$]. Note that $\xi' \in \mathbb{D}^{+}(\xi)$ if and only if $\xi \in \mathbb{D}^{-}(\xi')$ and similarly $\xi' \in \xi^{+}$ if and only if $\xi = (\xi')^{-}$.

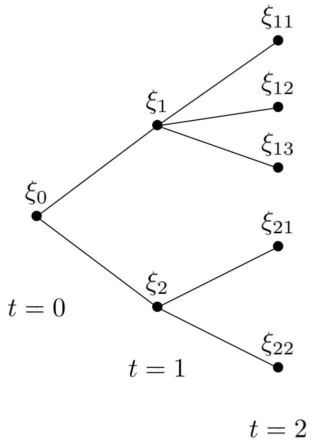

Figure 1: the tree $\mathbb{D}$

Here, $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}, \xi_{13}, \xi_{21}, \xi_{22}\}$, $T = 2$, the length of $\mathbb{D}$ is 3, $\mathbb{D}_2 = \{\xi_{11}, \xi_{12}, \xi_{13}, \xi_{21}, \xi_{22}\}$, $\xi_1^{+} = \{\xi_{11}, \xi_{12}, \xi_{13}\}$, $\mathbb{D}^{+}(\xi_2) = \{\xi_{21}, \xi_{22}\}$, $t(\xi_{11}) = t(\xi_{12}) = t(\xi_{13}) = t(\xi_{21}) = t(\xi_{22}) = 2$, $\mathbb{D}^{-}(\xi_{11}) = \{\xi_0, \xi_1\}$.

At each node $\xi \in \mathbb{D}$, there is a spot market on which a finite set $\mathbb{H} = \{1, \ldots, H\}$ of divisible and physical goods are exchanged. We assume that each good is perishable, that is, its life does not have more than one date. In this model, a commodity is a pair (h, ξ) of a physical good $h \in \mathbb{H}$ and the node $\xi \in \mathbb{D}$ at which the good is available. Then the commodity space is $\mathbb{R}^{\mathbb{L}}$, where $\mathbb{L} = \mathbb{H} \times \mathbb{D}$. An element $x \in \mathbb{R}^{\mathbb{L}}$ is called a consumption, that is to say $x = (x(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\mathbb{L}}$, where $x(\xi) = (x(h, \xi))_{h \in \mathbb{H}} \in \mathbb{R}^{\mathbb{H}}$ for each $\xi \in \mathbb{D}$.
We denote by $p = (p(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\mathbb{L}}$ the vector of spot prices and $p(\xi) = (p(h, \xi))_{h \in \mathbb{H}} \in \mathbb{R}^{\mathbb{H}}$ is called the spot price at node ξ. The spot price $p(h, \xi)$ is the price at the node ξ for immediate delivery of one unit of the physical good h. Thus the value of a consumption $x(\xi)$ at node $\xi \in \mathbb{D}$ (measured in unit account of the node ξ) is

$$p(\xi) \bullet_{\mathbb{H}} x(\xi) = \sum_{h \in \mathbb{H}} p(h, \xi)\, x(h, \xi).$$

2.2 The financial structure

The financial structure is constituted by a finite set of assets denoted $\mathcal{J} = \{1, \ldots, J\}$. An asset $j \in \mathcal{J}$ is a contract issued at a given and unique node in $\mathbb{D}$

denoted $\xi(j)$, called issuance node of j. Each asset is bought or sold only at its issuance node $\xi(j)$ and yields payoffs only at the successor nodes ξ' of $\mathbb{D}^+(\xi(j))$. To simplify the notation, we consider the payoff of asset j at every node $\xi \in \mathbb{D}$ and we assume that it is equal to zero if ξ is not a successor of the issuance node $\xi(j)$. The payoff may depend upon the spot price vector $p \in \mathbb{R}^{\mathbb{L}}$ and is denoted by $V_\xi^j(p)$. Formally, we assume that $V_\xi^j(p) = 0$ if $\xi \notin \mathbb{D}^+(\xi(j))$.

A portfolio $z = (z^j)_{j \in \mathcal{J}}$ is an element of $\mathbb{R}^{\mathcal{J}}$. If $z^j > 0$ [resp. $z^j < 0$], then $|z^j|$ is the quantity of asset j bought [resp. sold] at the issuance node $\xi(j)$.

To summarize a financial structure $\mathcal{F} = \left(\mathcal{J}, (\xi(j))_{j \in \mathcal{J}}, V\right)$ consists of

- a set of assets $\mathcal{J}$,
- a node of issuance $\xi(j)$ for each asset $j \in \mathcal{J}$,
- a payoff mapping $V : \mathbb{R}^{\mathbb{L}} \to \mathbb{R}^{\mathbb{D} \times \mathcal{J}}$ which associates to every spot price $p \in \mathbb{R}^L$ the $(\mathbb{D} \times \mathcal{J})$-payoff matrix $V(p) = \left(V_\xi^j(p)\right)_{\xi \in \mathbb{D}, j \in \mathcal{J}}$ and satisfies the condition $V_\xi^j(p) = 0$ si $\xi \notin \mathbb{D}^+(\xi(j))$.

The price of asset j is denoted by q_j; it is paid at its issuance node $\xi(j)$. We let $q = (q_j)_{j \in \mathcal{J}} \in \mathbb{R}^{\mathcal{J}}$ be the asset price vector.

The full payoff matrix $W(p, q)$ is the $(\mathbb{D} \times \mathcal{J})$-matrix with the following entries:

$$W_\xi^j(p, q) := V_\xi^j(p) - \delta_{\xi, \xi(j)} q_j,$$

where $\delta_{\xi,\xi'} = 1$ if $\xi = \xi'$ and $\delta_{\xi,\xi'} = 0$ otherwise.

So, given the prices (p, q), the full flow of payoffs for a given portfolio $z \in \mathbb{R}^{\mathcal{J}}$ is $W(p, q)z$ and the full payoff at node ξ is

$$\begin{aligned}[W(p,q)z](\xi) &:= W_\xi(p,q) \bullet_{\mathcal{J}} z = \textstyle\sum_{j \in \mathcal{J}} V_\xi^j(p) z^j - \sum_{j \in \mathcal{J}} \delta_{\xi,\xi(j)} q_j z^j \\ &= \textstyle\sum_{\{j \in \mathcal{J} \mid \xi(j) < \xi\}} V_\xi^j(p) z^j - \sum_{\{j \in \mathcal{J} \mid \xi(j) = \xi\}} q_j z^j.\end{aligned}$$

We are now able to define the set of marketable payoffs for (p, q) as:

$$H(p,q) = \{w \in \mathbb{R}^{\mathbb{D}} \mid \exists z \in \mathbb{R}^{\mathcal{J}}, w = W(p,q)z\}$$

which is the range of the matrix $W(p, q)$.

We now recall that for a given spot price p, the asset price q is an arbitrage free price if it does not exist a portfolio $z \in \mathbb{R}^{\mathcal{J}}$ such that $W(p,q)z > 0$. q is an arbitrage free price if and only if it exists a so-called state price vector $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$ such that ${}^tW(p,q)\lambda = 0$ (see, e.g. Magill-Quinzii [5]). Taken into account the particular structure of the matrix $W(p, q)$, this is equivalent to

$$\forall j \in \mathcal{J}, \lambda_{\xi(j)} q_j = \sum_{\xi \in \mathbb{D}^+(\xi(j))} \lambda_\xi V_\xi^j(p).$$

3 Sensitivity of the set of marketable payoffs

We study in this section the continuity of the set of marketable payoffs with respect to arbitrage free asset prices for given commodity spot prices. Given a commodity price p, we denote by Q_p the set of arbitrage free asset prices with respect to p. That is,

$$Q_p = \left\{ q \in \mathbb{R}^{\mathcal{J}}; \exists \lambda \in \mathbb{R}^{\mathbb{D}}_{++} \text{ satisfying } {}^tW(p,q)\lambda = 0 \right\}$$

The continuity of the set of marketable payoffs means that the correspondence $q \to H(p,q)$ from Q_p to $\mathbb{R}^{\mathbb{D}}$ has a closed graph and is lower semi-continuous. In other words, for all sequence (q^ν) of Q_p converging to $q \in Q_p$, for all sequence (w^ν) of $\mathbb{R}^{\mathbb{D}}$ converging to w, for all $w' \in H(p,q)$, $w \in H(p,q)$ if $w^\nu \in H(p,q^\nu)$ for all ν and there exists a sequence (w'^ν) of $\mathbb{R}^{\mathbb{D}}$ converging to w' and satisfying $w'^\nu \in H(p,q^\nu)$ for all ν.

In a two period economy, the assets being all issued at date 0, the structure of $V(p)$ and $W(p,q)$ are very closed since $W(p,q)$ is obtained by filling the first row with the opposite of the asset prices. So, it is quite easy as shown later that the set of marketable payoffs is then continuous. The proposition below states that in a multi-period economy, for each spot price p, the correspondence $H(p,.)$ is lower semi-continuous. This is a direct consequence of the fact that the matrix $W(p,q)$ depends continuously on q.

Proposition 3.1. *For each spot price vector $p \in \mathbb{R}^{\mathbb{L}}$, the correspondence $H(p,.)$ is lower semi-continuous.*

Proof of Proposition 3.1 Let $p \in \mathbb{R}^{\mathbb{D}}$ be a spot price vector. Let $\bar{q} \in \mathbb{R}^{\mathcal{J}}$ an asset price and let (q^ν) a sequence of Q_p, which converges to $\bar{q} \in Q_p$ and let $\bar{w} \in H(p,\bar{q})$. Let $\bar{z} \in \mathbb{R}^{\mathcal{J}}$ such that $\bar{w} = W(p,\bar{q})\bar{z}$. Then the sequence $(w^\nu = W(p,q^\nu)\bar{z})$ converges to $\bar{w}$ since the sequence of matrices $(W(p,q^\nu))$ converges to $W(p,\bar{q})$ and $w^\nu \in H(p,q^\nu)$ from the very definition of $H(p,\cdot)$. Hence the correspondance $H(p,.)$ is l.s.c. on Q_p. □

3.1 Closedness of the marketable payoff correspondence

The following example shows that the closedness of the graph of the set of marketable payoffs is not granted in a multi-period economy. In all our numerical examples, we assume that there is only one good at each node of the tree and the spot price of the unique good is equal to 1. Consequently, for the sake of simpler notations, we omit the price p and note the payoff matrix (resp. full payoff matrix) by V (resp. $W(q)$).

Let us consider the financial structure with $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_{21}, \xi_{22}\}$ as represented below, $\mathcal{F} = (\mathcal{J}, \mathbb{R}^{\mathcal{J}}, (\xi(j))_{j\in\mathcal{J}}, V)$ such that $\mathcal{J} = \{j_1, j_2, j_3, j_4\}$ and that the first two assets are issued at node ξ_0, The third asset is issued at node ξ_1 and the fourth is issued at node ξ_2.

The payoff matrix is:

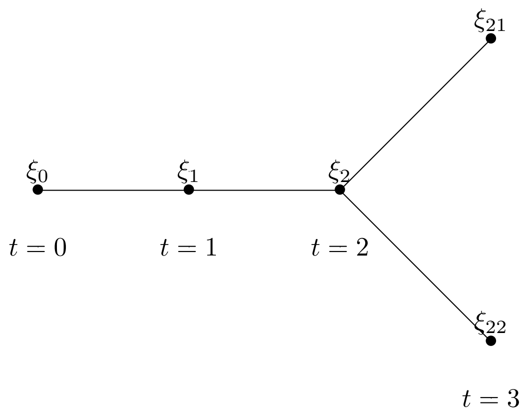

Figure 2: the tree $\mathbb{D}$

$$\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{21} \\ \xi_{22} \end{matrix}$$

We now exhibit a sequence of arbitrage free prices (q^ν) where the market is complete for each term of the sequence except at the limit $\bar{q}$. Then, we exhibit a payoff $\bar{w}$ such that $\bar{w}$ is marketable for each price q^ν and not at the limit.

