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Abstract: *In the present paper, we consider a two-country, two-good, two-factor general equilibrium model with CIES non-linear preferences, asymmetric technologies across countries and decreasing returns to scale. It is shown that aggregate instability and endogenous fluctuations may occur due to international trade. In particular, we prove that the integration into a common market on which countries trade the produced good and the capital input may lead to period-two cycles even when the closed-economy equilibrium is saddle-point stable in both countries.*

Keywords: *Perfect foresight dynamic general equilibrium model, international trade, aggregate instability, endogenous fluctuations, non-linear preferences*

Journal of Economic Literature Classification Numbers: C62, E32, F11, F43, O41.

1 Introduction

As is evidenced by the world financial crisis since 2008, globalization has closely interlinked business cycles in different countries with one another. While this interlinkage has extensively been studied in the literature on economics under uncertainty,[1] it has not been given much attention to in the literature on deterministic dynamic economics. In closed-economy models, in contrast, it has been established that endogenous fluctuations can be explained by the fundamental structure of production and consumption without uncertainty.[2] Despite this, little has been known on how trade affects deterministic-factor-driven periodic cycles in different countries. In this paper we provide such an analysis considering a two-country, two-good, two-factor general equilibrium model with CIES non-linear preferences, asymmetric technologies across countries and decreasing returns to scale in the production of all goods. We exhibit a global destabilization effect of international trade by proving that the openning of trade can create persistent endogenous fluctuations at the world level while the closed-economy equilibrium in each country is saddle-point stable.

Up to now, the lack of results in the literature is due to the difficulty that exists, in general, in making a comparative dynamic analysis for a dynamic general equilibrium model with heterogenous consumers. In order to compare pre- and post-trade endogenous cycles, it is necessary to make such a comparison in a dynamic general equilibrium model with heterogenous consumers (representing home and foreign consumers). Although, in the representative consumer case, country-wise endogenous cycles in a pre-trade equilibrium is determined by the representative agent's utility function, world periodic cycles in a post-trade equilibrium can be characterized only by a world social utility function.[3] The difficulty in characterizing this world social utility function has hampered a comparison between pre- and post-trade equilibrium paths in terms of the fundamental parameters of a model.

In the present study, we demonstrate that this difficulty can be overcome

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[1]See for instance Cole and Obstfeld [11], and Obstfeld [29].

[2]See Benhabib and Nishimura [5, 6], Boldrin and Montrucchio [8], Mitra and Sorger [22], Nishimura and Yano [28].

[3]See Negishi [23], Bewley [7].

by assuming non-linear CIES utility functions. This assumption enables us to characterize completely the local behavior around the steady state of pre- and post-trade equilibrium paths by means of the determinant and the trace of the matrix determining the linear approximation of the transition dynamics. Moreover, we apply for the first time in the international trade literature the geometrical methodology developed by Grandmont, Pintus and de Vilder [18] to analyse the local stability properties of a two-country, two-good, two-factor model. Such a methodology appears to be a crucial tool when the utility function is non-linear.

Our characterization allows to reveal what we call the macroeconomic destabilization effect of international trade. That is to say, the openning of trade can create persisting endogenous cycles in both countries, even if in each country's pre-trade equilibrium, endogenous fluctuations are eventually to disappear over time or even do not exist.[4] Although the possibility of the presence of this destabilization effect of international trade has been hinted in the work of Nishimura and Yano [26],[5] no proof has yet been known in the existing literature. Contrary to all the previous papers on trade in dynamic general equilibrium models in which the focus was on turnpike results,[6] we concentrate on the instability of the steady state and the occurrence of periodic cycles. Up to our knowledge, beside the paper by Bajona and Kehoe [2] in which it is shown that positive trade with endogenous cycles may occur within an Heckscher-Ohlin-Samuelson (HOS) infinite horizon model with two consumption goods, our methodology allows to provide the first proof of existence of endogenous fluctuations in a non-HOS optimal growth model of international trade with non-linear preferences.

We consider a two-country, two-sector model in which producers of one country differ from those of the other country in respect to the share of capital and labor in each sector. Contrary to Bajona and Kehoe [2], we then depart from the standard HOS formulation in which the technologies

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[4]In the recent literature studying the economic growth in the very long run, international trade is also shown to have asymmetrical effects on the evolution of industrial and non-industrial countries. As a result it may explain the "Great Divergence" in income per capita across countries in the last two centuries. See Galor and Mountford [15, 16].

[5]The existence of endogenous fluctuations is also studied but under the restrictive assumption of a linear utility function in both countries (see also Nishimura, Venditti and Yano [24, 25]).

[6]See for instance Yano [32, 33, 34, 35, 36, 37].

are symmetric across countries. In order to characterize the stability properties of free-trade equilibrium paths, we assume that the technologies are strictly concave Cobb-Douglas functions. Decreasing returns indeed allow to guarantee the diversification of production in each country and to get a non-degenerate social production function at the world level. Based on that, it is also a crucial assumption to generate the destabilization effect of international trade. We finally assume that labor is inelastic, and that the citizens of both countries have the same time discount factor. Although strong simplifying assumptions are considered, this model is sufficient to analyze the impact of a common market on the local stability properties of the steady state.[7]

We consider a trade reform which is based on two levels of integration. In the lowest level, countries are closed economies and the produced goods and inputs are traded only on domestic markets. The highest level is obtained after a trade agreement in which the countries trade the consumption good and capital on a common market with no transaction costs but labor is immobile. Such a sequence, built upon two polar cases, can be seen as extreme on many grounds, but we think that the message would not be significantly affected by generalizations leading to more complex configurations.

The geometrical analysis allows us to obtain several conclusions. First, we show that in a closed economy, the occurrence of endogenous fluctuations requires a sufficiently capital intensive consumption good sector to compensate for the degree of decreasing returns, and a large enough elasticity of intertemporal substitution in consumption. Second, and this is our main result, building on the same kind of restrictions for technologies and preferences, we prove the existence of configurations in which market integration leads to aggregate instability and endogenous fluctuations. Our main focus is a situation in which the closed-economy equilibrium of both countries is saddle-point stable and the reform, which consists of joining a common market for the consumption good and capital, leads to persistent endogenous cycles at the world level. It follows that international trade can promote aggregate instability. To our knowledge, this paper is the first to show that, in a non-HOS international trade model, opening to free-trade can have a global destabilizing effect on all trading partners.

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[7]General production functions could be considered without altering significantly our results.

Although our paper is mainly theoretical, it is worthwhile looking at empirical evidence. Building on some earlier studies by Easterly *et al.* [14] and Kose *et al.* [21], di Giovanni and Levchenko [12] have recently investigated in detail the relationship between trade openness and macroeconomic volatility using industry-level data. They clearly show that countries that trade more tend to be more volatile. This positive relationship indeed appears to be economically significant, even after controlling for the size of both countries and sectors using fixed effects. Building on the extreme comparison of the closed-economy and free-trade equilibria of two countries, we provide here a theoretical explanation of this empirical fact based on asymmetric technologies across countries and decreasing returns at the industry level.[8] This last non-standard assumption also appears to be empirically relevant. Indeed, using disaggregated US data, a number of empirical studies have shown that, while returns to scale appear to be roughly constant at the aggregate level, significantly decreasing returns cannot be rejected at the industry level.[9]

The present paper is related to Nishimura, Venditti and Yano [24, 25] and Ghiglino and Venditti [17]. In the first two papers, endogenous fluctuations are shown to occur in a similar model after a trade-agreement provided one of the two countries is characterized by persistent fluctuations in the closed-economy case. However, the authors do not address the issue of the global destabilizing effect of a common market. Another difference is that preferences are bound to be linear. Ghiglino and Venditti [17] consider a model like ours but with heterogeneous consumers instead of heterogeneous countries. Altough the framework is similar, they focus on a different question. The point is indeed to show that the distribution of capital shares matters in the stability properties of the steady state and that an increase of inequalities across agents may generate endogenous fluctuations. Such a correlation requires non standard utility functions with a strictly convex absolute risk tolerance and does not occur when the preferences are CIES. In the current paper, we focus instead on the consequences of international trade on the existence of endogenous fluctuations without considering any wealth inequalities across countries.

The paper is organized as follows: in Section 2 the model is introduced.

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[8]See also di Giovanni and Levchenko [13] for another theoretical explanation based on country size and firms heterogeneity.

[9]See Basu and Fernald [3], Burnside [9] and Burnside *et al.* [10].

Section 3 discusses the stability properties of the steady-state in a closed economy while Section 4 analyzes the effects of market integration on the occurrence of endogenous fluctuations. Section 5 concludes and the appendix contains all of the proofs.

2 The model

We consider an infinite horizon perfect foresight dynamic general equilibrium model with two countries, A and B, two factors, capital and labor, and two goods, consumption and investment. For the description of the model in the current Section and the analysis of the closed-economy equilibrium in the following Section we do not consider explicitly any particular notation for each country A and B. However, each time the two countries will be simultaneously considered within the free-trade equilibrium, all the symbols will be affected by a superscript A or B.

2.1 The production side

The pure consumption good, c, and the pure capital good, k are produced from capital and labor with a Cobb-Douglas technology. We denote by x and y the output of sectors c and k:

$$x = \mathcal{E}_c K_c^{\alpha_1} L_c^{\alpha_2}, \ y = \mathcal{E}_y K_y^{\beta_1} L_y^{\beta_2}$$

with $\mathcal{E}_c, \mathcal{E}_y > 0$ some normalization constants which will be used to modify the comparative advantages of the countries. We assume decreasing returns to scale in both sectors, i.e. $\beta_1 + \beta_2 \leq 1$ and $\alpha_1 + \alpha_2 \leq 1$.[10] Labor is normalized to one, $L_c + L_y = 1$, and the total stock of capital in country i is given by $K_c + K_y = k$. Moreover in order to simplify the analysis, we also assume that capital fully depreciates at each period.[11] Goods c and k are

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[10] A possible interpretation of decreasing returns is to assume the existence of a factor in fixed supply such as land in the technology, namely

$$x = \mathcal{E}_c K_c^{\alpha_1} L_c^{\alpha_2} \mathcal{L}_c^{1-\alpha_1-\alpha_2}, \ y = \mathcal{E}_y K_y^{\beta_1} L_y^{\beta_2} \mathcal{L}_y^{1-\beta_1-\beta_2}$$

Returns to scale are therefore constant when considering this factor but decreasing with respect to capital and labor. In such a case, the income of the representative consumer is increased by the rental rate of land. Our formulation implicitly assumes a normalization $\mathcal{L}_c = \mathcal{L}_y = 1$.

[11] See Baierl, Nishimura and Yano [1] for the dynamic analysis of a two-sector closed economy with partial depreciation.

assumed to be freely mobile between countries once trade opens, whereas labor is internationally immobile both before and after the openning of trade.

In each country, the optimal allocation of factors across sectors is obtained by solving the following program:

$$\max_{K_c, L_c, K_y, L_y} \quad \mathcal{E}_c K_c^{\alpha_1} L_c^{\alpha_2}$$
$$s.t. \quad y = \mathcal{E}_y K_y^{\beta_1} L_y^{\beta_2}, \ 1 = L_c + L_y \text{ and } k = K_c + K_y \tag{1}$$

Denote by q_t, p_t, ω_t and r_t respectively the prices of the consumption good and the capital good, the wage rate of labor and the rental rate of the capital good at time t. In free-trade equilibrium, $q_t^A = q_t^B$, $p_t^A = p_t^B$ and $r_t^A = r_t^B$ must hold. On the contrary, because labor is immobile across countries, ω_t may differ between countries even in the free-trade case. In the following we will choose the consumption good as numeraire and thus adopt the normalization $q_t^A = q_t^B = 1$.

For any (k_t, y_t), solving the first order conditions derived from program (1) gives inputs K_c, L_c, K_y and L_y as C^2 functions of (k_t, y_t), i.e. $\hat{K}_c(k_t, y_t)$, $\hat{L}_c(k_t, y_t)$, $\hat{K}_y(k_t, y_t)$ and $\hat{L}_y(k_t, y_t)$. We thus define the social production function as:

$$T(k_t, y_t) = \mathcal{E}_c \hat{K}_c(k_t, y_t)^{\alpha_1} \hat{L}_c(k_t, y_t)^{\alpha_2} \tag{2}$$

Using the envelope theorem we derive the equilibrium prices:

$$r_t = T_1(k_t, y_t), \quad p_t = -T_2(k_t, y_t) \tag{3}$$

where $T_1 = \partial T / \partial k$ and $T_2 = \partial T / \partial y$.

2.2 The consumption side

Each country is characterized by an infinitely-lived representative agent with single period CIES utility function given by

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

with c the consumption level and $\sigma \geq 0$ is the inverse of the elasticity of intertemporal substitution in consumption. We assume that the labor supply is inelastic.

Along a closed-economy equilibrium, the representative agent maximizes an infinite stream of discounted utilities subject to the market clearing conditions:

$$c_t = x_t, \quad k_{t+1} = y_t \tag{4}$$

Along a free-trade equilibrium, i.e. in an open economy, as the consumption and capital goods are assumed to be freely mobile between countries once

trade opens, the central planner maximizes a weighted sum of each country's discounted utilities subject to the following market clearing conditions:

$$c_t^A + c_t^B = x_t^A + x_t^B, \quad k_{t+1}^A + k_{t+1}^B = y_t^A + y_t^B \tag{5}$$

Since the technologies exhibit decreasing returns to scale, the competitive firms earn positive profits that have to be distributed back to the households who own physical capital. It can be shown that, with an identical CIES utility function in both countries, solving a planning problem in which the planner maximizes the discounted sum of utilities, under free-trade (as shown by Proposition 3)[12] or in the closed-economy case, subject to the social production function (2) for each country and the market clearing conditions (4) or (5), is equivalent to solving a decentralized problem in which the households maximize a discounted sum of utilitites subject to some budget constraint based on given sequences of prices and the distributed profits.

