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Per-Unit Royalty vs Fixed Fee: The Case of Weak Patents*

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Abstract

This paper explores a licensor's choice between charging a per-unit royalty or a fixed fee when her innovation is covered by a weak patent, i.e. a patent that is likely to be invalidated by a court if challenged. Using a general model where the nature of competition is not specified, we show that the patent holder prefers to use a per-unit royalty scheme if the *strategic* effect of an increase in a potential licensee's unit cost on the aggregate equilibrium profit is positive. To show the mildness of the latter condition, we establish that it holds in a Cournot (resp. Bertrand) oligopoly with homegenous (resp. heterogenous) products under very general assumptions on the demands faced by firms. As a byproduct of our analysis, we contribute to the oligopoly literature by offering some new insights of independent interest regarding the effects of cost variations on Cournot and Bertrand equilibria.

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1 Introduction

Early theoretical works on licensing (Kamien and Tauman 1984, 1986) have concluded that a patent holder should charge a fixed fee rather than per-unit royalties when licensing an innovation. This finding is however in sharp contrast with what is observed in the real world: Per-unit royalties seem to be more often used than fixed fees (see e.g Taylor and Silberstone, 1973; Rostoker, 1984).[1] Various reasons have been put forward to explain the prevalence of per-unit royalties in practice, including risk aversion (Bousquet et al. 1998), product differentiation (Muto, 1993; Wang and Yang, 1999; Poddar and Sinha, 2004; Stamatopoulos and Tauman, 2007), asymmetry of information (Gallini and Wright, 1990; Macho-Stadler and Pérez-Castrillo, 1991; Beggs, 1992; Poddar and Sinha, 2002; Sen, 2005), moral hazard (Macho-Stadler et al., 1996; Choi, 2001), incumbent innovator (Shapiro, 1985; Wang, 1998, 2002; Kamien and Tauman, 2002; Sen, 2002; Sen and Tauman, 2007), profits and sales objective through delegation (Saracho, 2002), leadership structure (Kabiraj, 2004) or variation in the quality of innovation (Rockett, 1990).

A common feature of all those papers is that patents are viewed as *ironclad rights,* the validity of which is unquestionable. However this contrasts with the observation that many patents have been challenged by third parties and actually invalidated by courts (Allison and Lemley, 1998). Furthermore, many authors have argued that patents should be considered as *probabilistic rights* because they do not give the right to exclude but rather a more limited right to "try to exclude" by asserting the patent in court (Ayres and Klemperer, 1999; Shapiro, 2003; Lemley and Shapiro, 2005).

Most commentators agree that many "innovations" are granted patent protection even though they probably do not meet patentability standards. Such issued patents are *weak* in the sense that they are likely to be invalidated by a court if challenged.[2] The proliferation of those questionable patents can be explained by several reasons. First, the major patent offices (USPTO, EPO and JPO) have insufficient resources to ensure an effective review process for the huge (and growing) number of patent applications (IDEI report, 2006).

[1] Available data on patent licensing is limited and scattered because the disclosure of licensing contracts crucially depends on firms' policy. Most firms elect not to make such information public. The empirical investigations by Annand and Khanna (2000) and Vonortas and Kim (2004) emphasize the factors that affect the likelihood of firms to engage in licensive agreements but are less informative on the choice of the licensing scheme.

[2] The notion of "weak patent" has at least two different meanings in the literature (see Ginarte and Park, 1997, van Pottelsberghe de la Potterie, 2010). A first meaning is that a patent is considered weak when it gives to its holder a low protection against imitators and other potential infringers, either because the patent's scope is badly defined or because the geographical protection extends to countries in which the enforcement of intellectual property rights is low. Another meaning is that a patent is weak when it is likely that it does not satisfy at least one of the patentability standards (novelty, non obviousness, utility, patentable subject matter). Therefore, it may be invalidated by a court if it is challenged by a third party. This paper focuses on that second meaning (Lemley and Shapiro, 2005).