Let $(q^\nu)_{\nu \in \mathbb{N}^*} = (1 - \frac{1}{2\nu}, \frac{1}{\nu}, \frac{1}{\nu}, \frac{1}{2} + \frac{1}{4\nu})_{\nu \in \mathbb{N}^*}$ a sequence of arbitrage free prices associated with the sequence of state prices $(\lambda^\nu)_{\nu \in \mathbb{N}^*} = (1, 1, 2, 1 - \frac{1}{2\nu}, 1 + \frac{1}{2\nu})_{\nu \in \mathbb{N}^*}$. For each $\nu \in \mathbb{N}^*$, the full payoff matrix is as follows:

$$\mathbf{W}(\mathbf{q}^\nu) = \begin{bmatrix} -1 + \frac{1}{2\nu} & -\frac{1}{\nu} & 0 & 0 \\ 0 & 0 & -\frac{1}{\nu} & 0 \\ 1 & 0 & 0 & -\frac{1}{2} - \frac{1}{4\nu} \\ 0 & -1 & -1 & 0 \\ -1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{21} \\ \xi_{22} \end{matrix}$$

For each $\nu \in \mathbb{N}^*$, the market is complete since the rank of the matrix $W(q^\nu)$ is 4. Indeed, the determinant of the square sub-matrix of size 4 constituted with the rows ξ_2, ξ_2, ξ_{21} and ξ_{22} and the columns of $W(q^\nu)$ is equal to $\frac{2\nu - 1}{4\nu^2} \neq 0$.

At the limit, $\bar{q} = (1, 0, 0, \frac{1}{2})$. $\bar{q}$ is a arbitrage free price associated to $\lambda = (1, 1, 2, 1, 1)$. We have

$$\mathbf{W}(\bar{\mathbf{q}}) = \begin{bmatrix} \mathbf{-1} & \mathbf{0} & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 \\ 1 & 0 & 0 & \mathbf{-1/2} \\ 0 & -1 & -1 & 0 \\ -1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{21} \\ \xi_{22} \end{matrix}$$

Here the rank of the matrix $W(\bar{q})$ is 3 because all entries of the second row are equal to zero and the square sub-matrix of size 3 formed with the rows $\xi_0, \xi_{21}, \xi_{22}$ and columns 1, 3 and 4 has a nonzero determinant. So the market is incomplete at the limit.

This drop of the rank of the full payoff matrix leads to the inability to cover some payoff at the limit. Indeed, let $w = (0, -1, \frac{1}{2}, 0, 0)$. For all ν, $w = W(q^\nu)z^\nu$ with $z^\nu = (\frac{2\nu}{2\nu-1}, -\nu, \nu, \frac{2\nu}{2\nu-1})$, hence w is marketable for q^ν but not for $\bar{q}$. Indeed, suppose that there exists $z \in \mathbb{R}^4$ such that $w = W(\bar{q})z$. Then,

$$(0, -1, \frac{1}{2}, 0, 0) = (-z_1, 0, z_1 - \frac{z_4}{2}, -z_2 - z_3, -z_1 + z_2 + z_3 + z_4) \Rightarrow 0 = -1$$

a contradiction.

We now provide a sufficient general condition under which the marketable payoff correspondence has a closed graph. This condition is based on the useless portfolios for a financial structure, that is a portfolio with zero returns at all nodes. Formally, for a spot price vector $p \in \mathbb{R}^{\mathbb{D}}$, a portfolio $z \in \mathbb{R}^{\mathcal{J}}$ is useless if $V(p)z = 0$, which in other words means that z belongs to the kernel of $V(p)$.

Proposition 3.2. *Let $p \in \mathbb{R}^{\mathbb{D}}$ be a given spot price vector. If for all arbitrage free asset price $q \in Q_p$ the kernel of $W(p,q)$ is equal to the kernel of $V(p)$ then the marketable payoff correspondence defined by*

$$q \to H(p,q) = \{w \in \mathbb{R}^{\mathbb{D}} \mid \exists z \in \mathbb{R}^{\mathcal{J}}, w = W(p,q)z\}$$

has a closed graph.

The condition on the kernel of $V(p)$ and $W(p,q)$ means that z is a useless portfolio if and only if the returns are all equal to 0 for the full payoff matrix. In other words, the definition of useless portfolios is not ambiguous since it does note depend on asset prices.

The equality of the kernels is obviously satisfied in a two period financial structure for a arbitrage free asset price but this is no longer true with long-term assets. Indeed, if we consider the same example as above and if we let $q^1 = (0, 0, 0, 1)$. q is arbitrage free price because ${}^tW\left(q^1\right)\lambda^1 = 0$ with $\lambda^1 = (1, 1, 1, 1, 1) \in \mathbb{R}^5_{++}$. The full payoff matrix associated with this price q^1 is the following:

$$\mathbf{W(q^1)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 \\ 1 & 0 & 0 & \mathbf{-1} \\ 0 & -1 & -1 & 0 \\ -1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{21} \\ \xi_{22} \end{matrix}$$

The rank of $W(q^1)$ is 2 since its kernel is of dimension 2 and equal to

$$\{z \in \mathbb{R}^4 \mid z_1 = z_4, z_2 + z_3 = 0\}$$

and it is not included in the kernel of V, which is equal to

$$\{z \in \mathbb{R}^4 \mid z_1 = z_4 = 0, z_2 + z_3 = 0\}$$

So the portfolio $z = (1, 1, -1, 1)$ has zero returns at all nodes for the full payoff matrix but is not useless since its return is $(0, 0, 1, 0, 0)$ for the payoff matrix.

Now, consider a second asset price $q^2 = (-1, 1, 1, 2)$. q^2 is arbitrage free price because ${}^tW\left(q^2\right)\lambda^2 = 0$ with $\lambda^2 = (1, 1, 1, 1, 2) \in \mathbb{R}^5_{++}$. The full payoff matrix associated with this price q^2 is the following:

$$\mathbf{W(q^2)} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & 0 & 0 \\ 0 & 0 & \mathbf{-1} & 0 \\ 1 & 0 & 0 & \mathbf{-2} \\ 0 & -1 & -1 & 0 \\ -1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{21} \\ \xi_{22} \end{matrix}$$

The rank of $W(q^2)$ is 4 because its kernel is reduced to $\{0\}$. Hence the kernel of $W(q^2)$ is included and not equal to the kernel of V. The portfolio $z' = (0, 1, -1, 0)$ is useless but its returns for the full payoff matrix are $(1, -1, 0, 0, 0)$.

Proof of Proposition 3.2 Let (q^ν) a sequence of Q_p converging to $\bar{q} \in Q_p$ and let (w^ν) a sequence of marketable payoffs converging to $\bar{w}$ such that, for each ν, $w^\nu \in H(p, q^\nu)$. We prove that $\bar{w} \in H(p, \bar{q})$.

For each ν there exists $z^\nu \in \mathbb{R}^{\mathbb{D}}$ such that $w^\nu = W(p, q^\nu)z^\nu$. Let $\hat{z}^\nu$ be the orthogonal projection of z^ν on $(\mathrm{Ker}V(p))^\perp$. Then $w^\nu = W(p, q^\nu)\hat{z}^\nu + W(p, q^\nu)(z^\nu - \hat{z}^\nu) = W(p, q^\nu)\hat{z}^\nu$ since $z^\nu - \hat{z}^\nu \in \mathrm{Ker}V(p) = \mathrm{Ker}W(p, q^\nu)$ by assumption.

We now prove that the sequence $(\hat{z}^\nu)$ is bounded. Indeed, suppose, by contradiction, that this is not true. Then, there exists a subsequence $(\hat{z}^{\phi(\nu)})$ such that $\left\|\hat{z}^{\phi(\nu)}\right\| \to +\infty$. For each ν, let $\zeta^\nu = \frac{\hat{z}^{\phi(\nu)}}{\left\|\hat{z}^{\phi(\nu)}\right\|}$

The sequence (ζ^ν) belongs to the unit sphere. So there exists a subsequence $(\zeta^{\psi(\nu)})$ of (ζ^ν) which converges to $\bar{\zeta}$. Clearly $\|\bar{\zeta}\| = 1$ and and $\bar{\zeta} \in (\mathrm{Ker}V)^\perp$ since $\hat{z}^\nu \in (\mathrm{Ker}V)^\perp$ for all ν. Thus, for each ν, we have

$$W(p, q^\nu)\frac{\hat{z}^{\psi\circ\phi(\nu)}}{\left\|\hat{z}^{\psi\circ\phi(\nu)}\right\|} = W(p, q^\nu)\zeta^{\psi(\nu)} \to W(p, \bar{q})\bar{\zeta}$$

and

$$W(p, q^\nu)\zeta^{\psi(\nu)} = W(p, q^\nu)\frac{\hat{z}^{\psi\circ\phi(\nu)}}{\left\|\hat{z}^{\psi\circ\phi(\nu)}\right\|} = \frac{w^{\psi\circ\phi(\nu)}}{\left\|\hat{z}^{\psi\circ\phi(\nu)}\right\|} \to 0$$

since (w^ν) is bounded and $\left\|\hat{z}^{\phi(\nu)}\right\| \to +\infty$. Thus $\bar{\zeta} \in \mathrm{Ker}W(p, \bar{q}) = \mathrm{Ker}V(p)$ by assumption and $\bar{\zeta} \in (\mathrm{Ker}V(p))^\perp$, so $\bar{\zeta} = 0$ which contradicts $\|\bar{\zeta}\| = 1$.

Since the sequence $(\hat{z}^\nu)$ is bounded, there exists a converging subsequence $(\hat{z}^{\varphi(\nu)})$ which converges to $\bar{z} \in (\mathrm{Ker}V)^\perp$ and we easily checks that

$$\bar{w} = \lim_{\nu\to+\infty} w^{\varphi(\nu)} = \lim_{\nu\to+\infty} W(p, q^{\varphi(\nu)})z^{\varphi(\nu)} = W(p, \bar{q})\bar{z} \in H(p, \bar{q}).$$

□

3.2 Equality between the kernels of payoff matrices

The previous result is based on the equality of the kernels of payoff matrices, which should be checked for each arbitrage free asset price. This kind of assumption exhibits the drawback that it involves an endogenous variable, the asset price, which is determined by the market mechanism. We now provide a sufficient condition (Assumption **R**) on the payoff matrix $V(p)$ to get the equality of the kernel for each arbitrage free price in presence of long-term assets. This generalizes the two-period and the short term asset cases. Hence, the continuity of the marketable payoffs can be determined on the fundamentals of the financial structure.

Furthermore, Assumption **R** allows also to check the completeness of the asset structure on $V(p)$ for all arbitrage free asset prices. Indeed, in the example given above, we remark that the market is complete in the sense that $W(q^2)$ has the maximal rank 4 for the arbitrage free price q^2, but the market is incomplete for the arbitrage free price q^1 since the rank of $W(q^1)$ is only 2. So, even if V has a rank equal to 3, the market is complete for some asset prices like q^2 and "more" incomplete than V for some other asset prices like q^1. We state below a proposition showing that, under Assumption **R**, if $V(p)$ has a maximal rank, then the markets are complete for all arbitrage free asset prices.

We first introduce some additional notations. For all $\xi \in \mathbb{D} \setminus \mathbb{D}_T$, $\mathcal{J}(\xi)$ is the set of assets issued at the node ξ, that is $\mathcal{J}(\xi) = \{j \in \mathcal{J} \mid \xi(j) = \xi\}$ and $n(\xi)$ is the cardinal of $\mathcal{J}(\xi)$. $\mathcal{J}(\mathbb{D}^-(\xi))$ is the set of assets issued at a predecessor of ξ, that is $\mathcal{J}(\mathbb{D}^-(\xi)) = \{j \in \mathcal{J} \mid \xi(j) < \xi\}$. For all $t \in \{0, \ldots, T-1\}$, we denote by $\mathcal{J}_t$ the set of assets issued at date t, that is, $\mathcal{J}_t = \{j \in \mathcal{J} \mid \xi(j) \in \mathbb{D}_t\}$.

Let $(\tau_1, \ldots, \tau_k)$ such that $0 \leq \tau_1 < \tau_2 < \ldots < \tau_k \leq T-1$ be the dates at which there is at least the issuance of one asset, that is $\mathcal{J}_{\tau_\kappa} \neq \emptyset$. For $\kappa = 1, \ldots, k$, let $\mathbb{D}^e_{\tau_\kappa}$ be the set of nodes at date τ_κ at which there is the issuance of at least one asset. $\mathbb{D}^e = \cup_{\kappa=1}^k \mathbb{D}^e_{\tau_\kappa}$ is the set of nodes at which there is the issuance of at least one asset. We remark that

$$\bigcup_{\tau \in \{0,\ldots,T-1\}} \mathcal{J}_\tau = \bigcup_{\kappa \in \{1,\ldots,k\}} \mathcal{J}_{\tau_\kappa} = \mathcal{J}, \quad J = \sum_{\kappa \in \{1,\ldots,k\}} \# \mathcal{J}_{\tau_\kappa}$$

and for all $\tau \in \{\tau_1, \ldots, \tau_k\}$, $\bigcup_{\xi \in \mathbb{D}_\tau} \mathcal{J}(\xi) = \mathcal{J}_\tau$.