3 Closed-economy equilibrium

In a closed economy the equilibrium is derived from the following optimization program:

$$\max_{y_t} \quad \sum_{t=0}^{+\infty} \rho^t \frac{T(k_t, y_t)^{1-\sigma}}{1-\sigma}$$

$$s.t. \quad k_{t+1} = y_t, \ k_0 \ given$$

with $\rho \in (0, 1)$ the discount factor. The corresponding Euler equation is

$$\left(\frac{c_{t+1}}{c_t}\right)^\sigma = -\rho \frac{T_1(k_{t+1}, k_{t+2})}{T_2(k_t, k_{t+1})} \tag{6}$$

A closed-economy steady state is defined by $c_t = c_{t+1}$, $k_t = k_{t+1} = y_t = \bar{k}$ and is obtained by solving $-T_1(k, k)/T_2(k, k) = 1/\rho$.

Proposition 1. *There exists a unique closed-economy steady state $\bar{k} > 0$.*

Note that the stationary closed-economy consumption level is given by

$$\bar{c} = c(\bar{k}, \bar{k}) = T(\bar{k}, \bar{k}) \tag{7}$$

As usual with Cobb-Douglas technologies, factor intensities are determined by the exponents of the functions: The investment (consumption)

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[12]See also Appendix 6.5 for additional details.

good sector of country i is capital intensive if and only if $\beta_1/\beta_2 > (<)\alpha_1/\alpha_2$. Building on the contribution of Benhabib and Nishimura [6], we know that the existence of endogenous fluctuations requires a negative cross derivative $T_{12}(\bar{k}, \bar{k})$. This property can be obtained when the following assumption is satisfied:[13]

Assumption 1. *The consumption good sector of country i is capital intensive with*

$$\alpha_1\beta_2 - \alpha_2\beta_1 > \frac{\alpha_1\beta_2(1-\alpha_1-\alpha_2)}{(1-\alpha_1)(1-\beta_1)} \tag{8}$$

Note that with respect to the case with constant returns to scale studied by Benhabib and Nishimura [6], we need here a sufficiently capital intensive consumption good sector to compensate for the degree of decreasing returns.

Linearizing the Euler equation (6) around $\bar{k}$ gives the characteristic polynomial

$$\mathcal{P}_c(x) = x^2 - \mathcal{T}_c(\sigma)x + \mathcal{D}_c = 0 \tag{9}$$

with $\mathcal{D}_c = \rho^{-1}$.[14] The local stability analysis is performed using a simplified version of the geometrical method provided by Grandmont, Pintus and de Vilder [18]. It is based on a particular property characterizing the product $(\mathcal{D}_c)$ and the sum $(\mathcal{T}_c)$ of characteristic roots. While $\mathcal{T}_c(\sigma)$ is a function of the inverse of the elasticity of intertemporal substitution in consumption σ, $\mathcal{D}_c$ is constant for any $\sigma \geq 0$. Considering the $(\mathcal{T}_c, \mathcal{D}_c)$ plane, it follows that for a given $\rho \in (0,1)$, when σ covers the interval $[0, +\infty)$, $\mathcal{T}_c(\sigma)$ varies along an horizontal line, called in what follows Δ_c. The starting point of Δ_c is obtained when $\sigma = 0$, and under Assumption 1 we get $\mathcal{T}_c(0) < 0$. The end point is obtained when $\sigma = +\infty$ and is characterized by $\mathcal{T}_c(+\infty) = 1+\rho^{-1} \geq 2$. We get in this case

$$1 - \mathcal{T}_c(+\infty) + \rho^{-1} = 0 \tag{10}$$

Based on these results, in order to locate the line Δ_c we finally need to study how $\mathcal{T}_c(\sigma)$ varies with σ. Lemma 6.3 in Appendix 6.3 exhibits a critical value $\sigma_c^* > 0$ such that starting from $(\mathcal{T}_c(0), 1/\rho)$, when σ increases, the point $(\mathcal{T}_c(\sigma), 1/\rho)$ decreases along the Δ_c line as $\sigma \in (0, \sigma_c^*)$, goes through $-\infty$ when $\sigma = \sigma_c^*$ and finally decreases from $+\infty$ as $\sigma > \sigma_c^*$ until it reaches the end point $((1+\rho)/\rho, 1/\rho)$ which is located on the line defined by (10). From all this, we easily conclude that the occurrence of period-two cycles

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[13]See Lemma 6.2 in Appendix 6.2.

[14]Details are given in Appendix 6.2.

through the existence of a flip bifurcation can be obtained if and only if the starting point $(\mathcal{T}_c(0), 1/\rho)$ satisfies

$$1 + \mathcal{T}_c(0) + \rho^{-1} > 0 \tag{11}$$

In such a case indeed, the Δ_c line is located as on the following Figure:
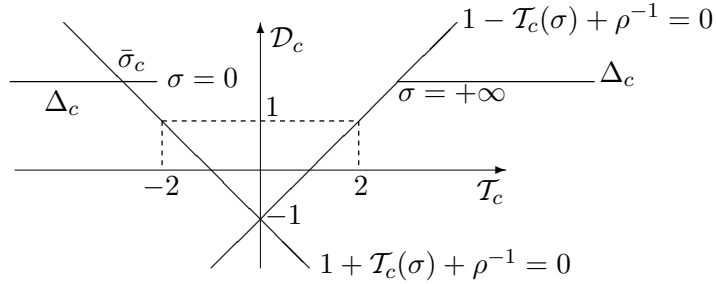

Figure 1: Flip bifurcation of closed-economy equilibrium.

Inequality (11) is satisfied if ρ is sufficiently lower than 1 and if the following assumption holds:

Assumption 2. *The capital share of country i satisfies*

$$\alpha_1 > \tfrac{1+\beta_1}{2(\beta_1+\beta_2)} \tag{12}$$

It is worth noting that Assumptions 1 and 2 are independent as none of them implies or is a consequence of the other. We derive therefore the following proposition:

Proposition 2. *Under Assumptions 1 and 2, there exist $\bar{\rho}_c \in (0,1)$ and $\bar{\sigma}_c \in (0, \sigma_c^*)$ such that for any given $\rho \in (0, \bar{\rho}_c)$, the closed-economy steady-state $\bar{k}$ is saddle-point stable when $\sigma \in (\bar{\sigma}_c, +\infty)$, undergoes a flip bifurcation when σ crosses $\bar{\sigma}_c$ from above and becomes locally unstable when $\sigma \in [0, \bar{\sigma}_c)$. Moreover there generically exist saddle-point stable (locally unstable) period-two cycles in a left (right) neighbourhood of $\bar{\sigma}_c$.*

It is worth noting that if Assumption 2 does not hold or if $\rho \in (\bar{\rho}_c, 1)$, inequality (11) is not satisfied and the closed-economy steady-state is saddle-point stable for all $\sigma \geq 0$.[15] However, damped fluctuations exist when $\sigma \in [0, \sigma_c^*)$ as both characteristic roots are negative while monotone convergence holds when $\sigma > \sigma_c^*$.[16] Graphically we obtain the following Figure:

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[15]We get in this case $\bar{\sigma}_c < 0$.

[16]If Assumption 1 does not hold, the closed-economy steady-state is saddle-point stable with monotone convergence. Indeed, σ_c^* becomes negative and the Δ_c line is only defined
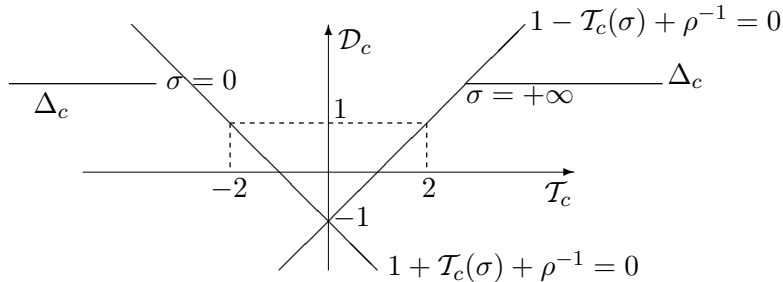
Figure 2: Oscillating/monotone convergence of closed-economy equilibrium.

The intuition of Proposition 2 can be summarized as follows. Consider an instantaneous increase in the capital stock k_t. This results in two opposing forces:

- Since the consumption good is more capital intensive than the investment good, the trade-off in production becomes more favorable to the consumption good. The Rybczinsky theorem thus implies a decrease of the output of the capital good y_t. This tends to lower the investment and the capital stock in the next period k_{t+1}.

- In the next period the decrease of k_{t+1} implies again through the Rybczinsky effect an increase of the output of the capital good y_{t+1}. Indeed, the decrease of k_{t+1} improves the trade-off in production in favor of the investment good which is relatively less intensive in capital. Therefore this tends to increase the investment and the capital stock in period $t+2$, k_{t+2}.[17] This mechanism requires a sufficiently capital intensive consumption good sector to compensate for the degree of decreasing returns and a large enough share of capital in the consumption good sector.

However the properties of preference also matter. The existence of persistent cycles requires first that the agents accept fluctuations in their consump-

in the positive orthant.

[17]The restriction on the capital intensity difference across sectors is compatible with recent empirical evidence. Building on aggregate Input-Output tables, Takahashi *et al.* [30] have shown that over the last 30 years the OECD countries have been characterized by a consumption good sector that is more capital-intensive than the investment good sector. See also Baxter [4] for similar results.

tion levels. This requires that the elasticity of intertemporal substitution in consumption $1/\sigma$ is large enough. But endogenous fluctuations require also that the oscillations in relative prices must not present intertemporal arbitrage opportunities. For instance, possible gains from postponing consumption from periods when the marginal rate of transformation between consumption and investment is high to periods when it is low must not be worth it. This configuration is obtained provided that the discount factor ρ is low enough.

It is worth noting at this point that, based on similar mechanisms, more complex behavior of optimal paths, i.e. periodic cycles of any period or even chaos in two-sector models,[18] and Hopf bifurcations in n-sector models with $n \geq 3$,[19] may occur. Our results are therefore compatible with more realistic aggregate fluctuations.

4 Free-trade equilibrium

Assume there exists a free-trade equilibrium for this economy. Denote by λ^i, $i = A, B$, the country i's marginal utility of wealth associated with the free-trade equilibrium. Given $\lambda = (\lambda^A, \lambda^B)$, let us define the following social welfare function

$$W(k_t, y_t; \lambda) = \max_{c_t^A, c_t^B, k_t^A, k_t^B, y_t^A, y_t^B} \quad \frac{1}{\lambda^A} \frac{(c_t^A)^{1-\sigma}}{(1-\sigma)} + \frac{1}{\lambda^B} \frac{(c_t^B)^{1-\sigma}}{(1-\sigma)}$$
$$s.t. \qquad c_t^A + c_t^B \leq T^A(k_t^A, y_t^A) + T^B(k_t^B, y_t^B) \qquad (13)$$
$$k_t^A + k_t^B \leq k_t \text{ and } y_t^A + y_t^B \leq y_t$$

On the one hand, the first welfare theorem allows to use this social planner's problem to solve for equilibrium. On the other hand, the Negishi [23] approach states that in an economy with heterogeneous agents, the set of Pareto allocations can be obtained by solving a social planner's problem with a utility function as given by (13). The competitive equilibrium is, then, the Pareto optimal allocation obtained using the "right" set of weights. However, under the assumption of CIES utility functions which are identical across the two countries, we can show that the free-trade allocation does not depend on the particular values of the weights (since certain marginal rates of substitution need to be equalized across countries).

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[18]See for instance Boldrin and Montrucchio [8] and more recently Yano [39].
[19]See Benhabib and Nishimura [5], Venditti [31].

Proposition 3. *An equilibrium aggregate consumption path under free-trade, c_t, is determined in such a way that it may solve the following maximization problem:*

$$\max \sum_{t=0}^{\infty} \rho^t (c_t)^{1-\sigma} \tag{14}$$

$$s.t. \ c_t \ \leq \ T^A(k_t^A, y_t^A) + T^B(k_t^B, y_t^B) \tag{15}$$

$$k_t^A + k_t^B \ \leq \ k_t \tag{16}$$

$$y_t^A + y_t^B \ \geq \ y_t \tag{17}$$

$$k_{t+1} \ = \ y_t \tag{18}$$

$$k_0 \ = \ k_0^A + k_0^B \ given \tag{19}$$

Once the aggregate consumption path, c_t, is determined, the two countries' consumption paths, c_t^A and c_t^B, are given by

$$(c_t^A, c_t^B) = \left(\frac{(1/\lambda^A)^\sigma}{(1/\lambda^A)^\sigma + (1/\lambda^B)^\sigma} c_t, \frac{(1/\lambda^B)^\sigma}{(1/\lambda^A)^\sigma + (1/\lambda^B)^\sigma} c_t \right). \tag{20}$$

This Proposition shows that an equilibrium aggregate consumption path under free-trade, c_t, is obtained independently of the marginal utilities of wealth of countries A and B, λ^A and λ^B, in that equilibrium. Once the aggregate consumption path, c_t, is determined, the two countries' consumption paths, c_t^A and c_t^B, are derived in such a way that the aggregate consumption good may be devided proportionately to

$$\frac{(1/\lambda^A)^\sigma}{(1/\lambda^A)^\sigma + (1/\lambda^B)^\sigma} \text{ and } \frac{(1/\lambda^B)^\sigma}{(1/\lambda^A)^\sigma + (1/\lambda^B)^\sigma}.$$

The first order conditions corresponding to the optimization program (14) give:

$$c_t^\sigma \left[T_1^A(k_t^A, y_t^A) - T_1^B(k_t^B, y_t^B) \right] = 0 = c_t^\sigma \left[T_2^A(k_t^A, y_t^A) - T_2^B(k_t^B, y_t^B) \right] \tag{21}$$

with $c_t = c_t^A + c_t^B$. Solving these equations gives $k_t^i = k^i(k_t, y_t)$ and $y_t^i = y^i(k_t, y_t)$ for $i = A, B$, and we get

$$V(k_t, y_t) = c_t^{1-\sigma} = \left[T^A(k^A(k_t, y_t), y^A(k_t, y_t)) + T^B(k^B(k_t, y_t), y^B(k_t, y_t)) \right]^{1-\sigma}$$

The social planner problem is then equivalent to maximizing a discounted value of the period function $V(k_t, y_t)$, and a free-trade equilibrium can then be obtained as an equilibrium path derived from the following optimization:

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$$\max_{y_t} \quad \sum_{t=0}^{+\infty} \rho^t V(k_t, y_t)$$

$$\text{s.t.} \quad k_{t+1} = y_t, \ k_0 = k_0^A + k_0^B \ given$$

From the first order conditions (21) and the envelope theorem we get the Euler equation

$$
\begin{aligned}
\left(\frac{c_{t+1}}{c_t}\right)^\sigma &= -\rho \frac{T_1^B(k^B(k_{t+1},k_{t+2}),y^B(k_{t+1},k_{t+2}))}{T_2^B(k^B(k_t,k_{t+1}),y^B(k_t,k_{t+1}))} \\
&= -\rho \frac{T_1^A(k^A(k_{t+1},k_{t+2}),y^A(k_{t+1},k_{t+2}))}{T_2^A(k^A(k_t,k_{t+1}),y^A(k_t,k_{t+1}))}
\end{aligned}
\tag{22}
$$

Let us denote k^* the solution of

$$-\frac{T_1^B(k^B(k^*,k^*),y^B(k^*,k^*))}{T_2^B(k^B(k^*,k^*),y^B(k^*,k^*))} = \frac{1}{\rho} = -\frac{T_1^A(k^A(k^*,k^*),y^A(k^*,k^*))}{T_2^A(k^A(k^*,k^*),y^A(k^*,k^*))} \tag{23}$$

The steady-state k^* gives the total stationary amount of capital at the world level. The allocation of capital across the two countries has to be determined.