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Second, the patentability standards, in particular novelty and non-obviousness, are difficult to assess especially for new patentable subject matters. Third, the incentives provided to patent office examiners are inadequate for making them reject the applications that do not meet the standards (Farrell and Merges, 2004; Langinier and Marcoul, 2009). Finally, the patentable subject matters are still debated and differ according to continents (Guellec and van Pottelsberghe, 2007).

In this paper, we compare the per-unit royalty scheme and the fixed fee scheme for the licensing of weak patents, hence extending the literature on the comparison between these two licensing mechanisms for ironclad patents. Our main contribution is to provide a novel justification, based on the uncertainty over patent validity, for the use of per-unit royalties instead of fixed fees in licensing agreements. More specifically, we establish that, for weak patents, the patent holder is better off using the optimal per-unit royalty licensing scheme deterring patent litigation rather than the optimal fixed fee licensing scheme deterring patent litigation if the *strategic* effect of an increase in one firm's marginal cost on the aggregate equilibrium profits is positive. We show that this condition is mild in that it holds under very general conditions for two of the most usual imperfect competition frameworks: Cournot oligopoly with homegenous goods and Bertrand oligopoly with differentiated goods. As a byproduct of our analysis, we contribute to the oligopoly literature by offering some new insights of independent interest regarding the effects of cost increases on Cournot and Bertrand equilibria.

The closest papers to ours are Farrell and Shapiro (2008) and Encaoua and Lefouili (2009) who also deal with the licensing of weak patents in the shadow of patent litigation, though with objectives that are different from the present paper's. In each of those two papers, the licensor is assumed to offer two-part tariff licensing contracts but some of the results are derived under a simplifying *ad hoc* assumption on the shape of the licensing revenue function which immediately ensures that, for weak patents, pure per-unit royalty licenses are optimal from the patent holder's perspective in the class of two-part tariff licensing contracts (with non-negative fixed fees) deterring litigation and, therefore, makes the analysis easier. Our main result may be seen as a corollary of the latter optimality statement, but in sharp contrast to the former papers, it is obtained under a weak condition which has a natural economic interpretation and is shown to hold with broad generality in standard oligopoly models with general demand functions (as are all the assumptions on the equilibrium profit functions made in our reduced-form model of competition). Therefore, our result that the licensor of a weak patent is better off charging per-unit royalties rather than fixed fees is arguably robust, especially as we also show that it holds in two extensions of the baseline model.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes the optimal license for each of the two schemes from the patent holder's

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perspective. In Section 4 we derive our main result on the comparison between the two licensing schemes for weak patents. In Section 5, we extend the analysis, first by including litigation costs borne by the challenger and, second, by considering a patent holder that is active on the downstream market. In section 6, we show that the general assumptions made on the equilibrium profits in our reduced-form model of competition and the (sufficient) condition under which the per-unit royalty scheme is preferred over the fixed fee scheme by the patent holder (be it an outsider or an insider) hold under mild conditions for both a Cournot oligopoly with homogenous goods and a Bertrand oligopoly with differentiated goods. Section 7 concludes.

2 The Model

We consider an industry consisting of $n \geq 2$ symmetric risk-neutral firms producing at a marginal cost $\bar{c}$ (fixed production costs are assumed to be zero). A firm P outside the industry holds a patent covering a technology that, if used, allows a firm to reduce its marginal cost from $\bar{c}$ to $\bar{c} - \epsilon$.

We consider the following three-stage game:

First stage: The patent holder P proposes to all firms a licensing contract whereby a licensee can use the patented technology against the payment of a per-unit royalty $r \in [0, \epsilon]$ or a fixed fee $F \geq 0$.[3]

Second stage: The n firms in the industry simultaneously and independently decide whether to purchase a license. If a firm does not accept the license offer, it can challenge the patent's validity before a court.[4] The outcome of such a trial is uncertain: with probability $\theta > 0$ the patent is upheld by the court and with probability $1 - \theta$ it is invalidated. Hence, the parameter θ may be interpreted as the patent's strength. If the patent is upheld, then a firm that does not purchase the license uses the old technology, thus producing at marginal cost c whereas a firm that accepted the license offer uses the new technology and pay the per-unit royalty r or the fixed fee F to the patent holder. If the patent is invalidated, all the firms, including those that accepted the license offer can use for free the new technology and their common marginal cost is $c - \epsilon$.[5]

[3] Following Farrell and Shapiro (2008) and Encaoua and Lefouili (2009), we focus on take-it-or-leave-it license offers.