3.2.1 The condition R

Let $p \in \mathbb{R}^{\mathbb{L}}$ be a given spot price vector. We now state our sufficient condition on the matrix $V(p)$.

Assumption R. $\forall \xi \in \mathbb{D}^e$,

$$\text{Vect}\left(V^{\mathcal{J}(\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}(p)\right) \bigcap \text{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}(p)\right) = \{0\}.$$

This assumption means that the returns of the assets issued at a node ξ are not redundant with the returns of the assets issued at a predecessor node of ξ.

So, the issuance of additional assets at ξ leads to a true financial innovation since the payoffs in the successors of ξ cannot be replicated by the payoffs of a portfolio built with the assets issued before ξ.

In the following lemma, we show that if Assumption **R** holds true for the financial structure $\mathcal{F}$, it is also true for any financial substructure $\mathcal{F}'$ of $\mathcal{F}$ obtained by considering only a subset $\mathcal{J}'$ of the set of assets $\mathcal{J}$.

Lemma 3.1. *Let*

$$\mathcal{F} = \left(\mathcal{J}, (\xi(j))_{j\in\mathcal{J}}, V\right) \quad \mathcal{F}' = \left(\mathcal{J}', (\xi(j))_{j\in\mathcal{J}'}, V'\right)$$

*two financial structures such as $\mathcal{J}' \subset \mathcal{J}$. If Assumption **R** holds true for the structure $\mathcal{F}$ then it holds also true for the structure $\mathcal{F}'$.*

Proof. The proof of Lemma 3.1 is given in Appendix.

Remark 3.1. The converse of Lemma 3.1 is not true. Let us consider an economy with three periods such as: $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}\}$.

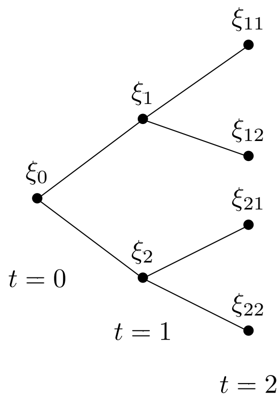

Figure 3: the tree $\mathbb{D}$

There are three assets issued at nodes ξ_0, ξ_1 and ξ_2. The return matrix is V and the one of the substructure where we keep only the two first assets is V':

$$V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{12} \\ \xi_{21} \\ \xi_{22} \end{matrix} \quad V' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{12} \\ \xi_{21} \\ \xi_{22} \end{matrix}$$

One remark that Assumption **R** is not satisfied for V for the node ξ_2 whereas it holds true for the reduced financial structure.

The next proposition provides some sufficient conditions under which Assumption **R** holds true.

Proposition 3.3. *Given a spot price vector $p \in \mathbb{R}^{\mathbb{L}}$. The return matrix $V(p)$ satisfies Assumption* **R** *if one of the following condition is satisfied:*

(i) For all $j \in \mathcal{J}$, asset j is a short term asset in the sense that $V^j_{\xi'}(p) = 0$ if $\xi' \notin \xi^+$.

(ii) All assets are issued at the same date τ_1.

(iii) For all $\xi \in \mathbb{D}^e$, $\mathbb{D}^+(\xi) \cap \mathbb{D}^e = \emptyset$, which means that if an asset is issued at node ξ, then no assets is issued at a successor of ξ.

(iv) For all $\xi \in \mathbb{D}$ and for all $j, \ell \in \mathcal{J}$, $V^j_\xi(p) V^l_\xi(p) = 0$ if $\xi(j) \neq \xi(l)$.

(v) For all $(\xi, \xi') \in (\mathbb{D}^e)^2$, if $\xi < \xi'$, then $V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi')} = 0$, which means that if an asset j is issued at node ξ and another one at a successor ξ', then the return of j at the successors of ξ' are equal to 0.

The proof of this proposition is left to the reader. It is a consequence of the fact that either $\mathcal{J}(\mathbb{D}^-(\xi))$ is an empty set or Vect $\left(V^{\mathcal{J}(\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}(p)\right) = \{0\}$. Note that Condition (iv) is introduced in [4] to study the structure of the arbitrage free asset prices.

Note that in a two period model, there are only short term assets, so all financial structures satisfy Assumption **R** in that case.

If the assets issued at each node are linearly independent, then Assumption **R** is derived from a slightly weaker assumption where we only deal with the returns at the immediate successors of a node ξ instead of looking at the whole returns for all successors.

Corollary 3.1. *Given a spot price vector $p \in \mathbb{R}^{\mathbb{L}}$. Let us assume that:*

1) $\forall \xi \in \mathbb{D}^e$, $\text{rank} V(p)^{\mathcal{J}(\xi)}_{\xi^+}(p) = n(\xi)$

and

2) $\text{Vect}\left(V^{\mathcal{J}(\mathbb{D}^-(\xi))}_{\xi^+}(p)\right) \bigcap \text{Vect}\left(V^{\mathcal{J}(\xi)}_{\xi^+}(p)\right) = \{0\}$.

Then Assumption **R** *is satisfied.*

The proof is given in Appendix.

Remark 3.2. In Magill and Quinzii [5], it is assumed that a long-term asset is re-traded at each nodes after its issuance node. In Angeloni and Cornet [1], it is shown that a financial structure with re-trading is equivalent to a financial without re-trading by considering that a re-trade is equivalent to the issuance of a new asset.

We remark that if the financial structure has long-term assets with re-trading, then Assumption **R** may hold true without re-trading but not with re-trading. Let us give an example. Let us consider the date-event tree $\mathbb{D}$ as above in Remark 3.1. Two assets are issued at ξ_0 with dividend processes

$$V^1 = (0, (0,0), (1,0,1,0)) \quad V^2 = (0, (0,0), (0,1,0,1))$$

so Assumption **R** is satisfied thanks to Proposition 3.3 (ii).

If these two assets are re-traded at each non-terminal node successor of ξ_0, for all arbitrage free price $q = (q_1(\xi_0), q_2(\xi_0), q_1(\xi_1), q_2(\xi_1), q_1(\xi_2), q_2(\xi_2))$, the full payoff matrix $\mathbf{W_{MQ}(q)}$ is:

$$\begin{bmatrix} -q_1(\xi_0) & -q_2(\xi_0) & 0 & 0 & 0 & 0 \\ q_1(\xi_1) & q_2(\xi_1) & -q_1(\xi_1) & -q_2(\xi_1) & 0 & 0 \\ q_1(\xi_2) & q_2(\xi_2) & 0 & 0 & -q_1(\xi_2) & -q_2(\xi_2) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{12} \\ \xi_{21} \\ \xi_{22} \end{matrix}$$

But if, following the methodology of Angeloni-Cornet [1], we consider an equivalent financial structure with 6 assets without re-trading, we obtain the following full payoff matrix with $\tilde{q} = (\tilde{q}_1, \tilde{q}_2, \tilde{q}_{11}, \tilde{q}_{12}, \tilde{q}_{21}, \tilde{q}_{22})$,

$$\mathbf{W_{AC}(\tilde{q})} = \begin{bmatrix} -\tilde{q}_1 & -\tilde{q}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\tilde{q}_{11} & -\tilde{q}_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\tilde{q}_{21} & -\tilde{q}_{22} \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{12} \\ \xi_{21} \\ \xi_{22} \end{matrix}$$

We remark that the two financial structures are equivalent when $q = \tilde{q}$ since, by performing elementary operations on the columns of $W_{AC}(q)$, we obtain $W_{MQ}(q)$. Assumption **R** is not satisfied because the returns of assets issued at nodes ξ_1 and ξ_2 are redundant with the return of assets issued at node ξ_0. As already remarked in Magill-Quinzii [5], the rank of the full payoff matrix $W_{MQ}(q)$, so the kernel, depends on the asset price vector q.

3.2.2 Equality of kernels under Condition R

Now, we state the main result of this section:

Proposition 3.4. *Given a spot price vector $p \in \mathbb{R}^{\mathbb{L}}$. If the return matrix $V(p)$ satisfies Assumption* **R***, then for all arbitrage free price q,* $\mathrm{Ker}V(p) = \mathrm{Ker}W(p,q)$.

Condition (i) of Proposition 3.3 shows that Proposition 3.5 is a generalization of Proposition 5.2. b) and c) in Angeloni-Cornet [1] and of Magill-Quinzii [5] where only short-term assets are considered.

Remark 3.3. For the following financial structure, Assumption **R** does not hold true and yet, for any (arbitrage free or not) price of assets q, $\mathrm{Ker}V(p) = \mathrm{Ker}W(p,q)$. So Assumption **R** is sufficient but not necessary. Let us consider the date-event tree $\mathbb{D}$ as above in Remark 3.1. Three assets are issued, two at

ξ_0 and one at ξ_1. For all asset price $q = (q_1, q_2, q_3)$, the return matrix and the full return matrix are:

$$\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{12} \\ \xi_{21} \\ \xi_{22} \end{matrix} \quad \text{and} \quad \mathbf{W}(\mathbf{q}) = \begin{bmatrix} -q_1 & -q_2 & 0 \\ 1 & 0 & -q_3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{12} \\ \xi_{21} \\ \xi_{22} \end{matrix}$$

One easily checks $\text{rank}V = \text{rank}W(q) = 3$ so $\text{Ker}V = \{0\} = \text{Ker}W(q)$ whatever is the asset price q.

The proof of Proposition 3.4 uses as a key step the following proposition on the equality of the rank of the payoff matrices.

Proposition 3.5. *Given a spot price vector $p \in \mathbb{R}^{\mathbb{L}}$. If the return matrix $V(p)$ satisfies Assumption* $\mathbf{R}$*, then for all arbitrage free price q,* $\text{rank}V(p) = \text{rank}W(p,q)$.

We first give the proof of Proposition 3.4 and then the one of Proposition 3.5.

Proof of Proposition 3.4. Let q be an arbitrage free price and let $\lambda = (\lambda_\xi) \in \mathbb{R}^{\mathbb{D}}_{++}$ such that ${}^tW(p,q)\lambda = 0$. From Proposition 3.5, $\text{rank}V(p) = \text{rank}W(p,q)$ and this implies that $\dim \text{Ker}V(p) = \dim \text{Ker}W(p,q)$. So, to get the equality of the kernels, it remains to show $\text{Ker}V(p) \subset \text{Ker}W(p,q)$.

Let $z = \left(z^j\right)_{j\in\mathcal{J}} \in \mathbb{R}^{\mathcal{J}}$ be an element of the kernel of the payoff matrice $V(p)$. So, $\sum_{j\in\mathcal{J}} z^j V^j(p) = 0$ which is equivalent to: for all $\xi \in \mathbb{D}, \sum_{j\in\mathcal{J}} z^j V^j_\xi(p) = 0$. Let us show that $z \in \text{Ker}W(p,q)$. We work by backward induction on $\kappa \in \{1, ..., k\}$.

For all $\xi \in \mathbb{D}^e_{\tau_k}$, $\sum_{j\in\mathcal{J}} z^j V^j(p) = 0$ implies that $\sum_{j\in\mathcal{J}} z^j V^j_{\mathbb{D}^+(\xi)}(p) = 0$. For all j such that $\xi(j) \notin \mathbb{D}^-(\xi)$, $V^j_{\mathbb{D}^+(\xi)}(p) = 0$. So we deduce that

$$\sum_{j\in\mathcal{J}(\xi)} z^j V^j_{\mathbb{D}^+(\xi)}(p) + \sum_{\xi'\in\mathbb{D}^-(\xi)} \sum_{j\in\mathcal{J}(\xi')} z^j V^j_{\mathbb{D}^+(\xi)}(p) = 0$$

From Assumption $\mathbf{R}$, $\text{Vect}\left(V^{\mathcal{J}(\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}(p)\right) \bigcap \text{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}(p)\right) = \{0\}$. From the above equality, $\sum_{j\in\mathcal{J}(\xi)} z^j V^j_{\mathbb{D}^+(\xi)}(p) = 0$ since $\sum_{j\in\mathcal{J}(\xi)} z^j V^j_{\mathbb{D}^+(\xi)}(p)$ belongs to $\text{Vect}\left(V^{\mathcal{J}(\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}(p)\right) \bigcap \text{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}(p)\right)$.