4.1 Stationary free-trade allocation

Let us denote by $\mathcal{X}_j^i$, $\mathcal{M}_j^i$ and $\mathcal{N}\mathcal{X}^i$ respectively the exports and imports of good $j = c, y$, and the balance of trade for country $i = A, B$. We focus on a free-trade allocation $k^* = k^{A*} + k^{B*}$ in which one country, say A, is characterized by net imports of capital, i.e. $k^{A*} > y^{A*}$, while country B is characterized by net exports of capital, i.e. $k^{B*} < y^{B*}$. The associated stationary consumption levels are then derived from the assumption that the balance of trade is in equilibrium in each country, namely

$$\mathcal{N}\mathcal{X}^A = \mathcal{X}_c^A - p\mathcal{M}_y^A = 0, \quad \mathcal{N}\mathcal{X}^B = p\mathcal{X}_y^B - \mathcal{M}_c^B = 0$$

and are such that country A is characterized by net exports of consumption, i.e. $c^{A*} < T^{A*}$, while country B is characterized by net imports of consumption, i.e. $c^{B*} > T^{B*}$. Let us consider a particular solution satisfying a symmetry property such that $k^{A*} = \theta y^{A*} > y^{A*}$ and $k^{B*} = y^{B*}/\theta < y^{B*}$ with $\theta > 1$ a given constant. In the rest of the paper, we will consider economies with parameter values for which such an equilibrium exists. We introduce the following restriction:

Assumption 3. $\alpha_1^B + \alpha_1^A \beta_1^B \geq \alpha_1^A + \alpha_1^B \beta_1^A$

Let us denote $\bar{\theta} = 1/\rho\beta_1^B$. The following Proposition provides conditions for the existence of this particular free-trade allocation:

Proposition 4. *Under Assumption 3, let $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$ and consider a constant $\theta \in (1, \bar{\theta})$. The free-trade allocation $k^* = k^{A*} + k^{B*}$ with*

$$k^{A*} = \theta \left(\frac{\alpha_1^A \beta_2^A}{\alpha_2^A \beta_1^A \theta + (\alpha_1^A \beta_2^A - \alpha_2^A \beta_1^A) \rho \beta_1^A} \right)^{\frac{\beta_2^A}{1-\beta_1^A}} \left[\mathcal{E}_y^A (\rho \beta_1^A)^{\beta_1^A + \beta_2^A} \right]^{\frac{1}{1-\beta_1^A}} = \theta y^{A*}$$

$$k^{B*} = \left(\frac{\alpha_1^B \beta_2^B}{\alpha_2^B \beta_1^B + (\alpha_1^B \beta_2^B - \alpha_2^B \beta_1^B) \rho \beta_1^B \theta} \right)^{\frac{\beta_2^B}{1-\beta_1^B}} \left[\frac{(\rho \beta_1^B \theta)^{\beta_1^B + \beta_2^B}}{\theta} \right]^{\frac{1}{1-\beta_1^B}} = y^{B*}/\theta$$

(24)

is a solution of equation (23) if and only if $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$ and $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$ with

$$\mathcal{E}_c^{A*} = \frac{\left[\alpha_2^A \beta_1^A \theta + (\alpha_1^A \beta_2^A - \alpha_2^A \beta_1^A) \rho \beta_1^A\right]^{\alpha_2^A} \alpha_1^B (\alpha_2^B \beta_1^B)^{\alpha_2^B} (1 - \rho \beta_1^B \theta)^{\alpha_1^B + \alpha_2^B - 1} (k^{B*})^{\alpha_1^B - \alpha_1^A}}{\alpha_1^A (\alpha_2^A \beta_1^A)^{\alpha_2^A} (\theta - \rho \beta_1^A)^{\alpha_1^A + \alpha_2^A - 1} \left[\alpha_2^B \beta_1^B + (\alpha_1^B \beta_2^B - \alpha_2^B \beta_1^B) \rho \beta_1^B \theta\right]^{\alpha_2^B}}$$

$$\mathcal{E}_y^{A*} = \frac{\left(\frac{\alpha_1^B \beta_2^B}{\alpha_2^B \beta_1^B + (\alpha_1^B \beta_2^B - \alpha_2^B \beta_1^B) \rho \beta_1^B \theta} \right)^{\frac{\beta_2^B (1-\beta_1^A)}{1-\beta_1^B}}}{(\rho \beta_1^A)^{\beta_1^A + \beta_2^A} \left(\frac{\alpha_1^A \beta_2^A}{\alpha_2^A \beta_1^A \theta + (\alpha_1^A \beta_2^A - \alpha_2^A \beta_1^A) \rho \beta_1^A} \right)^{\beta_2^A}} \left[\frac{(\rho \beta_1^B \theta)^{\beta_1^B + \beta_2^B}}{\theta} \right]^{\frac{1-\beta_1^A}{1-\beta_1^B}}$$

(25)

Moreover, the associated free-trade allocation of consumption is such that $c^{A*} = T^{B*}$ and $c^{B*} = T^{A*}$.

Proposition 4 shows that by changing the constants $\mathcal{E}_c^A$ and $\mathcal{E}_y^A$, we can construct economies satisfying the symmetry property for different values of $\theta \in (1, \bar{\theta})$.

4.2 Endogenous fluctuations under free-trade

We focus on local stability results when the consumption good is sufficiently capital intensive to compensate for the degree of decreasing returns. As in the closed-economy case, such a capital intensity configuration is associated with endogenous fluctuations. We introduce the following restrictions along the free-trade allocation:

Assumption 4. *In countries A and B, the consumption good is capital intensive with*

$$\alpha_1^A \beta_2^A - \alpha_2^A \beta_1^A > \frac{\alpha_1^A \beta_2^A \theta (1 - \alpha_1^A - \alpha_2^A)}{(1 - \alpha_1^A)(\theta - \beta_1^A)}$$

(26)

and

$$\alpha_1^B \beta_2^B - \alpha_2^B \beta_1^B > \frac{\alpha_1^B \beta_2^B (1 - \alpha_1^B - \alpha_2^B)}{(1 - \alpha_1^B)(1 - \theta \beta_1^B)}$$

(27)

Linearizing the Euler equation (22) around k^* gives the characteristic polynomial

$$\mathcal{P}_f(x) = x^2 - \mathcal{T}_f(\sigma)x + \mathcal{D}_f = 0 \tag{28}$$

with $\mathcal{D}_f = \rho^{-1}$.[20] As in the closed-economy case, the local stability analysis is performed using the fact that while $\mathcal{T}_f(\sigma)$ is a function of the inverse of the elasticity of intertemporal substitution in consumption σ, $\mathcal{D}_f$ is constant for any $\sigma \geq 0$. Based on Proposition 3, our methodology allows again to apply the geometrical method of Grandmont, Pintus and de Vilder [18] to the local stability analysis of a two-country general equilibrium trade model. Proceeding as in Section 3, we easily conclude that the occurrence of period-two cycles through the existence of a flip bifurcation can be obtained if and only if the starting point $(\mathcal{T}_f(0), 1/\rho)$ satisfies

$$1 + \mathcal{T}_f(0) + \rho^{-1} > 0 \tag{29}$$

Inequality (29) is satisfied if ρ is sufficiently lower than 1 and if the following assumption holds:

Assumption 5. *The capital shares of countries A and B satisfy*

$$\alpha_1^A > \frac{\theta(1 - \beta_1^A - \beta_2^A) + 2\beta_1^A + \beta_2^A}{2(\beta_1^A + \beta_2^A)} \tag{30}$$

and

$$\alpha_1^B > \frac{1 - \beta_1^B - \beta_2^B + \theta(2\beta_1^B + \beta_2^B)}{2\theta(\beta_1^B + \beta_2^B)} \tag{31}$$

We derive therefore the following Proposition:

Proposition 5. *Let $\theta \in (1, \bar{\theta})$. Under Assumptions 3-5, there exist $\bar{\rho}_f \in (0,1)$ and $\bar{\sigma}_f \in (0, \sigma_f^*)$ such that for any given $\rho \in (0, \bar{\rho}_f)$, the free-trade steady-state k^* is saddle-point stable when $\sigma \in (\bar{\sigma}_f, +\infty)$, undergoes a flip bifurcation when σ crosses $\bar{\sigma}_f$ from above, and becomes locally unstable when $\sigma \in [0, \bar{\sigma}_f)$. Moreover, the optimal path at the world level is generically characterized by saddle-point stable (locally unstable) period-two cycles in a left (right) neighbourhood of $\bar{\sigma}_f$.*

Proposition 5 provides conditions on the technologies of both countries for the existence of endogenous fluctuations at the free-trade steady-state which are similar to those given in Proposition 2. As our result imposes a lot of conditions on the parameters, one may wonder whether there exists a non empty set of parameters satisfying these conditions. The

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[20]Details are given in Appendix 6.7.

following numerical illustration allows to prove the non-emptiness of this set.

Example: Let us consider the following parameters' values: $\alpha_1^A = 0.66$, $\alpha_2^A = 0.33$, $\beta_1^A = 0.3$, $\beta_2^A = 0.69$, $\alpha_1^B = 0.67$, $\alpha_2^B = 0.32$, $\beta_1^B = 0.31$, $\beta_2^B = 0.68$. When $\theta \in (1, 1.67)$, Assumptions 3-5 hold with these values.

Proposition 5 shows that the sets of parameters values for which the closed economies and the free-trade economies have cycles are not the same, and therefore, the long run equilibrium behavior of the free-trade economy may be different from the equilibrium behavior of each of the economies in isolation. As a result, we derive the following two Theorems which are the first main results of the paper, and which prove that endogenous periodic cycles can occur at the world level in two different circumstances. It is worth noting that in the following, in order to compare the equilibria of the closed and free-trade economies, the *closed-economy steady state* refers to the steady state of the closed economy i with the same technological parameters $\mathcal{E}_c^i$ and $\mathcal{E}_y^i$ as the free-trade economy.

Theorem 1. *Let $\theta \in (1, \bar{\theta})$ and Assumption 3 hold. Assume that in country A, (30) in Assumption 5 holds, and, for any given $\rho \in (0, 1)$,*

$$\frac{\alpha_1^A \beta_2^A (1 - \alpha_1^A - \alpha_2^A)}{(1 - \alpha_1^A)(1 - \rho \beta_1^A)} > \alpha_1^A \beta_2^A - \alpha_2^A \beta_1^A > \frac{\alpha_1^A \beta_2^A \theta (1 - \alpha_1^A - \alpha_2^A)}{(1 - \alpha_1^A)(\theta - \rho \beta_1^A)} \tag{32}$$

If in country B, (27) in Assumption 4 holds and

$$\alpha_1^B > \frac{1 + \beta_1^B}{2(\beta_1^B + \beta_2^B)} \tag{33}$$

then there exist $\bar{\rho}_f \in (0, 1)$ and $\bar{\sigma}_f \in (0, \sigma_f^*)$ such that for any given $\rho \in (0, \bar{\rho}_f)$, the optimal path at the world level is generically characterized by saddle-point stable (locally unstable) period-two cycles in a left (right) neighbourhood of $\bar{\sigma}_f$, while the closed-economy steady-state in economy A is saddle-point stable with monotone convergence for any $\sigma \geq 0$. Business cycle fluctuations then translate across national boundaries as in economy B there generically exist period-two cycles in a neighborhood of $\bar{\sigma}_c^B$.

For a given value of $\theta \in (1, \bar{\theta})$, and thus given values of the technology parameters, Theorem 1 proves that endogenous fluctuations arise under free-trade while the capital importing country A is characterized by a saddle-point stable closed-economy steady-state with monotone convergence. However, the closed-economy equilibrium in country B is also characterized by

period-two cycles. International globalization and market integration then generate a contagion of the capital exporting country's endogenous fluctuations and have a destabilizing effect on the capital importing country. Note that decreasing returns in the consumption good sector of country A are necessary for this international transmission of cycles since condition (32) cannot hold if $1 - \alpha_1^A - \alpha_2^A = 0$. The Assumption $1 - \alpha_1^A - \alpha_2^A > 0$ means that our result requires large enough decreasing returns in the consumption good of country A.