[4] If the patent is granted by the European Patent Office (EPO), one may alternatively assume that a firm can challenge the patent's validity before the EPO. Indeed, any patent issued by the EPO can be opposed by a third party. The notice of opposition must be filed in writing at the EPO within nine months from the publication of the mention of the grant of the European patent.

[5] Following Farrell and Shapiro (2008) and Encaoua and Lefouili (2009), we ignore litigation costs in our baseline model. As argued in the former paper, understanding how the inclusion of litigation costs would affect the terms on which a patent is licensed in the shadow of litigation requires to extend the model to more general bargaining, which may be quite complex as we have multiple competing firms bargaining with the patent holder. However, in an extension to the baseline model we show that our main results hold - and are

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The n firms produce under the cost structure inherited from the second stage. We do not specify the type of competition that occurs. We only assume that there exists a unique equilibrium of the competition game for any cost structure and we set some general assumptions on the equilibrium profit functions. For this purpose, denote $\pi^e(k,c)$ (respectively $\pi^i(k,c)$) the equilibrium profit function, gross of any potential fixed cost (e.g fixed license fee), of a firm producing with marginal cost $c \leq \bar{c}$ (respectively with marginal cost $\bar{c}$) when $k \leq n$ firms produce at marginal cost c and the remaining $n-k$ firms produce at the marginal cost $\bar{c}$. We now make the following general assumptions for any given n and $k = 1, ..., n$.

A1. The equilibrium profits of an efficient firm and an inefficient firm, i.e. $\pi^e(k,c)$ and $\pi^i(k,c)$ respectively, are both continuously differentiable in c over the subset of $[0,\bar{c}]$ in which $\pi^i(c,k) > 0$. Furthermore, the function $c \to q^e(n,c)$ is continuously differentiable and strictly positive over $[0,\bar{c}]$.

A2. If the firms are symmetric (in terms of efficiency), an identical increase in all firms' marginal costs leads to a decrease in each firm's equilibrium profit: $\frac{\partial \pi^e}{\partial c}(n,c) < 0$.

A3. An inefficient firm's equilibrium profit is increasing in the efficient firms' marginal cost: If $\pi^i(k,c) > 0$ then $\frac{\partial \pi^i}{\partial c}(k,c) > 0$ and if $\pi^i(k,c) = 0$ then $\pi^i(k,c') = 0$ for any $c' < c$.

A4. A firm's profit is decreasing in the number of efficient firms in the industry: for any $c < \bar{c}$ and any $k < n$ it holds that $\pi^e(k,c) > \pi^e(k+1,c)$ and $\pi^i(k,c) \geq \pi^i(k+1,c)$.

A5. A firm's profit increases as it moves from the subgroup of inefficient firms to the subgroup of efficient firms: for any $c < \bar{c}$ and any $k < n$ it holds that $\pi^i(k,c) < \pi^e(k+1,c)$

As we shall argue in precise detail in Section 6, all these assumptions are satisfied with broad generality in the standard oligopoly models with general demand functions, the main restrictions being those needed to ensure existence and uniqueness of a pure-strategy equilibrium (such as Cournot or Bertrand equilibrium). In particular, these assumptions are clearly satisfied for instance for the widely used settings of Cournot competition with homogeneous goods and linear demand and Bertrand competition with differentiated goods and linear demands.