Moreover, by the fact that $V^j_{\xi'}(p) = 0$ for all $\xi' \notin \mathbb{D}^+(\xi)$, one deduces that $\sum_{j\in\mathcal{J}(\xi)} z^j V^j(p) = 0$.

For all $j \in \mathcal{J}(\xi)$ and for all $\eta \in \mathbb{D} \setminus \{\xi\}$, $V^j_\eta(p) = W^j_\eta(p,q)$. At the node ξ, $\sum_{j\in\mathcal{J}(\xi)} z^j W^j_\xi(p,q) = -\sum_{j\in\mathcal{J}(\xi)} z^j q_j$. But $q_j = (1/\lambda_\xi) \sum_{\xi'\in\mathbb{D}^+(\xi)} \lambda_{\xi'} V^j_{\xi'}(p)$. Hence,

$$\begin{aligned}\sum_{j\in\mathcal{J}(\xi)} z^j q_j &= (1/\lambda_\xi)\sum_{j\in\mathcal{J}(\xi)} z^j \left[\sum_{\xi'\in\mathbb{D}^+(\xi)}\lambda_{\xi'}V^j_{\xi'}(p)\right]\\ &= (1/\lambda_\xi)\sum_{\xi'\in\mathbb{D}^+(\xi)}\lambda_{\xi'}\left[\sum_{j\in\mathcal{J}(\xi)} z^j V^j_{\xi'}(p)\right]\\ &= 0\end{aligned}$$

So, we have proved that $\sum_{j\in\mathcal{J}(\xi)} z^j W^j(p,q)=0$, and since it holds true for all $\xi\in\mathbb{D}^e_{\tau_k}$, $\sum_{\xi\in\mathbb{D}^e_{\tau_k}}\sum_{j\in\mathcal{J}(\xi)} z^j W^j(p,q)=0$.

It remains to get that $\sum_{\kappa=1}^{k-1}\sum_{\xi\in\mathbb{D}^e_{\tau_\kappa}}\sum_{j\in\mathcal{J}(\xi)} z^j W^j(p,q)=0$. But we can repeat the same argument for the nodes $\xi\in\mathbb{D}^e_{\tau_{k-1}}$ since for all $j\in\cup_{\kappa=1}^{k-1}\mathcal{J}_{\tau_\kappa}$, $V^j_{\mathbb{D}^+(\xi)}(p)=0$ if $\xi(j)\notin\mathbb{D}^-(\xi)$. Hence, after a finite number of steps, we prove that $\sum_{j\in\mathcal{J}} z^j W^j(p,q)=0$ that is $z\in\mathrm{Ker}W(p,q)$. □

Proof of Proposition 3.5. For all $\xi\in\mathbb{D}^e$, we denote by $n(\xi)$ the number of assets issued at this node and by $\mathrm{rk}(\xi)$ the rank of $V^{\mathcal{J}(\xi)}_{\mathbb{D}}(p)$. We also simplify the notation by defining $V^{\mathcal{J}(\xi)}(p):=V^{\mathcal{J}(\xi)}_{\mathbb{D}}(p)$ and $W^{\mathcal{J}(\xi)}(p,q):=W^{\mathcal{J}(\xi)}_{\mathbb{D}}(p,q)$.

Step 1: For all $\xi\in\mathbb{D}^e$, $\mathrm{rank}W^{\mathcal{J}(\xi)}(p,q)=\mathrm{rk}(\xi)$.

If $\mathrm{rk}(\xi)=n(\xi)$, $\mathrm{rank}W^{\mathcal{J}(\xi)}(p,q)=n(\xi)$. Indeed, since $\mathrm{rk}(\xi)=n(\xi)$, there exists a regular $n(\xi)$ square sub-matrix of $V^{\mathcal{J}(\xi)}(p)$. Since $W^{\mathcal{J}(\xi)}(p,q)$ is obtained from $V^{\mathcal{J}(\xi)}(p)$ by replacing a zero row by the row of the opposite of asset prices issued at ξ, the regular $n(\xi)$ square sub-matrix is also a sub-matrix of $W^{\mathcal{J}(\xi)}(p,q)$, hence the rank of $W^{\mathcal{J}(\xi)}(p,q)$ is higher or equal to $n(\xi)$. But, since $n(\xi)$ is the number of columns of $W^{\mathcal{J}(\xi)}(p,q)$, then its rank is lower or equal to $n(\xi)$ so that we obtain the desired result[2].

If $\mathrm{rk}(\xi)<n(\xi)$, let us consider $\lambda=(\lambda_\xi)_{\xi\in\mathbb{D}}\in\mathbb{R}^{\mathbb{D}}_{++}$ such that ${}^tW(p,q)\,\lambda=0$. Such λ exists since q is an arbitrage free price.

For all $\xi\in\mathbb{D}^e$, let $\mathcal{J}'(\xi)\subset\mathcal{J}(\xi)$ such that $\#\mathcal{J}'(\xi)=\mathrm{rk}(\xi)$ and the family $(V^j_\xi(p))_{j\in\mathcal{J}'(\xi)}$ is linearly independent. By the same argument as above, $(W^j(p,q))_{j\in\mathcal{J}'(\xi)}$ are also linearly independent. Hence the rank of $W^{\mathcal{J}(\xi)}(p,q)$ is larger or equal to $\mathrm{rk}(\xi)$. Let us now prove that the rank of $W^{\mathcal{J}(\xi)}(p,q)$ is not strictly larger than $\mathrm{rk}(\xi)$. It suffices to prove that for all $j_0\notin\mathcal{J}'(\xi)$, $W^{j_0}(p,q)\in\mathrm{Vect}((W^j(p,q))_{j\in\mathcal{J}'(\xi)})$.

$V^{j_0}(p)$ is a linear combination of $(V^j(p))_{j\in\mathcal{J}'(\xi)}$ since the rank of $V^{\mathcal{J}(\xi)}(p)$ is $\mathrm{rk}(\xi)$. Hence there exists $(\alpha_j)_{j\in\mathcal{J}'(\xi)}$ such that $\sum_{j\in\mathcal{J}'(\xi)}\alpha_j V^j(p)=V^{j_0}(p)$. Since ${}^tW(p,q)\,\lambda=0$, $\lambda_\xi q_{j_0}=\sum_{\xi'\in\mathbb{D}^+(\xi)}\lambda_{\xi'}V^{j_0}_{\xi'}(p)$. Hence $\lambda_\xi q_{j_0}$ is equal to

$$\begin{aligned}\sum_{\xi'\in\mathbb{D}^+(\xi)}\left[\lambda_{\xi'}\sum_{j\in\mathcal{J}'(\xi)}\alpha_j V^j_{\xi'}(p)\right] &= \sum_{j\in\mathcal{J}'(\xi)}\left[\alpha_j\sum_{\xi'\in\mathbb{D}^+(\xi)}\lambda_{\xi'}V^j_{\xi'}(p)\right]\\ &= \sum_{j\in\mathcal{J}'(\xi)}[\alpha_j\lambda_\xi q_j]=\lambda_\xi\sum_{j\in\mathcal{J}'(\xi)}\alpha_j q_j\end{aligned}$$

Hence $q_{j_0}=\sum_{j\in\mathcal{J}'(\xi)}\alpha_j q_j$. Since $\sum_{j\in\mathcal{J}'(\xi)}\alpha_j V^j(p)=V^{j_0}(p)$, we obtain

$$\sum_{j\in\mathcal{J}'(\xi)}\alpha_j W^j(p,q)=W^{j_0}(p,q)$$

[2]Note that we do not use the fact that the asset price is an arbitrage free price in this part of the proof.

So $W^{j_0}(p,q)$ belongs to $\mathrm{Vect}(W^j(p,q))_{j\in\mathcal{J}'(\xi)})$.

For $\kappa = 1,\ldots,k$, we let $\mathrm{rk}_\kappa = \sum_{\xi\in\mathbb{D}^e_{\tau_\kappa}} \mathrm{rk}\,(\xi)$.

Step 2: $\forall\kappa\in\{1,\ldots,k\}$, $\mathrm{rank}V^{\mathcal{J}_{\tau_\kappa}}(p) = \mathrm{rk}_\kappa = \mathrm{rank}W^{\mathcal{J}_{\tau_\kappa}}(p,q)$.

If $\#\mathbb{D}^e_{\tau_\kappa} = 1$, this coincides with what is proved in Step 1. If $\#\mathbb{D}^e_{\tau_\kappa} > 1$, let $\xi \in \mathbb{D}^e_{\tau_\kappa}$. Then

$$\left[\sum_{\left\{\xi'\in\mathbb{D}^e_{\tau_\kappa}\setminus\{\xi\}\right\}} \mathrm{Vect}\left(V^{\mathcal{J}(\xi')}(p)\right)\right]\bigcap \mathrm{Vect}\left(V^{\mathcal{J}(\xi)}(p)\right) = \{0\}$$

Indeed, the return of the asset $j\in\mathcal{J}(\xi)$ can be non zero only on the subtree $\mathbb{D}^+(\xi)$, whereas for the asset $j\in\mathcal{J}(\xi')$ for $\xi'\in\mathbb{D}^e_{\tau_\kappa}\setminus\{\xi\}$, the returns on the subtree $\mathbb{D}^+(\xi)$ are identically equal to 0. This implies that the subspaces $\left(\mathrm{Vect}\left(V^{\mathcal{J}(\xi')}(p)\right)\right)_{\xi'\in\mathbb{D}^e_{\tau_\kappa}}$ are in direct sum so, using Step 1, we get the following formula for the dimensions:

$$\dim\mathrm{Vect}\left(V^{\mathcal{J}_{\tau_\kappa}}(p)\right) = \sum_{\xi\in\mathbb{D}^e_{\tau_\kappa}} \dim\mathrm{Vect}\left(V^{\mathcal{J}(\xi)}(p)\right) = \sum_{\xi\in\mathbb{D}^e_{\tau_\kappa}} \mathrm{rk}\,(\xi) = \mathrm{rk}_\kappa$$

For the matrix $W(p,q)$, the proof is the same as above if we remark that the full return of an asset $j\in\mathcal{J}(\xi)$ can be non zero only on the subtree $\xi\cup\mathbb{D}^+(\xi)$. Hence if ξ and ξ' are two different issuance nodes in $\mathbb{D}^e_{\tau_\kappa}$, there is no node ξ'' such that the coordinates of a column vectors of the matrix $W^{\mathcal{J}(\xi)}(p,q)$ and of a column vector of the matrix $W^{\mathcal{J}(\xi')}(p,q)$ are both non zero. Hence we get the following formula from which the result is a direct consequence of Step 1:

$$\left[\sum_{\left\{\xi'\in\mathbb{D}^e_{\tau_\kappa}\setminus\{\xi\}\right\}} \mathrm{Vect}\left(W^{\mathcal{J}(\xi')}(p,q)\right)\right]\bigcap \mathrm{Vect}\left(W^{\mathcal{J}(\xi)}(p,q)\right) = \{0\}$$

Step 3. $\mathrm{rank}V(p) = \sum_{\kappa=1}^k \mathrm{rk}_\kappa = \mathrm{rank}W\,(p,q)$.

We first remark that $\mathrm{Vect}\,(V(p)) = +_{\kappa=1}^k \mathrm{Vect}\left(V^{\mathcal{J}_{\tau_\kappa}}(p)\right)$ which implies using Step 2 that $\mathrm{rank}V(p) \leq \sum_{\kappa\in\{1,\ldots,k\}} \mathrm{rank}V^{\mathcal{J}_{\tau_\kappa}}(p) = \sum_{\kappa=1}^k \mathrm{rk}_\kappa$.

We remark that if $k=1$, then the result is obvious. If $k>1$, we will prove that the rank of $V(p)$ is equal to $\sum_{\kappa=1}^k \mathrm{rk}_\kappa$ by showing that a family of column vectors of $V(p)$ of cardinal $\sum_{\kappa=1}^k \mathrm{rk}_\kappa$ is linearly independent.

For all $\kappa = 1,\ldots,k$, $\mathcal{J}'_\kappa = \cup_{\xi\in\mathbb{D}^e_{\tau_\kappa}} \mathcal{J}'(\xi)$ and $\mathcal{J}' = \cup_{\kappa=1}^k \mathcal{J}'_\kappa$. We now prove that the family $(V^j(p))_{j\in\mathcal{J}'}$ is linearly independent.