Theorem 2. *Let $\theta \in (1, \bar{\theta})$ and Assumption 3 hold. Assume that in country A, (30) in Assumption 5 holds, and, for any given $\rho \in (0, 1)$,*

$$\frac{\alpha_1^A \beta_2^A (1 - \alpha_1^A - \alpha_2^A)}{(1 - \alpha_1^A)(1 - \rho \beta_1^A)} > \alpha_1^A \beta_2^A - \alpha_2^A \beta_1^A > \frac{\alpha_1^A \beta_2^A \theta (1 - \alpha_1^A - \alpha_2^A)}{(1 - \alpha_1^A)(\theta - \rho \beta_1^A)} \tag{34}$$

If in country B, (27) in Assumption 4 holds and

$$\frac{1 + \beta_1^B}{2(\beta_1^B + \beta_2^B)} > \alpha_1^B > \frac{1 - \beta_1^B - \beta_2^B + \theta(2\beta_1^B + \beta_2^B)}{2\theta(\beta_1^B + \beta_2^B)} \tag{35}$$

then there exist $\bar{\rho}_f \in (0, 1)$ and $\bar{\sigma}_f \in (0, \sigma_f^*)$ such that for any given $\rho \in (0, \bar{\rho}_f)$, the optimal path at the world level is generically characterized by saddle-point stable (locally unstable) period-two cycles in a left (right) neighbourhood of $\bar{\sigma}_f$, while the closed-economy steady-state in economy A is saddle-point stable with monotone convergence for any $\sigma \geq 0$. Opening to free-trade then creates persistent business cycle fluctuations as the closed-economy steady state in economy B is also saddle-point stable for any $\sigma \geq 0$.

Theorem 2 proves that endogenous fluctuations arise under free-trade even though saddle-point stability holds in both closed-economy countries. Now, international globalization and market integration create new opportunities for periodic cycles, and have a global destabilizing effect on all the trading countries. Note here that decreasing returns in the investment good sector of country B are also necessary for this global destabilizing effect since condition (35) cannot hold if $1 - \beta_1^B - \beta_2^B = 0$. The Assumption $1 - \beta_1^B - \beta_2^B > 0$ means that our result also requires large enough decreasing returns in the capital good of country B.

The intuition for the existence of persistent fluctuations at the world level is of course similar to the one derived in a closed economy. As suggested by Proposition 5, we need a sufficiently capital intensive aggregate consumption good sector and a large enough share of capital in the aggregate consumption

good sector to generate oscillations of the capital stock on the production side, a large enough elasticity of intertemporal substitution to allow the consumers accepting fluctuations in consumption, and a low enough discount factor to prevent intertemporal arbitrage opportunities. However, now the equilibrium is defined from the free-trade allocation across countries. As a result, the technological conditions may be less restrictive than in the closed-economy case.

Indeed, for country A which imports capital, the consumption good sector needs to be less capital intensive at the free-trade equilibrium than at the closed-economy equilibrium. This condition can then be satisfied while at the same time the consumption good sector in the closed-economy country A is not sufficiently capital intensive to generate oscillations of the capital stock. Moreover, for country B which exports capital, the share of capital in the consumption good sector needs to be lower at the free-trade equilibrium than at the closed-economy equilibrium. As a result this condition can be satisfied while at the same time in the consumption good sector of the closed-economy country B, the share of capital is not sufficiently large to generate persistent fluctuations of the capital stock.

In order to understand the transition between the closed-economy equilibrium and the free-trade equilibrium, assume that both countries are initially at their respective closed-economy steady state. When the countries open up to international trade, the new steady states are characterized by imports of capital good but exports of consumption good in country A, and conversely for country B. When saddle-point stable cycles occur at the free-trade equilibrium, each country will jump on the saddle-path that converges towards the period-two cycles.

Example: The following numerical illustration allows to prove the non-emptiness of the set of parameters satisfying the conditions of Theorem 2. Let us consider the parameters' values: $\alpha_1^A = 0.65$, $\alpha_2^A = 0.11$, $\beta_1^A = 0.28$, $\beta_2^A = 0.72$, $\alpha_1^B = 0.79$, $\alpha_2^B = 0.20$, $\beta_1^B = 0.25$, $\beta_2^B = 0.5$. All the conditions of Theorem 1 are satisfied when $\theta = 2.5$, and the closed-economy steady state is saddle-point stable in both countries while international trade generates a global destabilizing effect at the world level.

It is important to note here that our results fundamentaly rely on the fact that we depart from the standard HOS framework in which the technologies are symmetric across countries. In this case indeed we have $\mathcal{E}_c^A = \mathcal{E}_c^B$, $\mathcal{E}_y^A = \mathcal{E}_y^B$, $\alpha_i^A = \alpha_i^B$ and $\beta_i^A = \beta_i^B$, $i = 1, 2$, the free-trade stationary allocation is identical to the closed-economy steady-state so that there is no trade in the long-run, and Assumptions 4 and 5 imply Assumptions 1 and 2. As a result, persistent cycles at the world level are obtained provided persistent cycles also exist in the two closed economies.

Based on a similar argument, this property also explains why decreasing returns are also fundamental to get a destabilization effect of international trade. If we assume that the production functions are all linearly homogeneous, the production side of our model is a special case of the HOS model with freely internationally mobile capital. In the static context, this model was originally studied by Kemp [20] and Jones [19]. In the dynamic context, it was studied by Nishimura and Yano [27]. An important feature of this model is that, in general, one country completely specializes in one sector while the other country tends to be incompletely specialized. In such a framework, international trade does not modify the dynamic properties of the equilibrium with respect to the closed-economy case.

Theorems 1 and 2 show that opening to free-trade can have a destabilizing effect on both partner countries. However, it is important to note that under different conditions, opening to free-trade could also have a stabilizing effect. We derive indeed from Theorems 1 and 2:

Theorem 3. *Let $\theta \in (1, \bar{\theta})$ and Assumption 3 hold. Assume that in country A, Assumption 1 holds with*

$$\frac{\theta(1 - \beta_1^A - \beta_2^A) + 2\beta_1^A + \beta_2^A}{2(\beta_1^A + \beta_2^A)} > \alpha_1^A > \frac{1 + \beta_1^A}{2(\beta_1^A + \beta_2^A)} \tag{36}$$

and in country B, Assumption 2 holds with

$$\frac{\alpha_1^B \beta_2^B (1 - \alpha_1^B - \alpha_2^B)}{(1 - \alpha_1^B)(1 - \rho\beta_1^B\theta)} > \alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B > \frac{\alpha_1^B \beta_2^B (1 - \alpha_1^B - \alpha_2^B)}{(1 - \alpha_1^B)(1 - \rho\beta_1^B)} \tag{37}$$

for any given $\rho \in (0, 1)$, then both countries are characterized by the possible existence of period-two cycles in the closed-economy case, while saddle-point stability always holds at the world level.

Theorem 3 shows that there also exist technological configurations under which countries that would have cycles when closed experience a stable

steady state under free-trade.

To confirm that opening to free-trade can have a destabilizing effect on both trading countries, we may now focus on the impact of the elasticity of intertemporal substitution $1/\sigma$ on the existence of business cycle fluctuations at the world level. To simplify the formulation, let us assume that the returns to scale are constant in the consumption good sector of country B, i.e. $1 - \alpha_1^B - \alpha_2^B = 0$, and let us slightly modify Assumptions 4 and 5 as follows:

Assumption 6. *In countries A and B, the consumption good is capital intensive with*

$$\frac{\alpha_1^A \beta_2^A (1 - \alpha_1^A - \alpha_2^A)}{(1 - \alpha_1^A)(1 - \rho\beta_1^A)} > \alpha_1^A \beta_2^A - \alpha_2^A \beta_1^A > \frac{\alpha_1^A \beta_2^A \theta (1 - \alpha_1^A - \alpha_2^A)}{(1 - \alpha_1^A)(\theta - \rho\beta_1^A)}$$

for any given $\rho \in (0,1)$.

Assumption 7. *The capital shares of countries A and B satisfy*
$$\alpha_1^A > \frac{\theta(1 - \beta_1^A - \beta_2^A) + 2\beta_1^A + \beta_2^A}{2(\beta_1^A + \beta_2^A)}$$
and
$$\alpha_1^B \in \left(\frac{1 - \beta_1^B - \beta_2^B + \theta(2\beta_1^B + \beta_2^B)}{2\theta(\beta_1^B + \beta_2^B)}, \frac{3 + \beta_1^B}{4} \right) \tag{38}$$

Assumptions 6 still implies that in the closed economy A, the optimal path monotonically converges toward the closed-economy steady-state for any $\sigma \geq 0$. The modified condition (38) in Assumption 7 leads to a positive flip bifurcation value $\bar{\sigma}_c^B$ for country B in the closed-economy case so that the closed-economy steady-state is saddle-point stable for any $\sigma > \bar{\sigma}_c^B$. The following Theorem, which is the second main result of the paper, shows that $\bar{\sigma}_c^B$ may be as small as possible.

Theorem 4. *Let $\theta \in (1, \bar{\theta})$, $1 - \alpha_1^B - \alpha_2^B = 0$ and Assumptions 3, 6 and 7 hold. There exist $\epsilon > 0$, $\bar{\rho}_f \in (0,1)$ and $\bar{\sigma}_f > \bar{\sigma}_c^B > 0$ such that if*
$$\alpha_1^B \beta_2^B - \alpha_2^B \beta_1^B \in \left(\frac{(1 - \beta_1^B)\alpha_2^B}{2 + \rho(1 + \beta_1^B) - 2(1 + \rho)\alpha_1^B}, \frac{(1 - \beta_1^B)\alpha_2^B}{2 + \rho(1 + \beta_1^B) - 2(1 + \rho)\alpha_1^B} + \epsilon \right),$$
then for any given $\rho \in (0, \bar{\rho}_f)$, the optimal path at the world level is generically characterized by saddle-point stable (locally unstable) period-two cycles in a left (right) neighbourhood of $\bar{\sigma}_f$, while the closed-economy steady-state is saddle-point stable for any $\sigma > \bar{\sigma}_c^B$ in both economies. Persistent endogenous fluctuations at the world level occur for lower values of the elasticity of intertemporal substitution in consumption than in a closed economy.

This Theorem shows that for a given elasticity of intertemporal substitution in consumption such that $\sigma > \bar{\sigma}_c^B$ for which the autarky steady state is saddle-point stable in both countries, international trade and market integration can again create new opportunities for aggregate fluctuations by generating endogenous cycles at the world level when σ is in a left neighbourhood of $\bar{\sigma}_f$, i.e. with a lower elasticity of intertemporal substitution in consumption.

5 Concluding comments

In a perfect foresight model with two countries characterized by Cobb-Douglas technologies with decreasing returns and CIES non-linear preferences, we have investigated the way endogenous fluctuations may spread all over the world through international trade. Our main result shows that even in a situation in which the closed-economy equilibrium of both countries is saddle-point stable, a trade reform, which consists of joining a common market for the consumption good and capital, can lead to persistent endogenous cycles at the world level. To our knowledge, this paper is the first to show that, in a non-HOS international trade model, opening to free-trade can have a global destabilizing effect on all trading partners.

6 Appendix

6.1 Proof of Proposition 1

We start by characterizing the first partial derivatives of the social production function.

Lemma 6.1. *The first partial derivatives of $T(k,y)$ are given by:*

$$
\begin{aligned}
T_1(k,y) &= \mathcal{E}_c \alpha_1 (\alpha_2 \beta_1/\Delta)^{\alpha_2} (k-g)^{\alpha_1+\alpha_2-1} \\
T_2(k,y) &= -\tfrac{T_1(k,y)}{\mathcal{E}_y \beta_1}(\alpha_1 \beta_2/\Delta)^{-\beta_2} g^{1-\beta_1-\beta_2}
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta &= \alpha_2 \beta_1 k + (\alpha_1 \beta_2 - \alpha_2 \beta_1)g \\
g &= g(k,y) = \left\{ K_y \in [0,k] \ such \ that \ \ y = \frac{\mathcal{E}_y (\alpha_1 \beta_2)^{\beta_2} K_y^{\beta_1+\beta_2}}{[\alpha_2 \beta_1 k + (\alpha_1 \beta_2 - \alpha_2 \beta_1)K_y]^{\beta_2}} \right\}
\end{aligned}
$$

Proof: The Lagrangean corresponding to program (1) is:

$$\mathcal{L}_t = \mathcal{E}_c K_{ct}^{\alpha_1} L_{ct}^{\alpha_2} + p_t \left(\mathcal{E}_y K_{yt}^{\beta_1} L_{yt}^{\beta_2} - y_t \right)$$
$$+ \omega_t (1 - L_{ct} - L_{yt}) + r_t (k_t - K_{ct} - K_{yt}) \tag{39}$$

The first order conditions are such that:

$$\mathcal{E}_c \alpha_1 K_c^{\alpha_1 - 1} L_c^{\alpha_2} - r = 0, \quad \mathcal{E}_c \alpha_2 K_c^{\alpha_1} L_c^{\alpha_2 - 1} - \omega = 0 \tag{40}$$

$$\mathcal{E}_y p \beta_1 K_y^{\beta_1 - 1} L_y^{\beta_2} - r = 0, \quad \mathcal{E}_y p \beta_2 K_y^{\beta_1} L_y^{\beta_2 - 1} - \omega = 0 \tag{41}$$

Using $K_c = k - K_y$, $L_y = 1 - L_c$, and merging (40)-(41) we obtain:

$$L_c = \frac{\alpha_2 \beta_1 (k - K_y)}{(\alpha_1 \beta_2 - \alpha_2 \beta_1) K_y + \alpha_2 \beta_1 k} \tag{42}$$

$$L_y = \frac{\alpha_1 \beta_2 K_y}{(\alpha_1 \beta_2 - \alpha_2 \beta_1) K_y + \alpha_2 \beta_1 k} \tag{43}$$

$$K_c = k - K_y \tag{44}$$

$$K_y = g(k, y) \equiv g \tag{45}$$

where

$$g(k, y) = \left\{ K_y \in [0, k] \text{ such that } y = \frac{\mathcal{E}_y (\alpha_1 \beta_2)^{\beta_2} K_y^{\beta_1 + \beta_2}}{[\alpha_2 \beta_1 k + (\alpha_1 \beta_2 - \alpha_2 \beta_1) K_y]^{\beta_2}} \right\} \tag{46}$$