3 Optimal license deterring litigation

If litigation occurs then, with probability θ, the patent is upheld by the court (thus becoming an ironclad right) and, with probability $1-\theta$, it is invalidated and the technology can be used for free by all firms. Thus, if the patent holder expects its license offer to trigger litigation, it should make an offer that maximizes its revenues should the patent be ruled valid by the court. The patent holder would then essentially act as if the patent were ironclad, and the determination of the terms on which the technology is patented under each licensing scheme

actually strengthened - if we introduce small legal costs (or administrative fees) to be incurred by a challenger of the patent's validity.

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would amount to the analysis of licensing offers for ironclad patents, which has already been done extensively in the literature. We therefore consider in what follows only the class of license offers deterring litigation, for which the comparison of the two schemes cannot be derived trivially from that under ironclad patent protection.[6]

3.1 Optimal per-unit royalty license

Let us first examine a firm's incentives to challenge the patent's validity when the patent holder makes a license offer involving the payment of a per-unit royalty r. A firm that decides not to purchase a license is always (weakly) better off challenging the patent's validity: If no other firm challenges the patent's validity it gets a payoff $\theta\pi^i(n-1,\bar{c}-\epsilon+r)+(1-\theta)\,\pi^e(n,\bar{c}-\epsilon)$ which is strictly greater than the profit $\pi^i(n-1,\bar{c}-\epsilon+r)$ it would get by not challenging, and if some other firm challenges the patent's validity then it is indifferent between challenging and not. Thus, a situation where one or more firms do not buy a license and no firm challenges the patent's validity can never be an equilibrium of the second stage subgame. We can therefore state that a license offer will deter litigation if and only if it is accepted by all firms.

Let us now write the condition under which all firms accepting the license offer is an equilibrium of the second stage subgame. A firm anticipating that all other firms will purchase a license gets a profit equal to $\pi^e(n,\bar{c}-\epsilon+r)$ if it accepts the license offer. If it does not and challenges the patent's validity then with probability θ, the patent is upheld by the court and the challenger gets a profit equal to $\pi^i(n-1,\bar{c}-\epsilon+r)$ and, with probability $1-\theta$, the challenger gets a profit of $\pi^e(n,\bar{c}-\epsilon)$ (and so do all other firms). Thus, a firm challenging the patent's validity when all other firms accept the license offer, gets an expected profit of $\theta\pi^i(n-1,\bar{c}-\epsilon+r)+(1-\theta)\,\pi^e(n,\bar{c}-\epsilon)$. Therefore, all firms accepting the license offer is an equilibrium if and only if:

$$\pi^e(n,\bar{c}-\epsilon+r)\geq\theta\pi^i(n-1,\bar{c}-\epsilon+r)+(1-\theta)\,\pi^e(n,\bar{c}-\epsilon) \tag{1}$$

The equilibria of the second stage subgame for each value of r are presented in detail in the Appendix. In particular, it is shown that whenever Condition (1) holds, all firms buying a license is the *unique* equilibrium.[7] The next proposition characterizes the per-unit royalty license that maximizes the patent holder's licensing revenues subject to constraint (1).

Proposition 1 *For sufficiently weak patents, the optimal per-unit royalty license accepted by*

[6] In doing so, we follow Farrell and Shapiro (2008) who also focus on the licenses that are not litigated because they aim to investigate the social costs of the uncertainty over patent validity (which is resolved if litigation occurs). Moreover, there is empirical evidence that the vast majority of patent disputes are settled using licensing agreements before the court decides whether the patent is valid or not (see for instance Allison and Lemley, 1998 and Lemley and Shapiro, 2005)

[7] We assume throughout the paper that whenever a potential licensee is indifferent between buying a license or challenging the patent's validity, it buys a license.