Let $(\alpha_j)\in\mathbb{R}^{\mathcal{J}'}$ such that $\sum_{j\in\mathcal{J}'}\alpha_j V^j(p) = 0$. We work by backward induction on κ from k to 1.

For all $\xi \in \mathbb{D}^e_{\tau_k}$, $\sum_{j\in\mathcal{J}'} \alpha_j V^j_{\mathbb{D}^+(\xi)}(p) = 0$. Since $\tau_\kappa < \tau_k$ for all $\kappa = 1, \dots, k-1$, for all j such that $\xi(j) \notin \mathbb{D}^-(\xi) \cup \{\xi\}$, $V^j_{\mathbb{D}^+(\xi)}(p) = 0$. So, one gets

$$\sum_{j\in\mathcal{J}'(\xi)} \alpha_j V^j_{\mathbb{D}^+(\xi)}(p) + \sum_{\xi'\in\mathbb{D}^-(\xi)} \sum_{j\in\mathcal{J}'(\xi')} \alpha_j V^j_{\mathbb{D}^+(\xi)}(p) = 0$$

From Assumption **R**,

$$\text{Vect}\left(V^{\mathcal{J}(\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}(p)\right) \bigcap \text{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}(p)\right) = \{0\}.$$

From the above equality,

$$\sum_{j\in\mathcal{J}'(\xi)} \alpha_j V^j_{\mathbb{D}^+(\xi)}(p) \in \text{Vect}\left(V^{\mathcal{J}(\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}(p)\right) \bigcap \text{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}(p)\right)$$

hence $\sum_{j\in\mathcal{J}'(\xi)} \alpha_j V^j_{\mathbb{D}^+(\xi)}(p) = 0$.

By construction, the family $(V^j(p))_{j\in\mathcal{J}'(\xi)}$ is linearly independent and for all $\xi' \notin \mathbb{D}^+(\xi)$, $V^j_{\xi'}(p) = 0$, so the family $(V^j_{\mathbb{D}^+(\xi)}(p))_{j\in\mathcal{J}'(\xi)}$ is linearly independent. Hence, from above, one deduces that $\alpha_j = 0$ for all $j \in \mathcal{J}'(\xi)$. Since this is true for all $\xi \in \mathbb{D}^e_{\tau_k}$, one gets $\alpha_j = 0$ for all $j \in \mathcal{J}'_k$.

If $k = 2$, we are done since we have prove in Step 2 that the subspaces $\left(\text{Vect}\left(V^{\mathcal{J}(\xi)}(p)\right)\right)_{\xi\in\mathbb{D}^e_{\tau_1}}$ are in direct sum, so the family $\left(V^j(p)\right)_{j\in\mathcal{J}'_1}$ is linearly independent, hence for all $j \in \mathcal{J}'_1$, $\alpha_j = 0$.

If $k > 2$, we do again the same argument as above. Indeed, since we have proved that for all $j \in \mathcal{J}'_k$, $\alpha_j = 0$, for all $\xi \in \mathbb{D}^e_{\tau_{k-1}}$, $\sum_{j\in\mathcal{J}'} \alpha_j V^j_{\mathbb{D}^+(\xi)}(p) = 0$ implies

$$\sum_{j\in\mathcal{J}'(\xi)} \alpha_j V^j_{\mathbb{D}^+(\xi)}(p) + \sum_{\xi'\in\mathbb{D}^-(\xi)} \sum_{j\in\mathcal{J}'(\xi')} \alpha_j V^j_{\mathbb{D}^+(\xi)}(p) = 0.$$

Using again Assumption **R** and the linear independence of $(V^j_{\mathbb{D}^+(\xi)}(p))_{j\in\mathcal{J}'(\xi)}$, one then deduces that for all $j \in \mathcal{J}'_{k-1}$, $\alpha_j = 0$.

Consequently, after a finite number of steps, we deduce that all α_j are equal to 0, which implies that the family $(V^j(p))_{j\in\mathcal{J}'}$ is linearly independent.

To prove that the family $(W^j(p,q))_{j\in\mathcal{J}'}$ is linearly independent, we use a similar argument noticing that for all $\xi \in \mathbb{D}^e$, the family $(W^j(p,q))_{j\in\mathcal{J}'(\xi)}$ is linearly independent and for all $j \in \mathcal{J}'(\xi) \cup \left(\cup_{\xi'\in\mathbb{D}^-(\xi)} J'(\xi')\right)$, $V^j_{\mathbb{D}^+(\xi)}(p) = W^j_{\mathbb{D}^+(\xi)}(p,q)$. □

Remark 3.4. If the price q exhibits an arbitrage, then even under Assumption **R**, the rank of $V(p)$ and the rank of $W(p,q)$ may be different. With a three dates economy where $\mathbb{D} = \{\xi_0, \xi_1, \xi_2, \xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}\}$ as above in Remark 3.1, two assets issued at ξ_0 and one asset issued at ξ_1, the asset price $q = \left(1, \frac{3}{2}, 1\right)$, then

$$V = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{12} \\ \xi_{21} \\ \xi_{22} \end{matrix} \quad \text{and} \quad \mathbf{W}(\mathbf{q}) = \begin{bmatrix} -1 & -\frac{3}{2} & 0 \\ 1 & 2 & -1 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{12} \\ \xi_{21} \\ \xi_{22} \end{matrix}$$

We note that $\mathrm{rank}V = 2 < \mathrm{rank}W(q) = 3$. Nevertheless, the following result shows that if the payoff vectors are not redundant at each node, then the equality of ranks holds true even with an arbitrage price.

Proposition 3.6. *Given a spot price vector $p \in \mathbb{R}^{\mathbb{L}}$. Let us assume that V satisfies Assumption R.*

1) For all price $q \in \mathbb{R}^{\mathcal{J}}$, $\mathrm{rank}V(p) \leq \mathrm{rank}W(p,q)$.

2) Furthermore, if for all $\xi \in \mathbb{D}^e$, $\mathrm{rank}V^{\mathcal{J}(\xi)}(p) = n(\xi)$, the number of assets issued at this node, then $\mathrm{rank}V(p) = \mathrm{rank}W(p,q)$ for all price $q \in \mathbb{R}^{\mathcal{J}}$.

Proof. 1) The proof is just an adaptation of the proof of Proposition 3.5. In the first step, since the price q is not supposed to be a arbitrage free price, we get $\mathrm{rank}W^{\mathcal{J}(\xi)}(p,q) \geq \mathrm{rk}(\xi)$ instead of an equality. For the two next steps, the proofs never uses the fact that q is an arbitrage free price, so we can replicate them to obtain $\mathrm{rank}W(p,q) \geq \sum_{\xi \in \mathbb{D}^e} \mathrm{rk}(\xi) = \mathrm{rank}V(p)$.

2) If $\mathrm{rk}(\xi) = n(\xi)$ for all ξ, then $\sum_{\xi \in \mathbb{D}^e} \mathrm{rk}(\xi)$ is the cardinal of $\mathcal{J}$, which is the number of column of the matrix $W(p,q)$. So $\mathrm{rank}W(p,q) \leq \sum_{\xi \in \mathbb{D}^e} \mathrm{rk}(\xi) = \mathrm{rank}V(p)$. □

The following corollary is a generalization of Proposition 3.5, which could be interesting in presence of market restrictions.

Corollary 3.2. *Given a spot price vector $p \in \mathbb{R}^{\mathbb{L}}$. Let $\mathcal{F} = \left(\mathcal{J}, (\xi(j))_{j \in \mathcal{J}}, V\right)$ be a financial structure such that Assumption **R** is satisfied and let G be a linear subspace of $\mathbb{R}^{\mathcal{J}}$. Then for all arbitrage free price q, $\dim[W(p,q)G] = \dim[V(p)G]$.*

Proposition 3.5 is merely the case where $G = \mathbb{R}^{\mathcal{J}}$. The proof of Corollary 3.2 is deduced from Proposition 3.4 and the following lemma, the proof of which is given in Appendix.

Lemma 3.2. *Let E and F be two vector spaces and φ and ψ be two linear maps from E to F then $\mathrm{Ker}\varphi = \mathrm{Ker}\psi$ if and only if for all linear subspace G of E, $\dim \varphi(G) = \dim \psi(G)$.*

4 Existence of equilibrium

4.1 Financial exchange economy

We now consider a financial exchange economy, which is defined as the couple of an exchange economy $\mathcal{E}$ and a financial structure $\mathcal{F}$, which are linked by the portfolio sets of the consumers, which represent the sets of admissibles portfolios for each agent.

The stochastic exchange economy is described by a finite set of agent $\mathcal{I}$. Each agent $i \in \mathcal{I}$ has a consumption set $X_i \subset \mathbb{R}^{\mathbb{L}}$, which consists of all possible consumptions. An allocation is an element $x \in \prod_{i \in \mathcal{I}} X_i$ and we denote by x_i the consumption of agent i, which is the projection of x on X_i.

The tastes of each consumer $i \in \mathcal{I}$ are represented by a *strict preference correspondence* $P_i : \prod_{k \in \mathcal{I}} X_k \longrightarrow X_i$, where $P_i(x)$ defines the set of consumptions that are strictly preferred to x_i for agent i, given the consumption x_k for the other consumers $k \neq i$. P_i represents the consumer tastes, but also his behavior with respect to time and uncertainty, especially his impatience and attitude toward risk. If consumer preferences are represented by utility functions $u_i : X_i \longrightarrow \mathbb{R}$ for each $i \in \mathcal{I}$, the strict preference correspondence is defined by $P_i(x) = \{x'_i \in X_i | u_i(x'_i) > u_i(x_i)\}$.

For each node $\xi \in \mathbb{D}$, every consumer $i \in \mathcal{I}$ has a node endowment $e_i(\xi) \in \mathbb{R}^{\mathbb{H}}$ (contingent on the fact that ξ prevails) and we denote by $e_i = (e_i(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\mathbb{L}}$ the endowments for the whole set of nodes. The exchange economy $\mathcal{E}$ can be summarized by

$$\mathcal{E} = \left[\mathbb{D}, \mathbb{H}, \mathcal{I}, (X_i, P_i, e_i)_{i \in \mathcal{I}}\right].$$

We assume that each consumer i is endowed with a portfolio set $Z_i \subset \mathbb{R}^{\mathcal{J}}$. For a discussion on this concept we refer to Angeloni-Cornet [1], Aouani-Cornet [2] and the references therein. The financial exchange economy can thus be summarized by

$$(\mathcal{E}, \mathcal{F}) := \left[\mathbb{D}, \mathbb{H}, \mathcal{I}, (X_i, P_i, e_i)_{i \in \mathcal{I}}, \mathcal{J}, (Z_i)_{i \in \mathcal{I}}, (\xi(j))_{j \in \mathcal{J}}, V\right].$$

4.2 Financial equilibrium and arbitrage

Given the price $(p, q) \in \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$, the budget set of consumer $i \in \mathcal{I}$ is $B^i_{\mathcal{F}}(p, q)$ defined by[3]:

$$\{(x_i, z_i) \in X_i \times Z_i : \forall \xi \in \mathbb{D}, p(\xi) \bullet_{\mathbb{H}} [x_i(\xi) - e_i(\xi)] \leq [W(p, q) z_i](\xi)\}$$

or

$$\{(x_i, z_i) \in X_i \times Z_i : \; p \,\square\, (x_i - e_i) \leq W(p, q) z_i\}.$$

We now introduce the equilibrium notion:

Definition 4.1. *An equilibrium of the financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *is a list of strategies and prices* $(\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in \left(\mathbb{R}^{\mathbb{L}}\right)^{\mathcal{I}} \times \left(\mathbb{R}^{\mathcal{J}}\right)^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}} \setminus \{0\} \times \mathbb{R}^{\mathcal{J}}$ *such that*

(a) *for every* $i \in \mathcal{I}, (\bar{x}_i, \bar{z}_i)$ *maximizes the preferences* P_i *in the budget set* $B_{\mathcal{F}}(\bar{p}, \bar{q})$*, in the sense that*

$$(\bar{x}_i, \bar{z}_i) \in B^i_{\mathcal{F}}(\bar{p}, \bar{q}) \text{ and } [P_i(\bar{x}) \times Z_i] \bigcap B^i_{\mathcal{F}}(\bar{p}, \bar{q}) = \emptyset;$$

[3] For $x = (x(\xi))_{\xi \in \mathbb{D}}, p = (p(\xi))_{\xi \in \mathbb{D}}$ in $\mathbb{R}^{\mathbb{L}} = \mathbb{R}^{\mathbb{H} \times \mathbb{D}}$ (with $x(\xi), \; p(\xi)$ in $\mathbb{R}^{\mathbb{H}}$) we let $p \,\square\, x = (p(\xi) \bullet_{\mathbb{H}} x(\xi))_{\xi \in \mathbb{D}} \in \mathbb{R}^{\mathbb{D}}$.