To simplify notation let:

$$\Delta = \alpha_2 \beta_1 k + (\alpha_1 \beta_2 - \alpha_2 \beta_1) g \tag{47}$$

From (40), (42) and (44) we obtain:

$$T_1(k, y) = r = \mathcal{E}_c \alpha_1 (k - g)^{\alpha_1 + \alpha_2 - 1} (\alpha_2 \beta_1 / \Delta)^{\alpha_2} \tag{48}$$

and from (41), (43), (45) and (48):

$$-T_2(k, y) = p = \frac{\mathcal{E}_c \alpha_1}{\mathcal{E}_y \beta_1} \frac{(\alpha_2 \beta_1 / \Delta)^{\alpha_2}}{(\alpha_1 \beta_2 / \Delta)^{\beta_2}} (k - g)^{\alpha_1 + \alpha_2 - 1} g^{1 - \beta_1 - \beta_2} \tag{49}$$

From (48) and (49) we finally derive

$$T_2(k, y) = -\frac{T_1(k, y)}{\mathcal{E}_y \beta_1} (\alpha_1 \beta_2 / \Delta)^{-\beta_2} g^{1 - \beta_1 - \beta_2}$$

$\square$

We may now prove Proposition 1. Using (46) we derive

$$T_2(k, y) = -T_1(k, y) \frac{g}{y \beta_1} \tag{50}$$

It follows that at the closed-economy steady-state $K_y^* \equiv g = \rho \beta_1 y = \rho \beta_1 \bar{k}$. Solving $T_2(k, k) + \rho T_1(k, k) = 0$ finally gives the expression of the steady-state, namely

$$\bar{k} = \left(\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1 + (\alpha_1 \beta_2 - \alpha_2 \beta_1) \rho \beta_1} \right)^{\frac{\beta_2}{1 - \beta_1}} \left[\mathcal{E}_y (\rho \beta_1)^{\beta_1 + \beta_2} \right]^{\frac{1}{1 - \beta_1}}$$

$\square$

6.2 The characteristic polynomial in a closed economy

We start by characterizing the second partial derivatives of $T(k,y)$:

Lemma 6.2. *The second partial derivatives of $T(k,y)$ are given by:*

$$T_{11}(k,y) = \frac{T_1(k,y)}{k-g} \left\{ \frac{[(1-\alpha_1)(\alpha_1\beta_2-\alpha_2\beta_1)-\alpha_2\beta_2](k-g)-(1-\alpha_1-\alpha_2)\alpha_1\beta_2 k}{\alpha_2(\beta_1+\beta_2)k+(\alpha_1\beta_2-\alpha_2\beta_1)g} \right\}$$

$$T_{12}(k,y) = \frac{T_1(k,y)}{k-g}\frac{g}{y\beta_1} \left\{ \frac{-(1-\alpha_1)(\alpha_1\beta_2-\alpha_2\beta_1)(k-g)+(1-\alpha_1-\alpha_2)\alpha_1\beta_2 k}{\alpha_2(\beta_1+\beta_2)k+(\alpha_1\beta_2-\alpha_2\beta_1)g} \right\}$$

$$T_{22}(k,y) = -\frac{T_1(k,y)}{k-g}\left(\frac{g}{y\beta_1}\right)^2$$

$$\times \left\{ \frac{(1-\beta_1-\beta_2)(k/g)\alpha_2\beta_1(k-g)-(\beta_1-\alpha_1)(\alpha_1\beta_2-\alpha_2\beta_1)(k-g)+(1-\alpha_1-\alpha_2)\alpha_1\beta_2 k}{\alpha_2(\beta_1+\beta_2)k+(\alpha_1\beta_2-\alpha_2\beta_1)g} \right\}$$

with $|H(k,y)| \equiv T_{11}(k,y)T_{22}(k,y) - T_{12}(k,y)^2 > 0$.

Proof: Recall that by definition of g we have the identity:

$$y[\alpha_2\beta_1 k + (\alpha_1\beta_2 - \alpha_2\beta_1)g]^{\beta_2} = \mathcal{E}_y^i(\alpha_1\beta_2)^{\beta_2}g^{\beta_1+\beta_2} \tag{51}$$

Total differentiation gives after simplifications:

$$g\Big\{ dy\Delta + \beta_2 y[\alpha_2\beta_1 dk + (\alpha_1\beta_2 - \alpha_2\beta_1)dg]\Big\} = (\beta_1+\beta_2)y\Delta dg$$

We then get

$$\begin{aligned}
g_1 &= \frac{dg}{dk} = \frac{\alpha_2\beta_2 g}{\alpha_2(\beta_1+\beta_2)k+(\alpha_1\beta_2-\alpha_2\beta_1)g} \\
g_2 &= \frac{dg}{dy} = \frac{\Delta g}{y\beta_1[\alpha_2(\beta_1+\beta_2)k+(\alpha_1\beta_2-\alpha_2\beta_1)g]}
\end{aligned} \tag{52}$$

The second partial derivatives of $T(k,y)$ are obtained by differentiating (48) and (49):

$$T_{11}(k,y) = -\frac{(1-\alpha_1-\alpha_2)(1-g_1)T_1(k,y)}{k-g} - \frac{\alpha_2[\alpha_2\beta_1+(\alpha_1\beta_2-\alpha_2\beta_1)g_1]T_1(k,y)}{\Delta}$$

$$T_{12}(k,y) = \frac{(1-\alpha_1-\alpha_2)g_2 T_1(k,y)}{k-g} - \frac{\alpha_2(\alpha_1\beta_2-\alpha_2\beta_1)g_2 T_1(k,y)}{\Delta}$$

$$T_{22}(k,y) = \frac{(1-\alpha_1-\alpha_2)g_2 T_2(k,y)}{k-g} + \frac{(1-\beta_1-\beta_2)g_2 T_2(k,y)}{g}$$

$$+ \frac{(\beta_2-\alpha_2)(\alpha_1\beta_2-\alpha_2\beta_1)g_2 T_2(k,y)}{\Delta}$$

The final expressions of these derivatives are obtained after simplifications built on (50) and the fact that

$$\begin{aligned}
g - kg_1 &= \frac{\Delta g}{\alpha_2(\beta_1+\beta_2)k+(\alpha_1\beta_2-\alpha_2\beta_1)g} = y\beta_1 g_2 \\
1 - g_1 &= \frac{\alpha_2\beta_2(k-g)+\Delta}{\alpha_2(\beta_1+\beta_2)k+(\alpha_1\beta_2-\alpha_2\beta_1)g}
\end{aligned}$$

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Strict concavity of the production functions implies that the determinant of the Hessian matrix of $T(k,y)$ satisfies $|H(k,y)| \equiv T_{11}(k,y)T_{22}(k,y) - T_{12}(k,y)^2 > 0$. $\qquad\square$

From (42), (44) and (47), we get

$$c = \mathcal{E}_c(k-g)^{\alpha_1+\alpha_2}\left(\frac{\alpha_2\beta_1}{\Delta}\right)^{\alpha_2} \tag{53}$$

Consider the following notations

$$c_1 = \partial c(k,y)/\partial k, \quad c_2 = \partial c(k,y)/\partial y$$

We easily derive from the envelope theorem

$$c_1 = T_1(k,y), \quad c_2 = T_2(k,y) = -T_1(k,y)\frac{g}{y\beta_1} \tag{54}$$

Consider now the closed-economy steady-state with $g = \rho\beta_1 y = \rho\beta_1\bar{k}$. We then have $c_2 = -\rho T_1(k,y)$ and total differenciation of equation (6) gives

$$dk_{t+2}\left[1+\sigma\rho\frac{\bar{T}_1^2}{\bar{c}\bar{T}_{12}}\right] + dk_{t+1}\left[\frac{\bar{T}_{22}}{\rho\bar{T}_{12}} + \frac{\bar{T}_{11}}{\bar{T}_{12}} - \sigma(1+\rho)\frac{\bar{T}_1^2}{\bar{c}\bar{T}_{12}}\right] + dk_t\left[\frac{1}{\rho}+\sigma\frac{\bar{T}_1^2}{\bar{c}\bar{T}_{12}}\right] = 0$$

with $\bar{T}_1 = T_1(\bar{k},\bar{k})$ and $\bar{T}_{jk} = T_{jk}(\bar{k},\bar{k})$, $j,k = 1,2$. Straightforward computations finally give the characteristic polynomial:

$$\mathcal{P}_c(x) = x^2 - \mathcal{T}_c(\sigma)x + \mathcal{D}_c = 0 \tag{55}$$

with

$$\mathcal{D}_c = \frac{1}{\rho}, \quad \mathcal{T}_c(\sigma) = \frac{-\frac{\bar{T}_{11}}{\bar{T}_{12}} - \frac{\bar{T}_{22}}{\rho\bar{T}_{12}} + \sigma(1+\rho)\frac{\bar{T}_1^2}{\bar{c}\bar{T}_{12}}}{1+\sigma\rho\frac{\bar{T}_1^2}{\bar{c}\bar{T}_{12}}} \tag{56}$$

and

$$\frac{\bar{T}_{11}}{\bar{T}_{12}} \equiv \frac{T_{11}(\bar{k},\bar{k})}{T_{12}(\bar{k},\bar{k})} = -\frac{1}{\rho}\left[1 - \frac{\alpha_2\beta_2(1-\rho\beta_1)}{(1-\alpha_1)(\alpha_1\beta_2-\alpha_2\beta_1)(1-\rho\beta_1)-(1-\alpha_1-\alpha_2)\alpha_1\beta_2}\right]$$

$$\frac{\bar{T}_{22}}{\rho\bar{T}_{12}} \equiv \frac{T_{22}(\bar{k},\bar{k})}{\rho T_{12}(\bar{k},\bar{k})} = -1 + \frac{(1-\rho\beta_1)[\alpha_2(1-\beta_1-\beta_2)+\rho(1-\beta_1)(\alpha_1\beta_2-\alpha_2\beta_1)]}{\rho[(1-\alpha_1)(\alpha_1\beta_2-\alpha_2\beta_1)(1-\rho\beta_1)-(1-\alpha_1-\alpha_2)\alpha_1\beta_2]}$$

$$\frac{\bar{T}_1^2}{\bar{c}\bar{T}_{12}} \equiv \frac{T_1(\bar{k},\bar{k})^2}{\bar{c}T_{12}(\bar{k},\bar{k})} = -\frac{\alpha_1[\alpha_2(\beta_1+\beta_2)+(\alpha_1\beta_2-\alpha_2\beta_1)\rho\beta_1]}{\rho[(1-\alpha_1)(\alpha_1\beta_2-\alpha_2\beta_1)(1-\rho\beta_1)-(1-\alpha_1-\alpha_2)\alpha_1\beta_2]}$$

The last expressions are obtained from substituting $g = \rho\beta_1 y = \rho\beta_1\bar{k}$ into the second partial derivatives of $T(k,y)$ given in Lemma 6.2. $\qquad\square$

6.3 The function $\mathcal{T}_c(\sigma)$

Note first that under Assumption 1,

$$\mathcal{T}_c(0) = -\frac{\bar{T}_{11}}{\bar{T}_{12}} - \frac{\bar{T}_{22}}{\rho\bar{T}_{12}} < 0 \text{ and } \mathcal{T}_c(+\infty) = 1+\rho^{-1} \geq 2 \tag{57}$$

Lemma 6.3. *Under Assumption 1, there exists $\sigma_c^* \in (0, +\infty)$ such that $\mathcal{T}_c(\sigma)$ is continuous and monotonically decreasing in σ over $[0, \sigma_c^*) \cup (\sigma_c^*, +\infty)$. Moreover, $\lim_{\sigma \to \sigma_{c-}^*} \mathcal{T}_c(\sigma) = -\infty$ and $\lim_{\sigma \to \sigma_{c+}^*} \mathcal{T}_c(\sigma) = +\infty$.*

Proof: Straightforward computations from (89) give

$$
\begin{aligned}
\mathcal{T}_c'(\sigma) &= \frac{\bar{T}_1^2}{\bar{c}\bar{T}_{12}} \frac{1 + \rho\left(1 + \frac{\bar{T}_{11}}{\bar{T}_{12}} + \frac{\bar{T}_{22}}{\rho\bar{T}_{12}}\right)}{\left[1 + \sigma\rho\frac{\bar{T}_1^{*2}}{\bar{c}\bar{T}_{12}}\right]^2} \\[2mm]
&= -\frac{\frac{\alpha_1[\alpha_2(\beta_1+\beta_2)+(\alpha_1\beta_2-\alpha_2\beta_1)\rho\beta_1](1-\rho\beta_1)(1-\beta_1)[\alpha_2+\rho(\alpha_1\beta_2-\alpha_2\beta_1)]}{\rho[(1-\alpha_1)(\alpha_1\beta_2-\alpha_2\beta_1)(1-\rho\beta_1)-(1-\alpha_1-\alpha_2)\alpha_1\beta_2]^2}}{\left[1 + \sigma\rho\frac{\bar{T}_1^{*2}}{\bar{c}\bar{T}_{12}}\right]^2} < 0
\end{aligned}
$$

(58)

Consider now the expressions of $\mathcal{T}_c(\sigma)$ and $\bar{T}_1^2/\bar{c}\bar{T}_{12}$ given in Appendix 6.2. We first derive that $\bar{T}_{12} < 0$ and thus $\bar{T}_1^2/\bar{c}\bar{T}_{12} < 0$ if and only if the consumption good is capital intensive with

$$
\alpha_1\beta_2 - \alpha_2\beta_1 > \frac{1-\alpha_1-\alpha_2}{1-\alpha_1}\frac{\alpha_1\beta_2}{1-\rho\beta_1}
$$

(59)