(b) $\sum_{i\in\mathcal{I}} \bar{x}_i = \sum_{i\in\mathcal{I}} e_i$ *(Commodity market clearing condition);*

(c) $\sum_{i\in\mathcal{I}} \bar{z}_i = 0$ *(Portfolio market clearing condition).*

Angeloni-Cornet[1] noted that when portfolios may be constrained, the concept of arbitrage free has to be suitably modified. In particular, we shall make a distinction between the definitions of arbitrage free portfolio and arbitrage free financial structure.

Definition 4.2. *Given the financial structure $\mathcal{F} = (\mathcal{J}, (\xi(j))_{j\in\mathcal{J}}, V)$, and the portfolio sets $(Z_i)_{i\in\mathcal{I}}$, the portfolio $\bar{z}_i \in Z_i$ is said with no arbitrage opportunities or to be arbitrage free for agent $i \in \mathcal{I}$ at the price $(p,q) \in \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$ if there is no portfolio $z_i \in Z_i$ such that $W(p,q) z_i > W(p,q)(\bar{z}_i)$, that is, $[W(p,q) z_i](\xi) \geq [W(p,q)(\bar{z}_i)](\xi)$, for every $\xi \in \mathbb{D}$, with at least one strict inequality, or, equivalently, if:*

$$W(p,q)(Z_i - \bar{z}_i) \cap \mathbb{R}^{\mathbb{D}}_{+} = \{0\}.$$

The financial structure is said to be arbitrage free at (p,q) if there exists no portfolio $(z_i) \in \prod_{i\in\mathcal{I}} Z_i$ such that $W(p,q)\left(\sum_{i\in\mathcal{I}} z_i\right) > 0$, or, equivalently, if:

$$W(p,q)\left(\sum_{i\in\mathcal{I}} Z_i\right) \cap \mathbb{R}^{\mathbb{D}}_{+} = \{0\}.$$

Let the financial structure $\mathcal{F}$ be arbitrage free at (p,q), and let $(\bar{z}_i) \in \prod_{i\in\mathcal{I}} Z_i$ such that $\sum_{i\in\mathcal{I}} \bar{z}_i = 0$. Then, for every $i \in \mathcal{I}$, $\bar{z}_i$ is arbitrage free at (p,q). The converse is true, for example, when some agent's portfolio set is unconstrained, that is, when $Z_i = \mathbb{R}^{\mathcal{J}}$ for some $i \in \mathcal{I}$.

The following characterization of arbitrage free portfolio is taken from Angeloni-Cornet [1].

Proposition 4.1. *Let $\mathcal{F} = \left(\mathcal{J}, (\xi(j))_{j\in\mathcal{J}}, V\right)$ and the portfolio set $(Z_i)_{i\in\mathcal{I}}$, let $(p,q) \in \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}}$, for $i \in \mathcal{I}$, let $z_i \in Z_i$, assume that Z_i is convex and consider the following statements:*

(i) *There exists $\lambda_i \in \mathbb{R}^{\mathbb{D}}_{++}$ such that ${}^tW(p,q)\lambda_i \in N_{Z_i}(z_i)$*[4]*, or, equivalently, there exists $\eta \in N_{Z_i}(z_i)$ such that for every $j \in \mathcal{J}$,*

$$\lambda_{i\xi(j)} q_j = \sum_{\xi > \xi(j)} \lambda_{i\xi} V^j_{\xi}(p) - \eta_j.$$

(ii) *The portfolio z_i is arbitrage free for agent $i \in \mathcal{I}$ at price (p,q).*

The implication $[(i) \Rightarrow (ii)]$ always holds and the converse is true under the additional assumption that Z_i is a polyhedral[5] *set.*

[4] we recall that $N_{Z_i}(z_i)$ is the normal cone to Z_i at z_i, which is defined as $N_{Z_i}(z_i) = \{\eta \in \mathbb{R}^{\mathcal{J}} : \eta \bullet_{\mathcal{J}} z_i \geq \eta \bullet_{\mathcal{J}} z'_i, \forall z'_i \in Z_i\}$.

[5] A subset $C \subset \mathbb{R}^n$ is said to be polyhedral if it is the intersection of finitely many closed half-spaces, namely $C = \{x \in \mathbb{R}^n : Ax \leq b\}$, where A is a real $(m \times n)$-matrix , and $b \in \mathbb{R}^m$. Note that polyhedral sets are always closed and convex and that the empty set and the whole space $\mathbb{R}^n$ are both polyhedral.

We recall that equilibrium portfolios are arbitrage free under the following non-satiation assumption:

Assumption NS

(i) (Non-Saturation at Every Node.) For every $\bar{x} \in \prod_{i\in\mathcal{I}} X_i$ if $\sum_{i\in\mathcal{I}} \bar{x}_i = \sum_{i\in\mathcal{I}} e_i$, then, for every $i \in \mathcal{I}$, for every $\xi \in \mathbb{D}$, there exists $x_i \in X_i$ such that, for each $\xi' \neq \xi$, $x_i(\xi') = \bar{x}_i(\xi')$ and $x_i \in P_i(\bar{x})$;

(ii) if $x_i \in P_i(\bar{x})$, then $]\bar{x}_i, x_i] \subset P_i(\bar{x})$.

Proposition 4.2. *Under Assumption (NS), if $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium of the economy $(\mathcal{E}, \mathcal{F})$, then $\bar{z}_i$ is arbitrage free at price $(\bar{p}, \bar{q})$ for every $i \in \mathcal{I}$.*

The proof is given in Angeloni-Cornet [1].

4.3 The existence result

From now on, we only consider nominal asset financial structure for which V does not depend on the spot price p.

We introduce the following assumptions on the consumers and the financial structure. They are borrowed from Angeloni-Cornet [1] and Cornet-Gopalan [3]. In the following $\mathcal{Z}_{\mathcal{F}}$ is the linear space spanned by $\cup_{i\in\mathcal{I}} Z_i$.

Assumption C (Consumption Side) For all $i \in \mathcal{I}$ and all $\bar{x} \in \prod_{i\in\mathcal{I}} X_i$,

(i) X_i is a closed, convex and bounded below subset of $\mathbb{R}^{\mathbb{L}}$;

(ii) the preference correspondence P_i, from $\prod_{k\in\mathcal{I}} X_k$ to X_i, is lower semicontinuous[6] and $P_i(\bar{x})$ is convex;

(iii) for every $x_i \in P_i(\bar{x})$ for every $x'_i \in X_i$, $x'_i \neq x_i$, $[x'_i, x_i[\cap P_i(\bar{x}) \neq \emptyset$[7];

(iv) (Irreflexivity) $\bar{x}_i \notin P_i(\bar{x})$;

(v) (Non-Saturation of Preferences at Every Node) if $\sum_{i\in\mathcal{I}} \bar{x}_i = \sum_{i\in\mathcal{I}} e_i$, for every $i \in \mathcal{I}$, for every $\xi \in \mathbb{D}$, there exists $x_i \in X_i$ such that, for each $\xi' \neq \xi$, $x_i(\xi') = \bar{x}_i(\xi')$ and $x_i \in P_i(\bar{x})$;

(vi) (Strong Survival Assumption) $e_i \in \text{int} X_i$.

Note that these assumptions on P_i are satisfied when agents' preferences are represented by a continuous, strongly monotone and quasi-concave utility function.

Assumption F (Financial Side)

(i) for every $i \in \mathcal{I}$, Z_i is a closed, convex subset of $\mathbb{R}^{\mathcal{J}}$ containing 0;

[6] A correspondence $\phi : X \longrightarrow Y$ is said lower semicontinuous at $x_0 \in X$ if, for every open set $V \subset Y$ such that $V \cap \phi(x_0)$ is nonempty, there exists a neighborhood U of x_0 in X such that, for all $x \in U$, $V \cap \phi(x)$ is nonempty. The correspondence ϕ is said to be lower semicotinuous if it is lower semicontinuous at each point of X.

[7] This is satisfied, in particular, when $P_i(\bar{x})$ is open in X_i (for its relative topology).

(ii) there exists $i_0 \in \mathcal{I}$ such that $0 \in \mathrm{ri}_{\mathcal{Z}_{\mathcal{F}}}(Z_{i_0})$[8].

Note that we slightly weaken the assumption of Angeloni-Cornet [1] since we consider the linear space $\mathcal{Z}_{\mathcal{F}}$ instead of $\mathbb{R}^{\mathcal{J}}$ for the relative interior. Nevertheless, Assumption F is stronger than the corresponding one in Cornet-Gopalan [3] (Assumption FA), which is that the closed cone spanned by $\cup_{i\in\mathcal{I}} W(q)(Z_i)$ is a linear space. Indeed, if $0 \in \mathrm{ri}_{\mathcal{Z}_{\mathcal{F}}}(Z_{i_0})$, then the cone spanned by $W(q)(Z_{i_0})$ is equal to $W(q)(\mathcal{Z}_{\mathcal{F}})$, which is a linear space and since $Z_i \subset \mathcal{Z}_{\mathcal{F}}$ for all i, $W(q)(Z_i) \subset W(q)(\mathcal{Z}_{\mathcal{F}})$. Hence, the cone spanned by $\cup_{i\in\mathcal{I}} W(q)(Z_i)$ is equal to $W(q)(\mathcal{Z}_{\mathcal{F}})$, which is a linear space.

Our main existence result is the following:

Proposition 4.3. *Let*

$$(\mathcal{E},\mathcal{F}) := \left[\mathbb{D},\mathbb{H},\mathcal{I},(X_i,P_i,e_i)_{i\in\mathcal{I}},\mathcal{J},(Z_i)_{i\in\mathcal{I}},(\xi(j))_{j\in\mathcal{J}},V\right]$$

be a financial economy with nominal assets satisfying Assumptions **C**, **R**, **F** *and such that* $\mathrm{Ker}V \cap \mathcal{Z}_{\mathcal{F}} = \{0\}$. *Then, for any given* $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$, *there exists a financial equilibrium* $(\bar{x},\bar{z},\bar{p},\bar{q})$ *where* $\bar{q}$ *satisfies*

$${}^tW(\bar{q})\lambda \in N_{Z_{i_0}}(\bar{z}_{i_0})$$

Our contribution is to obtain an existence result with long-term assets with assumptions only on the fundamentals of the economy, namely the payoff matrix V and the portfolio sets Z_i, regardless of the arbitrage free price. Note that in Cornet-Gopalan [3], Assumption FA depends on the asset price q, which is an endogenous variable.

Remark 4.1. $\mathcal{Z}_{\mathcal{F}} \cap \mathrm{Ker}V = \{0\}$ is a slight weakening of the usual assumption of no redundant assets. To remove this assumption, as in [2] in a two-period model, we need to consider an auxiliary economy with a projection of the portfolio sets, which is a topic of further research.

The proof of our existence result is based upon Theorem 3.1 of Angeloni-Cornet [1]. To state this theorem, we need to introduce the set $B^{\delta}(\lambda)$ of δ-admissible consumptions and portfolios for a given state price $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$, that is, the set of consumption-portfolio pair $(x,z) \in \prod_{i\in\mathcal{I}} X_i \times \prod_{i\in\mathcal{I}} Z_i$ such that there exists a commodity-asset price pair $(p,q) \in \bar{B}_{\mathbb{L}}(0,1) \times \mathbb{R}^{\mathcal{J}}$ satisfying:

$$\begin{cases} {}^tW(p,q)\lambda \in \bar{B}_{\mathcal{J}}(0,\delta), \\ (x_i,z_i) \in B^i_{\mathcal{F}}(p,q)\, \forall i \in \mathcal{I}, \\ \sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} e_i, \\ \sum_{i\in\mathcal{I}} z_i = 0 \end{cases}$$

The standard existence result requires that the set $B^1(\lambda)$ is bounded but the proof can be easily adapted to the case where $B^{\delta}(\lambda)$ is bounded for some $\delta > 0$ (see the proof of Proposition 4.3 below). In [1], it is proved that this holds true if

[8]Let Z a nonempty subset of $\mathbb{R}^{\mathcal{J}}$ and let H a subspace of $\mathbb{R}^{\mathcal{J}}$ such that $Z \subset H$. We call relative interior of Z with respect to H denoted $\mathrm{ri}_H(Z)$ the set $\{z \in \mathbb{R}^{\mathcal{J}} \mid \exists r > 0; B(z,r) \cap H \subset Z\}$.

the assets are all short-term and rank$V = \#\mathcal{J}$ or, if there are long-term assets, that rank$W(p,q) = \#\mathcal{J}$ for all $(p,q,\eta) \in B_L(0,1) \times \mathbb{R}^{\mathcal{J}} \times B_J(0,1)$ such that ${}^tW(p,q)\lambda = \eta$. Note that $B^1(\lambda)$ may be not bounded under the assumptions of Proposition 4.3 but Assumption R.

Let a financial structure with the same tree $\mathbb{D}$ as in Remark 3.1 above. At each non-terminal node, two assets are issued, hence $J = 6$. The return matrix V is constant and equal to

$$\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{12} \\ \xi_{21} \\ \xi_{22} \end{matrix}$$

One remarks that the rank of the matrix V is 6. So the kernel of V is reduced to $\{0\}$ hence Ker$V \cap \mathcal{Z}_{\mathcal{F}} = \{0\}$ whatever is the linear space $\mathcal{Z}_{\mathcal{F}}$. We now consider the asset price $q = (7,7,2,1,1,1)$. q is an arbitrage free price since ${}^tW(q)\lambda = 0$ with $\lambda = (1,1,1,1,1,1,1) \in \mathbb{R}^7_{++}$. Hence the full-return matrix is

$$\mathbf{W}(\mathbf{q}) = \begin{bmatrix} -7 & -7 & 0 & 0 & 0 & 0 \\ 1 & 2 & -2 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_{11} \\ \xi_{12} \\ \xi_{21} \\ \xi_{22} \end{matrix}$$

The rank of $W(q)$ is 5 since the dimension of the kernel of $W(q)$ is 1.

Let us consider

$$(\mathcal{E},\mathcal{F}) := \left[\mathbb{D}, \mathbb{H}, \mathcal{I}, (X_i, P_i, e_i)_{i\in\mathcal{I}}, \mathcal{J}, (Z_i)_{i\in\mathcal{I}}, (\xi(j))_{j\in\mathcal{J}}, V\right]$$

a financial exchange economy satisfying Assumption C with $X_i = \mathbb{R}^7_+$, $\mathcal{I} = 2$, $\mathbb{H}$ is a singleton and $Z_i = \mathbb{R}^6$.

Let (z_i^ν) be a sequence of elements of $\mathbb{R}^6$ such that

$$z_1^\nu = -z_2^\nu = \nu(1,-1,-1,1,0,0)$$

The spot price is $p = (1,1,1,1,1,1,1)$. Let $\hat{x}_1 = e_1 = (3,3,3,3,3,3,3) = \hat{x}_2 = e_2$. Clearly, for all ν, $(\hat{x}, z^\nu) \in B^1(\lambda)$ since for all $\nu \in \mathbb{N}$, ${}^t[W(q)z_1^\nu] = 0 = {}^t[W(q)z_2^\nu]$. Hence $B^1(\lambda)$ is not bounded since (z^ν) is not bounded.

The proof of Proposition 4.3 is divided into two steps. We first prove that the set $B^\delta(\lambda)$ is bounded for $\delta > 0$ small enough (Proposition 4.4). Then, we deduce the existence of an equilibrium under this additional assumption from Theorem 3.1 of Angeloni-Cornet [1] with a slight adaptation of the proof to deal with the space $\mathcal{Z}_{\mathcal{F}}$ instead of $\mathbb{R}^{\mathcal{J}}$ and $B^\delta(\lambda)$ instead of $B^1(\lambda)$.

Proposition 4.4. *Let*

$$(\mathcal{E},\mathcal{F}) := \left[\mathbb{D},\mathbb{H},\mathcal{I},(X_i,P_i,e_i)_{i\in\mathcal{I}},\mathcal{J},(Z_i)_{i\in\mathcal{I}},(\xi(j))_{j\in\mathcal{J}},V\right]$$

be a financial economy satisfying for all $i \in \mathcal{I}$, X_i is bounded below, $\mathcal{F}$ consists of nominal assets and satisfies Assumption **(F)**. *Let $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$ and q be the unique asset price such that ${}^tW(q)\lambda = 0$. If $\mathcal{Z}_{\mathcal{F}} \cap \mathrm{Ker}W(q) = \{0\}$, there exists $\delta > 0$ such that $B^{\delta}(\lambda)$ is bounded.*

The proof of Proposition 4.4 is in Appendix.

Proof of Proposition 4.3 Let $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$. Thanks to Assumption **R**, Proposition 3.4 and Proposition 4.4, all assumptions of Theorem 3.1 of Angeloni-Cornet [1] are satisfied but the fact that $0 \in \mathrm{ri}_{\mathcal{Z}_{\mathcal{F}}}(Z_{i_0})$ instead of $0 \in \mathrm{int}Z_{i_0}$ and $B^{\delta}(\lambda)$ is bounded instead of $B^1(\lambda)$. To complete the proof, we now show how to adapt the proof of Angeloni-Cornet to these slightly more general conditions.

In the preliminary definitions, η is chosen in $\mathcal{Z}_{\mathcal{F}}$ instead of $\mathbb{R}^{\mathcal{J}}$. Then the set B is replaced by

$$B^{\delta} = \{(p,\eta) \in \mathbb{R}^{\mathbb{L}} \times \mathcal{Z}_{\mathcal{F}} \mid \|p\| \le 1, \|\eta\| \le \delta\}$$

and the function ρ is defined by $\rho(p,\eta) = \max\{0, 1-\|p\|-(1/\delta)\|\eta\|\}$. This choice of the set B^{δ} allows us to conclude in Sub-sub-section 4.1.3 that $(\bar{x},\bar{z},\bar{p},\bar{q})$ is an equilibrium and furthermore $(\bar{x},\bar{z})$ belongs to $B^{\delta}(\lambda)$, which is used in the proof of Proposition 4.2 on page 22. In Step 2 of the proof of Claim 4.1, if $\eta \neq 0$, we obtain $0 < \max\{\eta \bullet_{\mathcal{J}} z_{i_0} \mid z_{i_0} \in Z_{i_0}\}$ since Z_{i_0} is included in $\mathcal{Z}_{\mathcal{F}}$, $\eta \in \mathcal{Z}_{\mathcal{F}}$ and $0 \in \mathrm{ri}_{\mathcal{Z}_{\mathcal{F}}}(Z_{i_0})$, so $r\eta \in Z_{i_0}$ for $r > 0$ small enough. In Claim 4.3 of Sub-sub-section 4.1.3, the argument holds true since $\bar{z}_i \in \mathcal{Z}_{\mathcal{F}}$ for all i and so, $(\delta/\|\sum_{i\in\mathcal{I}}\bar{z}_i\|)\sum_{i\in\mathcal{I}}\bar{z}_i$ belongs to $\mathcal{Z}_{\mathcal{F}}$. The equality $\sum_{i\in I}(\bar{x}_i - e_i) = 0$ is obtained by the same argument. Indeed, since $\lambda_{\xi} > 0$ for all ξ, $p \to (\lambda \Box p) \bullet_{\mathbb{L}} \sum_{i\in I}(\bar{x}_i - e_i)$ is a non zero linear mapping if $\sum_{i\in I}(\bar{x}_i - e_i) \neq 0$ so its maximum on the ball is positive and reached on the boundary of $\bar{B}_{\mathbb{L}}(0,1)$, which implies that $\|\bar{p}\| = 1$ and $\rho(\bar{p},\bar{\eta}) = 0$.

In Sub-sub-section 4.2.2, to show that $0 \in \mathrm{ri}_{\mathcal{Z}_{\mathcal{F}}}(Z_{i_0r})$ in the truncated economy, it suffices to remark that there exists $r' > 0$ such that $B_{\mathcal{J}}(0,r') \cap \mathcal{Z}_{\mathcal{F}} \subset Z_{i_0}$, hence, $B_{\mathcal{J}}(0,\min\{r,r'\}) \cap \mathcal{Z}_{\mathcal{F}} \subset Z_{i_0r}$, which means that 0 belongs to the relative interior of Z_{i_0r} with respect to $\mathcal{Z}_{\mathcal{F}}$. $\Box$

5 Appendix

Proof of Corollary 3.1 First, we remark that $V^{\mathcal{J}(\xi)}_{\xi^+}(p)$ is a sub-matrix of $V^{\mathcal{J}(\xi)}(p)$, so $n(\xi) = \mathrm{rank}V^{\mathcal{J}(\xi)}_{\xi^+}(p) \le \mathrm{rank}V^{\mathcal{J}(\xi)}(p)$. On the other hand, $\mathrm{rank}V^{\mathcal{J}(\xi)}(p) \le n(\xi)$ since the number of column of $V^{\mathcal{J}(\xi)}(p)$ is $n(\xi)$. Hence $n(\xi) = \mathrm{rank}V^{\mathcal{J}(\xi)}(p)$.

We now prove that Assumption **R** is satisfied. Let $\kappa \in \{2,\dots,k\}$, $\xi \in \mathbb{D}^e_{\tau_\kappa}$ and $y \in \mathbb{R}^{\mathbb{D}^+(\xi)} \setminus \{0\}$ such that

$$y \in \mathrm{Vect}\left(V^{\mathcal{J}(\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}(p)\right) \cap \mathrm{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}(p)\right)$$

Then, there exists $(a_j) \in \mathbb{R}^{\mathcal{J}(\xi)}$ such that $y = \sum_{j\in\mathcal{J}(\xi)} a_j V^j_{\mathbb{D}^+(\xi)}(p)$ and there exists $(b_j) \in \mathbb{R}^{\mathcal{J}(\mathbb{D}^-(\xi))}$ such that $y = \sum_{j\in\mathcal{J}(\mathbb{D}^-(\xi))} b_j V^j_{\mathbb{D}^+(\xi)}(p)$. Restricting the above equality to the coordinates in ξ^+, one gets $y_{\xi^+} = \sum_{j\in\mathcal{J}(\xi)} a_j V^j_{\xi^+}(p) = \sum_{j\in\mathcal{J}(\mathbb{D}^-(\xi))} b_j V^j_{\xi^+}(p)$. From our second assumption, this implies that $y_{\xi^+} = 0$. From the first assumption, since the vectors $(V^j_{\xi^+})_{j\in\mathcal{J}(\xi)}(p)$ are of maximal rank hence linearly independent, this implies that $a_j = 0$ for all $j \in \mathcal{J}(\xi)$. Hence, $y = 0$, which proves that $\text{Vect}\left(V^{\mathcal{J}(\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}(p)\right) \bigcap \text{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}(p)\right) = \{0\}$. Consequently Assumption **R** is satisfied. □

Proof of lemma 3.1. Let us denote by k [resp. k'] the number of dates where there are issuance of at least one asset for the financial structure $\mathcal{F}$ [resp. $\mathcal{F}'$]. It is clear that $k' \leq k$.

By Assumption **R**, we have: for all $\kappa \in \{2, \ldots, k\}$ and for all $\xi \in \mathbb{D}^e_{\tau_\kappa}$,

$$\text{Vect}\left(V^{\mathcal{J}(\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}\right) \bigcap \text{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}\right) = \{0\}.$$

Since $\mathcal{J}' \subset \mathcal{J}$, $\text{Vect}\left(V^{\mathcal{J}'((\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}\right) \subset \text{Vect}\left(V^{\mathcal{J}((\mathbb{D}^-(\xi))}_{\mathbb{D}^+(\xi)}\right)$ and $\text{Vect}\left(V^{\mathcal{J}'(\xi)}_{\mathbb{D}^+(\xi)}\right) \subset \text{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}\right)$. So,

$$\text{Vect}\left(V^{\mathcal{J}'(\xi^-)}_{\mathbb{D}^+(\xi)}\right) \cap \text{Vect}\left(V^{\mathcal{J}'(\xi)}_{\mathbb{D}^+(\xi)}\right) \subset \text{Vect}\left(V^{\mathcal{J}(\mathbb{D}^-(\xi)}_{\mathbb{D}^+(\xi)}\right) \bigcap \text{Vect}\left(V^{\mathcal{J}(\xi)}_{\mathbb{D}^+(\xi)}\right) = \{0\}.$$

hence the financial structure $\mathcal{F}'$ satisfies Assumption R. □

Proof of lemma 3.2. We first show that the equality of the kernels implies the equality of dimensions of the images. Let G be a linear subspace of E and let φ_G (resp. ψ_G) be the restriction of φ (resp. ψ) at G. We have $\varphi(G) = \text{Im}\varphi_G$[9] and $\dim \text{Im}\varphi_G = \dim G - \dim(\text{Ker}\varphi_G)$. As $\text{Ker}\varphi = \text{Ker}\psi$ we have $\text{Ker}\varphi_G = (\text{Ker}\varphi) \cap G = (\text{Ker}\psi) \cap G = \text{Ker}\psi_G$ hence $\dim \varphi(G) = \dim \psi(G)$.