As the right-hand-side is an increasing function of ρ, $\bar{T}_1^2/\bar{c}\bar{T}_{12} < 0$ for any $\rho \in (0,1)$ if

$$
\alpha_1\beta_2 - \alpha_2\beta_1 > \frac{\alpha_1\beta_2(1-\alpha_1-\alpha_2)}{(1-\alpha_1)(1-\beta_1)}
$$

This gives Assumption 1. Under this condition, there exists a critical value σ_c^* as defined by

$$
\sigma_c^* = -\frac{\bar{c}\bar{T}_{12}}{\rho\bar{T}_1^2} = \frac{(1-\alpha_1)(\alpha_1\beta_2-\alpha_2\beta_1)(1-\rho\beta_1)-(1-\alpha_1-\alpha_2)\alpha_1\beta_2}{\alpha_1[\alpha_2(\beta_1+\beta_2)+(\alpha_1\beta_2-\alpha_2\beta_1)\rho\beta_1]}
$$

(60)

such that the denominator of $\mathcal{T}_c(\sigma)$ is equal to zero when $\sigma = \sigma_c^*$. Moreover, when $\sigma = \sigma_c^*$, the numerator of $\mathcal{T}_c(\sigma)$ is equal to

$$
-\frac{\bar{T}_{11}}{\bar{T}_{12}} - \frac{\bar{T}_{22}}{\rho\bar{T}_{12}} - \frac{1+\rho}{\rho} = -\frac{(1-\rho\beta_1)(1-\beta_1)[\alpha_2+\rho(\alpha_1\beta_2-\alpha_2\beta_1)]}{\rho[(1-\alpha_1)(\alpha_1\beta_2-\alpha_2\beta_1)(1-\rho\beta_1)-(1-\alpha_1-\alpha_2)\alpha_1\beta_2]}
$$

which is negative under Assumption 1. It follows that $\lim_{\sigma \to \sigma_{c-}^*} \mathcal{T}_c(\sigma) = -\infty$ and $\lim_{\sigma \to \sigma_{c+}^*} \mathcal{T}_c(\sigma) = +\infty$. $\qquad\square$

6.4 Proof of Proposition 2

Based on Figure 1, we only need to show that $1 + \mathcal{T}_c(0) + 1/\rho > 0$, i.e.

$$
\begin{aligned}
h_c(\rho) &= \frac{(1+\rho)}{\rho} - \frac{\bar{T}_{11}}{\bar{T}_{12}} - \frac{\bar{T}_{22}}{\rho\bar{T}_{12}} \\[2mm]
&= \frac{2(1+\rho)}{\rho} - \frac{(1-\rho\beta_1)(1-\beta_1)[\alpha_2+\rho(\alpha_1\beta_2-\alpha_2\beta_1)]}{\rho[(1-\alpha_1)(\alpha_1\beta_2-\alpha_2\beta_1)(1-\rho\beta_1)-(1-\alpha_1-\alpha_2)\alpha_1\beta_2]} > 0
\end{aligned}
$$

Consider the second partial derivatives of $T(k,y)$ evaluated at the closed-economy steady-state given in Appendix 6.2. When $\rho = 1$, we have by definition

$$h_c(1) = \frac{2\bar{T}_{12} - \bar{T}_{11} - \bar{T}_{22}}{\bar{T}_{12}} = -\frac{\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \bar{T}_{11} & \bar{T}_{12} \\ \bar{T}_{12} & \bar{T}_{22} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\bar{T}_{12}}$$

Under Assumption 1 we have $\bar{T}_{12} < 0$ and thus $h_c(1) < 0$. On the other side, we get $\lim_{\rho \to 0} h_c(\rho) = +\infty$ if and only if $2\alpha_1(\beta_1 + \beta_2) - 1 - \beta_1 > 0$. This gives Assumption 2. Therefore, under this condition there exists $\bar{\rho}_c \in (0,1)$ such that $h_c(\rho) > 0$ for any $\rho \in (0, \bar{\rho}_c)$. We then conclude from Figure 1 that when $\rho \in (0, \bar{\rho}_c)$, there exists $\bar{\sigma}_c \in (0, \sigma_c^*)$ such that the closed-economy steady-state $\bar{k}$ is saddle-point stable for all $\sigma \in (\bar{\sigma}_c, +\infty)$. The existence of period-two cycles when σ crosses $\bar{\sigma}_c$ from above follows from the flip bifurcation theorem. The bifurcation value $\bar{\sigma}_c$ is obtained as the solution of $1 + \mathcal{T}_c(\sigma) + 1/\rho = 0$ and is equal to

$$\bar{\sigma}_c = -\frac{\frac{1+\rho}{\rho} - \frac{\bar{T}_{11}}{\bar{T}_{12}} - \frac{\bar{T}_{22}}{\rho \bar{T}_{12}}}{2(1+\rho)\frac{\bar{T}_1^2}{\bar{c}\bar{T}_{12}}} = \frac{2(1+\rho)[(1-\alpha_1)(\alpha_1\beta_2 - \alpha_2\beta_1)(1-\rho\beta_1) - (1-\alpha_1-\alpha_2)\alpha_1\beta_2]}{2(1+\rho)\alpha_1[\alpha_2(\beta_1+\beta_2) + (\alpha_1\beta_2 - \alpha_2\beta_1)\rho\beta_1]}$$
$$- \frac{(1-\rho\beta_1)(1-\beta_1)[\alpha_2 + \rho(\alpha_1\beta_2 - \alpha_2\beta_1)]}{2(1+\rho)\alpha_1[\alpha_2(\beta_1+\beta_2) + (\alpha_1\beta_2 - \alpha_2\beta_1)\rho\beta_1]} \tag{61}$$

$\square$

6.5 Proof of Proposition 3

In a dynamic equilibrium, as is well known, a weighed sum of all consumers' utilities is maximized over all feasible paths, and the weight attached to each consumer's utility is equal to the inverse of the marginal utility of wealth (see Negishi [23], Bewley [7], and Yano [35] and [38]). This implies that

$$\frac{1}{\lambda^A} \sum_{t=0}^{\infty} \rho^t u^A(c_t^A) + \frac{1}{\lambda^B} \sum_{t=0}^{\infty} \rho^t u^B(c_t^B) \tag{62}$$

is maximized over all feasible path, that is to say, the following holds (see Yano [35] and [38] for a proof).

Lemma 6.4. *An equilibrium consumption path under free trade, (c_t^A, c_t^B), solves the following maximization problem:*

$$\max \left[\frac{1}{\lambda^A} \sum_{t=0}^{\infty} \rho^t u^A(c_t^A) + \frac{1}{\lambda^B} \sum_{t=0}^{\infty} \rho^t u^B(c_t^B) \right] \tag{63}$$
$$s.t.\ c_t^A + c_t^B \leq T^A(k_t^A, y_t^A) + T^B(k_t^B, y_t^B) \tag{64}$$
$$k_t^A + k_t^B \leq k_t \tag{65}$$
$$y_t^A + y_t^B \geq y_t \tag{66}$$

for given k_0.

Since all resources are utilitized efficiently along an equilibrium, Lemma 6.4 implies that, in each period, the two countries' consumption path (c_t^A, c_t^B) maximizes

$$\frac{1}{\lambda^A}\frac{(c_t^A)^{1-\sigma}}{1-\sigma} + \frac{1}{\lambda^B}\frac{(c_t^B)^{1-\sigma}}{1-\sigma} \tag{67}$$

subject to

$$c_t^A + c_t^B = c_t, \tag{68}$$

given that c_t is the world consumption path in equilibrium. Since we assume $u^A(c^A) = \frac{(c^A)^{1-\sigma}}{1-\sigma}$ and $u^B(c^B) = \frac{(c^B)^{1-\sigma}}{1-\sigma}$, the Lagrangean associated with this periodwise optimization problem can be, after dropping subscript t, written as

$$L = \frac{1}{\lambda^A}\frac{(c^A)^{1-\sigma}}{1-\sigma} + \frac{1}{\lambda^B}\frac{(c^B)^{1-\sigma}}{1-\sigma} + \gamma(c^A + c^B - c). \tag{69}$$

From this, we obtain the FOC as follows:

$$\frac{1}{\lambda^A}\left(c^A\right)^{-\sigma} = \gamma, \tag{70}$$

$$\frac{1}{\lambda^B}\left(c^B\right)^{-\sigma} = \gamma, \tag{71}$$

and

$$c^A + c^B = c. \tag{72}$$

This implies that

$$\gamma^\sigma = \frac{1}{(1/\lambda^A)^\sigma + (1/\lambda^B)^\sigma}c, \tag{73}$$

$$c^A = \frac{\left(1/\lambda^A\right)^\sigma}{(1/\lambda^A)^\sigma + (1/\lambda^B)^\sigma}c, \tag{74}$$

and

$$c^B = \frac{\left(1/\lambda^B\right)^\sigma}{(1/\lambda^A)^\sigma + (1/\lambda^B)^\sigma}c. \tag{75}$$

By solving these expressions, we obtain

$$\frac{1}{\lambda^A}u^A(c^A) + \frac{1}{\lambda^B}u^B(c^B) = \mu c^{1-\sigma}, \tag{76}$$

where

$$\mu = \frac{1}{1-\sigma}\left(\frac{1}{\lambda^A}\left(\frac{(1/\lambda^A)^\sigma}{(1/\lambda^A)^\sigma + (1/\lambda^B)^\sigma}\right)^{1-\sigma} + \frac{1}{\lambda^B}\left(\frac{(1/\lambda^B)^\sigma}{(1/\lambda^A)^\sigma + (1/\lambda^B)^\sigma}\right)^{1-\sigma}\right).$$

This implies that, under free trade, an equilibrium aggregate consumption path maximizes

$$\sum_{t=0}^\infty \rho^t (c_t)^{1-\sigma} \tag{77}$$

over all feasible aggregate consumption paths from k_0. Once the aggregate consumption path, c_t, is determined, an equilibrium consumption path for the countries, (c_t^A, c_t^B), is determined by the above FOC. In other words, each country's consumption path depends on the marginal utilities of wealth, λ^A and λ^B.

$\square$

6.6 Proof of Proposition 4

A steady-state is obtained as a solution (k^A, k^B, y^A, y^B, k) of the following system

$$T_2^A(k^A, y^A) + \rho T_1^A(k^A, y^A) = 0 \tag{78}$$

$$T_2^B(k^B, y^B) + \rho T_1^B(k^B, y^B) = 0 \tag{79}$$

$$T_1^A(k^A, y^A) - T_1^B(k^B, y^B) = 0 \tag{80}$$

$$T_2^A(k^A, y^A) - T_2^B(k^B, y^B) = 0 \tag{81}$$

$$k^A + k^B = y^A + y^B = k \tag{82}$$

with $c^A + c^B = T^A(k^A, k^A) + T^B(k^B, k^B)$. We get from equations (78)-(79) the following property for a steady-state under free-trade:

Lemma 6.5. *At a steady-state under free-trade, the following relationship holds:*
$$\frac{g^A}{y^A \beta_1^A} = \frac{g^B}{y^B \beta_1^B} = \rho$$

Proof: The first order conditions (80)-(81) show that $T_j^A(k^A, y^A) = T_j^B(k^B, y^B)$, $j = 1, 2$. Since $T_2(k, y) = -T_1(k, y)g/y\beta_1$, we derive that $g^A/y^A \beta_1^A = g^B/y^B \beta_1^B$. Consider now the Euler equation (22) evaluated at a steady-state under free-trade. We get $-T_2(k, y) = \rho T_1(k, y)$ and the result follows. $\qquad\square$

We may now prove Proposition 4. Consider equations (78)-(82). We know from Lemma 6.5 that equations (78) and (79) imply $g = \rho \beta_1 y$, $i = A, B$. Assume then that $k^A = \theta y^A$ and $k^B = y^B/\theta$ with $\theta > 1$ some constant. It follows from (82) that $k^A = \theta k^B$, $y^B = \theta y^A$ and $g^A = \rho \beta_1^A k^A/\theta$, $g^B = \rho \beta_1^B k^B \theta$. Substituting these expressions into (46) with $K_y = g$ and solving for k, $i = A, B$, gives

$$k^{A*} = \theta \left(\frac{\alpha_1^A \beta_2^A}{\Phi_\theta^A} \right)^{\frac{\beta_2^A}{1-\beta_1^A}} \left[\mathcal{E}_y^A (\rho \beta_1^A)^{\beta_1^A + \beta_2^A} \right]^{\frac{1}{1-\beta_1^A}}$$

$$k^{B*} = \left(\frac{\alpha_1^B \beta_2^B}{\Phi_\theta^B} \right)^{\frac{\beta_2^B}{1-\beta_1^B}} \left[\frac{\mathcal{E}_y^B (\rho \beta_1^B \theta)^{\beta_1^B + \beta_2^B}}{\theta} \right]^{\frac{1}{1-\beta_1^B}} \tag{83}$$

with

$$\Phi_\theta^A = \alpha_2^A \beta_1^A \theta + (\alpha_1^A \beta_2^A - \alpha_2^A \beta_1^A) \rho \beta_1^A \text{ and } \Phi_\theta^B = \alpha_2^B \beta_1^B + (\alpha_1^B \beta_2^B - \alpha_2^B \beta_1^B) \rho \beta_1^B \theta$$

Assuming $\mathcal{E}_y^B = 1$, we derive from (83) that $k^{A*} = \theta k^{B*}$ if and only if $\theta \in (1, 1/\rho \beta_1^B)$ and $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$ with

$$\mathcal{E}_y^{A*} = \frac{\left(\alpha_1^B \beta_2^B / \Phi_\theta^B\right)^{\frac{\beta_2^B(1-\beta_1^A)}{1-\beta_1^B}}}{(\rho\beta_1^A)^{\beta_1^A + \beta_2^A}\left(\alpha_1^A \beta_2^A / \Phi_\theta^A\right)^{\beta_2^A}} \left[\frac{(\rho\beta_1^B\theta)^{\beta_1^B + \beta_2^B}}{\theta}\right]^{\frac{1-\beta_1^A}{1-\beta_1^B}} \tag{84}$$

Considering $T_1(k,y)$ in Lemma 6.1 with $\mathcal{E}_c^B = 1$, equation (80) is satisfied if and only if $\theta \in (1, 1/\rho\beta_1^B)$ and $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$ with

$$\mathcal{E}_c^{A*} = \frac{(\Phi_\theta^A)^{\alpha_2^A}\alpha_1^B(\alpha_2^B\beta_1^B)^{\alpha_2^B}(1-\rho\beta_1^B\theta)^{\alpha_1^B + \alpha_2^B - 1}(k^{B*})^{\alpha_1^B - \alpha_1^A}}{\alpha_1^A(\alpha_2^A\beta_1^A)^{\alpha_2^A}(\theta-\rho\beta_1^A)^{\alpha_1^A + \alpha_2^A - 1}(\Phi_\theta^B)^{\alpha_2^B}} \tag{85}$$

Then, since from (78) and (79) we have $T_1(k,y) = T_2(k,y)/\rho$, $i = A, B$, equation (81) also holds.