Let us show the converse implication. If $\text{Ker}\varphi \neq \text{Ker}\psi$, then there exists $u \in \text{Ker}\varphi$ such that $u \notin \text{Ker}\psi$ or there exists $u \in \text{Ker}\psi$ such that $u \notin \text{Ker}\varphi$. In the first case, with $G = \text{Ker}\varphi$, we have $\varphi(G) = \{0\} \neq \psi(G)$, hence $\dim \varphi(G) = 0 < \dim \psi(G)$. In the second case, we obtain the same inequality with $G = \text{Ker}\psi$. So the equality of the dimension of $\varphi(G)$ and $\psi(G)$ for all linear subspace G implies the equality of kernels. □

Proof of Proposition 4.4 For every $\delta > 0$, for every $i \in \mathcal{I}, \lambda \in \mathbb{R}^{\mathbb{D}}_{++}$, we let $\hat{X}^\delta_i(\lambda)$ and $\hat{Z}^\delta_i(\lambda)$ be the projections of $B^\delta(\lambda)$ on X_i and Z_i, that is respectively:

$$\hat{X}^\delta_i(\lambda) := \left\{ x_i \in X_i \mid \exists\, (x_k)_{k\neq i} \in \prod_{k\neq i} X_k, \exists z \in \prod_{k\in\mathcal{I}} Z_k, (x,z) \in B^\delta(\lambda) \right\}$$

$$\hat{Z}^\delta_i(\lambda) := \left\{ z_i \in Z_i \mid \exists\, (z_k)_{k\neq i} \in \prod_{k\neq i} Z_k,\ \exists x \in \prod_{k\in\mathcal{I}} X_k, (x,z) \in B^\delta(\lambda) \right\}.$$

[9] Let γ be a linear map from E to F. We denote its image by $\text{Im}\gamma := \{y \in F \mid \exists z \in E; y = \gamma(z)\}$.

It suffices to prove that $\hat{X}_i^\delta(\lambda)$ and $\hat{Z}_i^\delta(\lambda)$ are bounded sets for every i to show that $B^\delta(\lambda)$ is bounded

Step 1. Let us show that: for all $\delta \geq 0$, and for all $i \in \mathcal{I}$, $\hat{X}_i^\delta(\lambda)$ is bounded.

Indeed, let $\delta \geq 0$, and $i \in \mathcal{I}$, since the sets X_i are bounded below, there exists $\underline{x}_i \in \mathbb{R}^{\mathbb{L}}$ such that $X_i \subset \{\underline{x}_i\} + \mathbb{R}_+^{\mathbb{L}}$. If $x_i \in \hat{X}_i(\lambda)$, there exists $x_k \in X_k$, for every $k \neq i$, such that $x_i + \sum_{k \neq i} x_k = \sum_{\kappa \in \mathcal{I}} e_\kappa$. Consequently,

$$\underline{x}_i \leq x_i = -\sum_{k \neq i} x_k + \sum_{\kappa \in \mathcal{I}} e_\kappa \leq -\sum_{k \neq i} \underline{x}_k + \sum_{\kappa \in \mathcal{I}} e_\kappa$$

and so $\hat{X}_i^\delta(\lambda)$ is bounded. So for all $\delta \geq 0$ and $i \in \mathcal{I}$, $\hat{X}_i^\delta(\lambda)$ is bounded.

Step 2. Let us show that for all $i \in \mathcal{I}$, $\hat{Z}_i^0(\lambda)$ is bounded.

For all $i \in \mathcal{I}$ and for every $z_i \in \hat{Z}_i^0(\lambda)$, there exists $(z_k)_{k \neq i} \in \prod_{k \neq i} Z_k$, $x \in \prod_{\kappa \in \mathcal{I}} X_\kappa$ and $p \in \bar{B}_{\mathbb{L}}(0,1)$, such that $z_i + \sum_{k \neq i} z_k = 0$ and $(x_\kappa, z_\kappa) \in B_{\mathcal{F}}^\kappa(p,q)$ for every $\kappa \in \mathcal{I}$. As $(x_\kappa, z_\kappa) \in B_{\mathcal{F}}^\kappa(p,q)$ and $(x_\kappa, p) \in \hat{X}_j^0(\lambda) \times \bar{B}_{\mathbb{L}}(0,1)$, a compact set, there exists $\alpha_j \in \mathbb{R}^{\mathbb{D}}$ such that

$$\alpha_j \leq p_\square(x_\kappa - e_\kappa) \leq W(q) z_\kappa.$$

Using the fact that $\sum_{\kappa \in \mathcal{I}} z_\kappa = 0$, we have

$$\alpha_i \leq W(q) z_i = W(q)\left(-\sum_{k \neq i} z_k\right) \leq -\sum_{k \neq i} \alpha_k,$$

hence there exists $r > 0$ such that $W(q) z_i \in \bar{B}_{\mathbb{D}}(0,r)$.

By assumption, $\mathcal{Z}_{\mathcal{F}} \cap \mathrm{Ker} W(q) = \{0\}$. So, the linear mapping $W(q)_{|\mathcal{Z}_{\mathcal{F}}}$ from $\mathcal{Z}_{\mathcal{F}}$ to $W(q)\mathcal{Z}_{\mathcal{F}}$ is an isomorphism. Since we have proved that for every $z_i \in \hat{Z}_i^0(\lambda)$, $W(q) z_i \in \bar{B}_{\mathbb{D}}(0,r)$ and $W(q) z_i$ obviously belongs to $W(q)\mathcal{Z}_{\mathcal{F}}$, we can conclude that $\hat{Z}_i^0(\lambda) \subset [W(q)_{|\mathcal{Z}_{\mathcal{F}}}]^{-1}(\bar{B}_{\mathbb{D}}(0,r) \cap W(q)\mathcal{Z}_{\mathcal{F}})$, a bounded subset, so $\hat{Z}_i^0(\lambda)$ is a bounded subset of $\mathcal{Z}_{\mathcal{F}}$.

Let $M \in \mathbb{R}_+^*$ such that for all $(x,z) \in B^0(\lambda)$, $\|z\| < M$.

Step 3. There exists $\delta > 0$ such that $B^\delta(\lambda)$ is bounded.

By contradiction. Suppose that for all $\delta > 0$, $B^\delta(\lambda)$ is not bounded. This implies that for all $\nu \in \mathbb{N}^*$, $B^{1/\nu}(\lambda)$ is not bounded. We build a sequence $(x^\nu, z^\nu)_{\nu \in \mathbb{N}^*}$ in $\prod_{i \in \mathcal{I}} X_i \times \prod_{i \in \mathcal{I}} Z_i$ by induction in the following way: $(x^1, z^1) \in B^1(\lambda)$ such that $\left\|z^1\right\| > M + 1$ and for all $\nu \in \mathbb{N}^*$, $(x^{\nu+1}, z^{\nu+1}) \in B^{\frac{1}{\nu+1}}(\lambda)$ and $\left\|z^{\nu+1}\right\| > \|z^\nu\| + 1$. So $\|z^\nu\|$ converges to $+\infty$.

Since for all $\nu \in \mathbb{N}^*$, $(x^\nu, z^\nu) \in B^{1/\nu}(\lambda)$, there exists a sequence $(p^\nu, q^\nu)_{\nu \in \mathbb{N}^*}$ such that for all $\nu \in \mathbb{N}^*$, $\|p^\nu\| \leq 1$, $p^\nu{}_\square(x_i^\nu - e_i) \leq W(q^\nu) z_i^\nu$ and $0 \leq \|{}^t W(q^\nu)\lambda\| \leq \frac{1}{\nu}$. We remark that for all $\nu \in \mathbb{N}^*$, $B^{\frac{1}{\nu+1}}(\lambda) \subset B^{\frac{1}{\nu}}(\lambda)$ so the sequence $(x^\nu, z^\nu) \subset B^1(\lambda)$. By Step 1, the sequence (x^ν) is bounded. For each $\nu \in \mathbb{N}^*$, let $\zeta^\nu = M \frac{z^\nu}{\|z^\nu\|}$. ${}^t W(q^\nu)\lambda \in \bar{B}_{\mathcal{J}}(0, \frac{1}{\nu})$ implies that for all $\nu \in \mathbb{N}^*$ and for all $j \in \mathcal{J}$ there exists $\eta^\nu \in \bar{B}_{\mathcal{J}}(0, \frac{1}{\nu})$ such that $\lambda_{\xi(j)} q^{\nu j} = \sum_{\xi \in \mathbb{D}^+(\xi(j))} \lambda_\xi V_\xi^j + \eta^{\nu j}$. Hence the sequence $(q^{\nu j})$ is bounded for all j. Consequently the sequence $(x^\nu, \zeta^\nu, p^\nu, q^\nu)$

is bounded so it has a subsequence $(x^{\phi(\nu)}, \zeta^{\phi(\nu)}, p^{\phi(\nu)}, q^{\phi(\nu)})$, which converges to $(\bar{x}, \bar{\zeta}, \bar{p}, \bar{q})$.

Let us now show that $(\bar{x}, \bar{\zeta}) \in B^0(\lambda)$.

• ${}^tW(\bar{q})\lambda = 0$ since $\|{}^tW(q^{\phi(\nu)})\lambda\| \leq \frac{1}{\phi(\nu)}$ for all ν. For all i, $\bar{x}_i \in X_i$ because X_i is a closed. For all i, $\bar{\zeta}_i \in Z_i$. Indeed, Z_i is closed and $\zeta_i^{\phi(\nu)} = M \frac{z_i^{\phi(\nu)}}{\|z^{\phi(\nu)}\|} \in Z_i$ since $z_i^{\phi(\nu)} \in Z_i$, $0 \in Z_i$, $0 < \frac{M}{\|z^{\phi(\nu)}\|} < 1$ and Z_i is convex.

• For all i, $(\bar{x}_i, \bar{\zeta}_i) \in B^i_{\mathcal{F}}(0, \bar{q})$. Indeed, for all $\nu \in \mathbb{N}^*$, $(x^\nu, z^\nu) \in B^1(\lambda)$, hence $p^{\phi(\nu)} \square (x_i^{\phi(\nu)} - e_i) \leq W(q^{\phi(\nu)}) z_i^{\phi(\nu)}$. So $\left(\frac{M}{\|z^{\phi(\nu)}\|} p^{\phi(\nu)}\right) \square (x_i^{\phi(\nu)} - e_i) \leq W(q^{\phi(\nu)}) \left(\frac{M}{\|z^{\phi(\nu)}\|} z_i^{\phi(\nu)}\right)$. At the limit, since $\zeta_i^{\phi(\nu)} = \frac{M z^{\phi(\nu)}}{\|z^{\phi(\nu)}\|}$ and $\frac{M}{\|z^{\phi(\nu)}\|}$ converges to 0, one gets, $0 \leq W(\bar{q})\bar{\zeta}_i$, which means that $(\bar{x}_i, \bar{\zeta}_i) \in B^i_{\mathcal{F}}(0, \bar{q})$.

• $\sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} e_i$ and $\sum_{i \in \mathcal{I}} \bar{\zeta}_i = 0$ since for all ν, $\sum_{i \in \mathcal{I}} x_i^{\phi(\nu)} = \sum_{i \in \mathcal{I}} e_i$ and $\sum_{i \in \mathcal{I}} \zeta_i^{\phi(\nu)} = \sum_{i \in \mathcal{I}} M \frac{z_i^{\phi(\nu)}}{\|z^{\phi(\nu)}\|} = 0$.

Hence, one gets a contradiction since $(\bar{x}, \bar{\zeta}) \in B^0(\lambda)$ and $\|\bar{\zeta}\| = M$ whereas we have chosen M large enough so that for all $(x, z) \in B^0(\lambda)$, $\|z\| < M$. □

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