Country i's production of the consumption good is derived from (2) as $T^{i*} = T(k^{i*}, k^{i*})$. We know that country A imports capital goods while country B exports capital goods: $\mathcal{M}_y^A = (\theta - 1)y^{A*}$ and $\mathcal{X}_y^B = \left(\frac{\theta-1}{\theta}\right)y^{B*}$. In order to have a balance of trade in equilibrium, country A has to export consumption goods while country B has to import consumption goods. Let $\eta > 1$ be such that

$$c^{A*} = \frac{T^{A*}}{\eta} < T^{A*}, \quad c^{B*} = \eta T^{B*} > T^{B*}$$

It follows that

$$\mathcal{X}_c^A = \left(\frac{\eta-1}{\eta}\right)T^{A*}, \quad \mathcal{M}_c^B = (\eta - 1)T^{B*}$$

Therefore, the balance of trade is in equilibrium in each country if $\mathcal{NX}^A = \mathcal{X}_c^A - p\mathcal{M}_y^A = 0$ and $\mathcal{NX}^B = p\mathcal{X}_y^B - \mathcal{M}_c^B = 0$, or equivalently

$$(\theta - 1)py^{A*} = \left(\frac{\eta-1}{\eta}\right)T^{A*}, \quad \left(\frac{\theta-1}{\theta}\right)py^{B*} = (\eta - 1)T^{B*}$$

with p the relative price of the investment good. Taking the ratio of these expressions yields $T^{A*}/T^{B*} = \eta$. Now let $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$ and $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$, $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$ as given by (84) and (85), and consider $T(k,y)$ as defined by (2). We get

$$T^{A*} = \frac{\alpha_1^B(\theta - \rho\beta_1^A)}{\alpha_1^A(1 - \rho\beta_1^B\theta)}T^{B*} \equiv \eta T^{B*} \tag{86}$$

with $\eta = \alpha_1^B(\theta - \rho\beta_1^A)/\alpha_1^A(1 - \rho\beta_1^B\theta) > 1$ for any $\theta > 1$ under Assumption 3. Therefore, we conclude that

$$c^{A*} = T^{B*}, \quad c^{B*} = \eta T^{A*}$$

The result follows assuming $\theta \in (1, \bar{\theta})$ with $\bar{\theta} = 1/\rho\beta_1^B$.

$\square$

6.7 The characteristic polynomial under free-trade

At a steady-state under free-trade we have $y^* = k^*$. Let us introduce the following notations:

$$\mathcal{T}(k,y) = T^A(k^A(k,y), y^A(k,y)) + T^B(k^B(k,y), y^B(k,y)) \qquad (87)$$

and $\mathcal{T}_{ij}^* = \mathcal{T}_{ij}(k^*, k^*)$, $i,j = 1,2$. We thus have $c^* = \mathcal{T}(k^*, k^*)$ and

$$V(k_t, y_t) = \frac{\mathcal{T}(k_t, y_t)^{1-\sigma}}{1-\sigma}$$

Total differenciation of equation (22) gives

$$dk_{t+2}\left[1 + \sigma\rho\frac{\mathcal{T}_1^{*2}}{c^*\mathcal{T}_{12}^*}\right] + dk_{t+1}\left[\frac{\mathcal{T}_{22}^*}{\rho\mathcal{T}_{12}^*} + \frac{\mathcal{T}_{11}^*}{\mathcal{T}_{12}^*} - \sigma(1+\rho)\frac{\mathcal{T}_1^{*2}}{c^*\mathcal{T}_{12}^*}\right] + dk_t\left[\frac{1}{\rho} + \sigma\frac{\mathcal{T}_1^{*2}}{c^*\mathcal{T}_{12}^*}\right] = 0$$

Straightforward computations give the characteristic polynomial

$$\mathcal{P}_f(x) = x^2 - \mathcal{T}_f(\sigma)x + \mathcal{D}_f = 0 \qquad (88)$$

with

$$\mathcal{D}_f = \frac{1}{\rho}, \quad \mathcal{T}_f(\sigma) = \frac{-\frac{\mathcal{T}_{11}^*}{\mathcal{T}_{12}^*} - \frac{\mathcal{T}_{22}^*}{\rho\mathcal{T}_{12}^*} + \sigma(1+\rho)\frac{\mathcal{T}_1^{*2}}{c^*\mathcal{T}_{12}^*}}{1 + \sigma\rho\frac{\mathcal{T}_1^{*2}}{c^*\mathcal{T}_{12}^*}} \qquad (89)$$

and $\mathcal{T}_1^* = T_1^A(k^A(k^*, k^*), y^A(k^*, k^*)) = T_1^B(k^B(k^*, k^*), y^B(k^*, k^*))$. We know from Nishimura and Yano [26] that:

Lemma 6.6. *Along a free-trade equilibrium, the second partial derivatives of $\mathcal{T}(k,y)$ satisfy the following:*

$$\mathcal{T}_{11}(k,y) = \frac{1}{\Theta}\left[T_{11}^A(k^A, y^A)|H^B(k^B, y^B)| + T_{11}^B(k^B, y^B)|H^A(k^A, y^A)|\right]$$

$$\mathcal{T}_{12}(k,y) = \frac{1}{\Theta}\left[T_{12}^A(k^A, y^A)|H^B(k^B, y^B)| + T_{12}^B(k^B, y^B)|H^A(k^A, y^A)|\right]$$

$$\mathcal{T}_{22}(k,y) = \frac{1}{\Theta}\left[T_{22}^A(k^A, y^A)|H^B(k^B, y^B)| + T_{22}^B(k^B, y^B)|H^A(k^A, y^A)|\right]$$

where

$$|H^i(k^i, y^i)| \equiv T_{11}^i(k^i, y^i)T_{22}^i(k^i, y^i) - T_{12}^i(k^i, y^i)^2 > 0$$

$$\Theta \equiv \left[T_{11}^A(k^A, y^A) + + T_{11}^B(k^B, y^B)\right]\left[T_{22}^A(k^A, y^A) + T_{22}^B(k^B, y^B)\right]$$

$$- \left[T_{12}^A(k^A, y^A) + T_{12}^B(k^B, y^B)\right]^2 > 0$$

Final simplifications give:

$$\frac{\mathcal{T}_{11}^*}{\mathcal{T}_{12}^*} = \frac{T_{11}^A(k^A, y^A)|H^B(k^B, y^B)| + T_{11}^B(k^B, y^B)|H^A(k^A, y^A)|}{T_{12}^A(k^A, y^A)|H^B(k^B, y^B)| + T_{12}^B(k^B, y^B)|H^A(k^A, y^A)|}$$

$$\frac{\mathcal{T}_{22}^*}{\rho\mathcal{T}_{12}^*} = \frac{T_{22}^A(k^A, y^A)|H^B(k^B, y^B)| + T_{22}^B(k^B, y^B)|H^A(k^A, y^A)|}{\rho\left[T_{12}^A(k^A, y^A)|H^B(k^B, y^B)| + T_{12}^B(k^B, y^B)|H^A(k^A, y^A)|\right]}$$

$$\frac{\mathcal{T}_1^{*2}}{c^*\mathcal{T}_{12}^*} = \frac{\Theta T_1^A(k^A(k^*, k^*), y^A(k^*, k^*))^2}{c^*\left[T_{12}^A(k^A, y^A)|H^B(k^B, y^B)| + T_{12}^B(k^B, y^B)|H^A(k^A, y^A)|\right]}$$

$\square$

6.8 The function $\mathcal{T}_f(\sigma)$

Note first that under Assumption 4

$$\mathcal{T}_f(0) = -\frac{T_{11}^*}{T_{12}^*} - \frac{T_{22}^*}{\rho T_{12}^*} < 0 \text{ and } \mathcal{T}_f(+\infty) = 1 + \rho^{-1} \geq 2$$

Lemma 6.7. *Under Assumption 4, there exists $\sigma_f^* \in (0, +\infty)$ such that $\mathcal{T}_f(\sigma)$ is continuous and monotonically decreasing in σ over $[0, \sigma_f^*) \cup (\sigma_f^*, +\infty)$. Moreover, $\lim_{\sigma \to \sigma_{f-}^*} \mathcal{T}_c(\sigma) = -\infty$ and $\lim_{\sigma \to \sigma_{f+}^*} \mathcal{T}_c(\sigma) = +\infty$.*

Proof: Straightforward computations from (89) give

$$\mathcal{T}_f'(\sigma) = \frac{T_1^{*2}}{c^* T_{12}^*} \frac{1 + \rho\left(1 + \frac{T_{11}^*}{T_{12}^*} + \frac{T_{22}^*}{\rho T_{12}^*}\right)}{\left[1 + \sigma\rho\frac{T_1^{*2}}{c^* T_{12}^*}\right]^2}$$

$$= \frac{\Theta T_1^{*2}}{c^*\left[T_{12}^A|H^B| + T_{12}^B|H^A|\right]^2} \frac{T_{12}^A|H^B|\left[1 + \rho\left(1 + \frac{T_{11}^A}{T_{12}^A} + \frac{T_{22}^A}{\rho T_{12}^A}\right)\right] + T_{12}^B|H^A|\left[1 + \rho\left(1 + \frac{T_{11}^B}{T_{12}^B} + \frac{T_{22}^B}{\rho T_{12}^B}\right)\right]}{\left[1 + \sigma\rho\frac{T_1^{*2}}{c^* T_{12}^*}\right]^2}$$

with $|H^A| > 0$ and $|H^B| > 0$. Under Assumption 4 we get for any $\rho \in (0, 1)$:

$$T_{12}^A = -\frac{T_1^A(k^A/\theta)\rho}{k^A - g^A} \left\{\frac{(1-\alpha_1^A)(\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A)(\theta - \rho\beta_1^A) - (1-\alpha_1^A - \alpha_2^A)\alpha_1^A\beta_2^A\theta}{\theta\alpha_2^A(\beta_1^A + \beta_2^A) + (\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A)\rho\beta_1^A}\right\} < 0$$

$$T_{12}^B = -\frac{T_1^B k^B \rho}{k^B - g^B} \left\{\frac{(1-\alpha_1^B)(\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B)(1 - \rho\beta_1^B\theta) - (1-\alpha_1^B - \alpha_2^B)\alpha_1^B\beta_2^B}{\alpha_2^B(\beta_1^B + \beta_2^B) + (\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B)\rho\beta_1^B\theta}\right\} < 0$$

and

$$1 + \rho\left(1 + \frac{T_{11}^A}{T_{12}^A} + \frac{T_{22}^A}{\rho T_{12}^A}\right) = (\theta - \rho\beta_1^A)\frac{\rho(1-\beta_1^A)(\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A) + \alpha_2^A[\beta_2^A + (1-\beta_1^A-\beta_2^A)\theta]}{(1-\alpha_1^A)(\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A)(\theta - \rho\beta_1^A) - (1-\alpha_1^A-\alpha_2^A)\alpha_1^A\beta_2^A\theta} > 0$$

$$1 + \rho\left(1 + \frac{T_{11}^B}{T_{12}^B} + \frac{T_{22}^B}{\rho T_{12}^B}\right) = (1 - \rho\beta_1^B\theta)\frac{\rho(1-\beta_1^B)(\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B) + \frac{\alpha_2^B}{\theta}(\theta\beta_2^B + 1 - \beta_1^B - \beta_2^B)}{(1-\alpha_1^B)(\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B)(1 - \rho\beta_1^B\theta) - (1-\alpha_1^B-\alpha_2^B)\alpha_1^B\beta_2^B} > 0$$

It follows that $\mathcal{T}_f'(\sigma) < 0$. Moreover, there exists a critical value σ_f^* as defined by

$$\sigma_f^* = -\frac{c^* T_{12}^*}{\rho T_1^{*2}} = -\frac{c^*\left[T_{12}^A|H^B| + T_{12}^B|H^A|\right]}{\rho\Theta(T_1^A)^2} \tag{90}$$

such that the denominator of $\mathcal{T}_f(\sigma)$ is equal to zero when $\sigma = \sigma_f^*$. When $\sigma = \sigma_f^*$, the numerator of $\mathcal{T}_f(\sigma)$ is equal to

$$-\frac{T_{11}^*}{T_{12}^*} - \frac{T_{22}^*}{\rho T_{12}^*} - \frac{1+\rho}{\rho} = -\frac{1}{\rho}\left[1 + \rho\left(1 + \frac{T_{11}^*}{T_{12}^*} + \frac{T_{22}^*}{\rho T_{12}^*}\right)\right]$$

$$= -\frac{1}{\rho}\frac{T_{12}^A|H^B|\left[1 + \rho\left(1 + \frac{T_{11}^A}{T_{12}^A} + \frac{T_{22}^A}{\rho T_{12}^A}\right)\right] + T_{12}^B|H^A|\left[1 + \rho\left(1 + \frac{T_{11}^B}{T_{12}^B} + \frac{T_{22}^B}{\rho T_{12}^B}\right)\right]}{T_{12}^A|H^B| + T_{12}^B|H^A|}$$

which is negative for any $\rho \in (0, 1)$ under Assumption 4. It follows that $\lim_{\sigma \to \sigma_{f-}^*} \mathcal{T}_f(\sigma) = -\infty$ and $\lim_{\sigma \to \sigma_{f+}^*} \mathcal{T}_f(\sigma) = +\infty$. $\qquad\square$

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6.9 Proof of Proposition 5

Proceeding as in the case of the closed economy, we only need to show that $1 + \mathcal{T}_f(0) + 1/\rho > 0$, i.e.

$$h_f(\rho) = \frac{(1+\rho)}{\rho} - \frac{\mathcal{T}_{11}^*}{\mathcal{T}_{12}^*} - \frac{\mathcal{T}_{22}^*}{\rho \mathcal{T}_{12}^{i*}}$$

$$= \frac{T_{12}^A |H^B| \left[1 + \rho \left(1 - \frac{T_{11}^A}{T_{12}^A} - \frac{T_{22}^A}{\rho T_{12}^A}\right)\right] + T_{12}^B |H^A| \left[1 + \rho \left(1 - \frac{T_{11}^B}{T_{12}^B} - \frac{T_{22}^B}{\rho T_{12}^B}\right)\right]}{\rho \left[T_{12}^A |H^B| + T_{12}^B |H^A|\right]} > 0$$

When $\rho = 1$, we have

$$h_f(1) = \frac{2\mathcal{T}_{12}^* - \mathcal{T}_{11}^* - \mathcal{T}_{22}^*}{\mathcal{T}_{12}^*} = -\frac{\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \mathcal{T}_{11}^* & \mathcal{T}_{12}^* \\ \mathcal{T}_{12}^* & \mathcal{T}_{22}^* \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\mathcal{T}_{12}^*}$$

Under Assumption 4 we have $\mathcal{T}_{12}^* < 0$ and the function $\mathcal{T}(k, y)$ as defined by (87) is obviously strictly concave so that $h_f(1) < 0$. We also get:

$$1 + \rho \left(1 - \frac{T_{11}^A}{T_{12}^A} - \frac{T_{22}^A}{\rho T_{12}^A}\right) = 2(1+\rho)$$
$$- \frac{(\theta - \rho\beta_1^A)\left[\rho(1-\beta_1^A)(\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A) + \alpha_2^A[\beta_2^A + (1-\beta_1^A-\beta_2^A)\theta]\right]}{(1-\alpha_1^A)(\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A)(\theta - \rho\beta_1^A) - (1-\alpha_1^A-\alpha_2^A)\alpha_1^A\beta_2^A\theta}$$

$$1 + \rho \left(1 - \frac{T_{11}^B}{T_{12}^B} - \frac{T_{22}^B}{\rho T_{12}^B}\right) = 2(1+\rho)$$
$$- \frac{(1-\rho\beta_1^B\theta)\left[\rho(1-\beta_1^B)(\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B) + \frac{\alpha_2^B}{\theta}(\theta\beta_2^B + 1 - \beta_1^B - \beta_2^B)\right]}{(1-\alpha_1^B)(\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B)(1-\rho\beta_1^B\theta) - (1-\alpha_1^B-\alpha_2^B)\alpha_1^B\beta_2^B}$$

and thus under Assumptions 4 and 5

$$\lim_{\rho \to 0} 1 + \rho \left(1 - \frac{T_{11}^A}{T_{12}^A} - \frac{T_{22}^A}{\rho T_{12}^A}\right) = \frac{\alpha_2^A\left[2\alpha_1^A(\beta_1^A + \beta_2^A) - 2\beta_1^A - \beta_2^A - (1 - \beta_1^A - \beta_2^A)\theta\right]}{(1-\alpha_1^A)(\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A) - (1-\alpha_1^A-\alpha_2^A)\alpha_1^A\beta_2^A} > 0$$

$$\lim_{\rho \to 0} 1 + \rho \left(1 - \frac{T_{11}^B}{T_{12}^B} - \frac{T_{22}^B}{\rho T_{12}^B}\right) = \frac{2\alpha_1^B\theta(\beta_1^B + \beta_2^B) - \theta(2\beta_1^B + \beta_2^B) - (1 - \beta_1^B - \beta_2^B)}{\theta\left[(1-\alpha_1^B)(\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B) - (1-\alpha_1^B-\alpha_2^B)\alpha_1^B\beta_2^B\right]} > 0$$

It follows that $\lim_{\rho \to 0} h_f(\rho) = +\infty$. Therefore, there exists $\bar{\rho}_f \in (0, 1)$ such that $h_f(\rho) > 0$ for any $\rho \in (0, \bar{\rho}_f)$. We then conclude that when $\rho \in (0, \bar{\rho}_f)$, there exists $\bar{\sigma}_f \in (0, \sigma_f^*)$ such that the free-trade steady-state k^* is saddle-point stable for all $\sigma \in (\bar{\sigma}_f, +\infty)$. The existence of period-two cycles when σ crosses $\bar{\sigma}_f$ from above follows from the flip bifurcation theorem. The bifurcation value $\bar{\sigma}_f$ is obtained as the solution of $1 + \mathcal{T}_f(\sigma) + 1/\rho = 0$ and is equal to

$$\bar{\sigma}_f = -\frac{\frac{1+\rho}{\rho} - \frac{\mathcal{T}_{11}^*}{\mathcal{T}_{12}^*} - \frac{\mathcal{T}_{22}^*}{\rho \mathcal{T}_{12}^*}}{2(1+\rho)\frac{\mathcal{T}_1^{*2}}{c^* \mathcal{T}_{12}^*}}$$

$$= -\frac{c^*\left[T_{12}^A |H^B| \left(\frac{1+\rho}{\rho} - \frac{T_{11}^A}{T_{12}^A} - \frac{T_{22}^A}{\rho T_{12}^A}\right) + T_{12}^B |H^A| \left(\frac{1+\rho}{\rho} - \frac{T_{11}^B}{T_{12}^B} - \frac{T_{22}^B}{\rho T_{12}^B}\right)\right]}{2(1+\rho)\Theta(T_1^A)^2}$$

(91)

$\square$

6.10 Proof of Theorem 1

Let us introduce the following notations:

$$\frac{\alpha_1\beta_2(1-\alpha_1-\alpha_2)}{(1-\alpha_1)(1-\rho\beta_1)} \equiv \mathcal{Z}_1^i, \quad \frac{\alpha_1^A\beta_2^A\theta(1-\alpha_1^A-\alpha_2^A)}{(1-\alpha_1^A)(\theta-\beta_1^A)} \equiv \mathcal{Z}_2^A, \quad \frac{\alpha_1^B\beta_2^B(1-\alpha_1^B-\alpha_2^B)}{(1-\alpha_1^B)(1-\theta\beta_1^B)} \equiv \mathcal{Z}_2^B$$

$$\frac{1+\beta_1^B}{2(\beta_1^B+\beta_2^B)} \equiv \mathcal{Z}_3^B, \quad \frac{1-\beta_1^B-\beta_2^B+\theta(2\beta_1^B+\beta_2^B)}{2\theta(\beta_1^B+\beta_2^B)} \equiv \mathcal{Z}_4^B$$

In the proof of Proposition 2, assume that inequality (59) applied to country A is not satisfied, i.e. $\alpha_1^A\beta_2^A - \alpha_2^A\beta_1^A < \mathcal{Z}_1^A$ for any given $\rho \in (0,1)$. It follows that the closed-economy steady-state of country A is saddle-point stable with monotone convergence for any given $\rho \in (0,1)$. Consider then condition (26) in Assumption 4. Straightforward computations give $\mathcal{Z}_1^A > \mathcal{Z}_2^A$. It follows that all the conditions of Proposition 5 for country A may be satisfied for the free-trade steady-state while the closed-economy steady-state is characterized by monotone convergence for any $\sigma \geq 0$.

For country B we have $\mathcal{Z}_1^B < \mathcal{Z}_2^B$ for any given $\rho \in (0,1)$ and $\mathcal{Z}_3^B > \mathcal{Z}_4^B$ since $\theta > 1$. It follows that if $\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B > \mathcal{Z}_2^B$ and $\alpha_1^B > \mathcal{Z}_3^B$, then all the conditions of Proposition 5 for country B are satisfied along the free-trade steady-state while the closed-economy optimal path is also characterized by endogenous fluctuations (period-2 cycles) in the neighborhood of the flip bifurcation value $\bar{\sigma}_c^B$. As a result endogenous fluctuations arise at the world level while country A is characterized by monotone convergence in the closed-economy case. $\square$

6.11 Proof of Theorem 2

Consider the same characterization for country A as in the proof of Theorem 1. Then all the conditions of Proposition 5 for country A are satisfied for the free-trade steady-state while the closed-economy steady-state is characterized by monotone convergence for any $\sigma \geq 0$.

If for country B, $\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B > \mathcal{Z}_2^B$ and $\mathcal{Z}_3^B > \alpha_1^B > \mathcal{Z}_4^B$, then all the conditions of Proposition 5 for country B are satisfied along the free-trade steady-state while the closed-economy optimal path is characterized by saddle-point stability with damped fluctuations or monotone convergence depending on whether $\sigma \lessgtr \sigma_c^{B*}$. As a result endogenous fluctuations arise at the world level while both closed-economy countries are characterized by saddle-point stability. $\square$

6.12 Proof of Theorem 4

Consider the bound $\bar{\sigma}_c^B$ and $\bar{\sigma}_f$ as defined by (61) and (91) respectively. We get $\bar{\sigma}_f > \bar{\sigma}_c^B$ if and only if

$$-\frac{c^*\left[T_{12}^A|H^B|\left(\frac{1+\rho}{\rho}-\frac{T_{11}^A}{T_{12}^A}-\frac{T_{22}^A}{\rho T_{12}^A}\right)+T_{12}^B|H^A|\left(\frac{1+\rho}{\rho}-\frac{T_{11}^B}{T_{12}^B}-\frac{T_{22}^B}{\rho T_{12}^B}\right)\right]}{\Theta(T_1^A)^2}$$
$$> -\frac{\bar{c}^B\bar{T}_{12}^B\left(\frac{1+\rho}{\rho}-\frac{\bar{T}_{11}^B}{\bar{T}_{12}^B}-\frac{\bar{T}_{22}^B}{\rho\bar{T}_{12}^B}\right)}{(\bar{T}_1^B)^2} \tag{92}$$

Using the fact that along the free-trade steady-state with $\mathcal{E}_c^B = \mathcal{E}_y^B = 1$, $\mathcal{E}_c^A = \mathcal{E}_c^{A*}$, $\mathcal{E}_y^A = \mathcal{E}_y^{A*}$ and $\theta \in (1,\bar{\theta})$, we have $k^{A*} = \theta y^{A*}$, $k^{B*} = y^{B*}/\theta$, $k^{A*} = \theta k^{B*}$, $c^* = c^{A*} + c^{B*} = T^{A*} + T^{B*}$ and $T_j^A(k^A,y^A) = T_j^B(k^B,y^B)$, $j = 1,2$, we derive after tedious computations that when $1 - \alpha_1^B - \alpha_2^B = 0$, inequality (92) is equivalent to the following

$$\frac{1}{\Theta}\left(\frac{\theta-\rho\beta_1^A}{\alpha_1^A}+\frac{1-\rho\beta_1^B\theta}{\alpha_1^B}\right)\left\{\frac{|H^B|}{(\theta-\rho\beta_1^A)}\left(\frac{2(1+\rho)\left[(1-\alpha_1^A)(\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)(\theta-\rho\beta_1^A)-(1-\alpha_1^A-\alpha_2^A)\alpha_1^A\beta_2^A\theta\right]}{\theta\alpha_2^A(\beta_1^A+\beta_2^A)+(\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)\rho\beta_1^A}\right.\right.$$

$$\left.-\frac{(\theta-\rho\beta_1^A)\left[\rho(1-\beta_1^A)(\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)+\alpha_2^A[\beta_2^A+(1-\beta_1^A-\beta_2^A)\theta]\right]}{\theta\alpha_2^A(\beta_1^A+\beta_2^A)+(\alpha_1^A\beta_2^A-\alpha_2^A\beta_1^A)\rho\beta_1^A}\right)$$

$$\left.+|H^A|\left(\frac{2(1+\rho)\alpha_2^B(\alpha_1^B\beta_2^B-\alpha_2^B\beta_1^B)-\left[\rho(1-\beta_1^B)(\alpha_1^B\beta_2^B-\alpha_2^B\beta_1^B)+\frac{\alpha_2^B}{\theta}(\theta\beta_2^B+1-\beta_1^B-\beta_2^B)\right]}{\alpha_2^B(\beta_1^B+\beta_2^B)+(\alpha_1^B\beta_2^B-\alpha_2^B\beta_1^B)\rho\beta_1^B\theta}\right)\right\} >$$

$$(1-\rho\beta_1^B)\frac{(\alpha_1^B\beta_2^B-\alpha_2^B\beta_1^B)\left[2+\rho(1+\beta_1^B)-2(1+\rho)\alpha_1^B\right]-(1-\beta_1^B)\alpha_2^B}{\alpha_1^B\left[\alpha_2^B(\beta_1^B+\beta_2^B)+(\alpha_1^B\beta_2^B-\alpha_2^B\beta_1^B)\rho\beta_1^B\right]}$$

The modified condition (38) in Assumption 7 implies that $2 + \rho(1 + \beta_1^B) - 2(1+\rho)\alpha_1^B > 0$ for any $\rho \in (0,1)$. Moreover, there exists $\epsilon > 0$ such that if

$$\alpha_1^B\beta_2^B - \alpha_2^B\beta_1^B \in \left(\frac{(1-\beta_1^B)\alpha_2^B}{2+\rho(1+\beta_1^B)-2(1+\rho)\alpha_1^B}, \frac{(1-\beta_1^B)\alpha_2^B}{2+\rho(1+\beta_1^B)-2(1+\rho)\alpha_1^B}+\epsilon\right),$$

the right-hand-side is positive but lower than the left-hand-side. The result follows. $\square$

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