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all firms, thus deterring any litigation, involves the payment of the royalty rate $r(\theta)$ defined as the unique solution in r to the following equation:

$$\pi^e(n, \bar{c} - \epsilon + r) = \theta \pi^i(n-1, \bar{c} - \epsilon + r) + (1-\theta)\, \pi^e(n, \bar{c} - \epsilon)$$

Proof. See Appendix. ■

3.2 Optimal fixed fee license

For a license offer involving the payment of a fixed fee F to be accepted by all firms, the following condition must hold:

$$\pi^e(n, \bar{c} - \epsilon) - F \geq \theta \pi^i(n-1, \bar{c} - \epsilon) + (1-\theta)\, \pi^e(n, \bar{c} - \epsilon)$$

which can be rewritten as:

$$F \leq \theta \left[\pi^e(n, \bar{c} - \epsilon) - \pi^i(n-1, \bar{c} - \epsilon)\right] \tag{2}$$

The equilibria of the second stage subgame for each value of F are presented in precise detail in the Appendix. In particular, it is shown that under condition (2) all firms accepting the license offer is the *unique* equilibrium for sufficiently weak patents. The next proposition characterizes the optimal fixed fee license deterring litigation.

Proposition 2 *The optimal fixed fee license accepted by all firms, thus deterring any litigation, involves the payment of the fee* $F(\theta) = \theta \left[\pi^e(n, \bar{c} - \epsilon) - \pi^i(n-1, \bar{c} - \epsilon)\right]$.

Proof. See Appendix. ■

4 Optimal licensing scheme

Before stating formally our main result let us explain why one could expect a high probability of invalidation (i.e. small θ) to increase the attractiveness of per-unit royalties with respect to fixed fees relative to the case of ironclad patents (i.e. $\theta = 1$). As shown in Farrell and Shapiro (2008) and Encaoua and Lefouili (2009), the use of per-unit royalties may allow the patent holder to deter litigation while getting licensing revenues which are higher than the expected licensing revenues it would earn if the patent validity issue were resolved before licensing. The reason is that the free rider problem arising from the public good nature of patent invalidation combined with the non-linearity of equilibrium profits with respect to per-unit royalties can make the licensing revenues grow more than proportionally with respect to the patent strength θ for small values of this parameter (i.e. for weak patents). Proposition 2

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however shows that this cannot happen if a license involving the payment of a fixed fee only is used instead: the optimal fixed fee deterring litigation is *linear* in the patent strength θ which makes the expected licensing revenues the patent holder derives from the optimal fixed fee license deterring litigation also linear in θ. In the light of these remarks, one might expect that even in a setting where a patent holder would have used the fixed fee licensing scheme if its patent were certainly valid, it could prefer to use the per-unit royalty scheme in case its patent is weak.

The next proposition provides a sufficient condition for the per-unit royalty scheme to be preferred over the fixed fee scheme by the holder of a weak patent.

Proposition 3 *The optimal per-unit royalty scheme deterring litigation provides higher licensing revenues than the optimal fixed fee scheme deterring litigation for sufficiently weak patents if :*

$$\frac{\partial \pi^e}{\partial c}(n, \bar{c} - \epsilon) > -q^e(n, \bar{c} - \epsilon) \tag{3}$$

Moreover, the reverse holds if the reverse strict inequality is verified.

Proof. See Appendix. ■

To understand why condition (3) matters for the comparison of the licensing revenues under the two schemes, it is useful to consider the sharing of the total revenues generated by the innovation under each licensing scheme between the patent holder and the licensees. Denote $\tilde{P}_r(\theta)$ (resp. $\tilde{P}_F(\theta)$) the patent holder's revenues under the optimal per-unit royalty (resp. fixed fee) license deterring litigation and $\tilde{\Delta}_r(\theta)$ (resp. $\tilde{\Delta}_F(\theta)$) the difference between pre-innovation and post-innovation profits for each licensee under the optimal per-unit royalty (resp. fixed fee) license. The equilibrium total revenues generated by the innovation are $\tilde{V}_r(\theta) = \tilde{P}_r(\theta) + n\tilde{\Delta}_r(\theta)$ (resp. $\tilde{V}_F(\theta) = \tilde{P}_F(\theta) + n\tilde{\Delta}_F(\theta)$) under the per-unit royalty scheme (resp. the fixed fee scheme).
Let us first compare $\tilde{V}_r(\theta)$ and $\tilde{V}_F(\theta)$. We have:

$$\begin{aligned}
\tilde{V}_r(\theta) - \tilde{V}_F(\theta) &= n\left[\pi^e(n, \bar{c} - \epsilon + r(\theta)) + r(\theta)\, q^e(n, \bar{c} - \epsilon + r(\theta)) - \pi^e(n, \bar{c})\right] - n\left[\pi^e(n, \bar{c} - \epsilon) - \pi^e(n, \bar{c})\right] \\
&= n\left[\pi^e(n, \bar{c} - \epsilon + r(\theta)) - \pi^e(n, \bar{c} - \epsilon) + r(\theta)\, q^e(n, \bar{c} - \epsilon + r(\theta))\right]
\end{aligned}$$

The latter is positive if the loss in market profits $\pi^e(n, \bar{c} - \epsilon + r(\theta)) - \pi^e(n, \bar{c} - \epsilon)$ is overcompensated by the per-firm licensing revenues $r(\theta)\, q^e(n, \bar{c} - \epsilon + r(\theta))$. Since $\tilde{V}_r(0) - \tilde{V}_F(0) = 0$, a sufficient condition for $\tilde{V}_r(\theta) - \tilde{V}_F(\theta)$ to be positive for θ sufficiently small is that its derivative at $\theta = 0$ is positive. Since:

$$\frac{d}{d\theta}\left[\tilde{V}_r(\theta) - \tilde{V}_F(\theta)\right]\Big|_{\theta=0} = nr'(0)\left[\frac{\partial \pi^e}{\partial c}(n, \bar{c} - \epsilon) + q^e(n, \bar{c} - \epsilon)\right]$$

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and $r'(0) > 0$, condition (3) ensures that the total revenues $\tilde{V}_r(\theta)$ generated by the optimal per-unit royalty license are higher than the total revenues $\tilde{V}_F(\theta)$ generated by the optimal fixed fee license for θ sufficiently small.

To derive the comparison between the licensing revenues $\tilde{P}_r(\theta)$ and $\tilde{P}_F(\theta)$ under the two schemes, we need to compare $\tilde{\Delta}_r(\theta)$ and $\tilde{\Delta}_F(\theta)$ as well. We have:

$$\begin{aligned}
\tilde{\Delta}_r(\theta) - \tilde{\Delta}_F(\theta) &= [\pi^e(n, \bar{c} - \epsilon + r(\theta)) - \pi^e(n, \bar{c})] - [\pi^e(n, \bar{c} - \epsilon) - F(\theta) - \pi^e(n, \bar{c})] \\
&= \theta\pi^i(n-1, \bar{c} - \epsilon + r(\theta)) + (1-\theta)\pi^e(n, \bar{c} - \epsilon) - \pi^e(n, \bar{c} - \epsilon) \\
&\quad + \theta\left[\pi^e(n, \bar{c} - \epsilon) - \pi^i(n-1, \bar{c} - \epsilon)\right] \\
&= \theta\left[\pi^i(n-1, \bar{c} - \epsilon + r(\theta)) - \pi^i(n-1, \bar{c} - \epsilon)\right] > 0
\end{aligned}$$

Thus, each licensee is better off under the per-unit royalty scheme, which makes the comparison between $\tilde{P}_r(\theta) = \tilde{V}_r(\theta) - n\tilde{\Delta}_r(\theta)$ and $\tilde{P}_F(\theta) = \tilde{V}_F(\theta) - n\tilde{\Delta}_F(\theta)$ a *priori* ambiguous. Note however that the positive effect of the patent strength θ on $\tilde{\Delta}_r(\theta) - \tilde{\Delta}_F(\theta)$ is second order for θ sufficiently small (since $\frac{d}{d\theta}\left[\tilde{\Delta}_r(\theta) - \tilde{\Delta}_F(\theta)\right]\Big|_{\theta=0} = 0$) while the effect of θ on $\tilde{V}_r(\theta) - \tilde{V}_F(\theta)$ is positive and first order under condition (3). Thus, for θ sufficiently small, the licensing revenues generated by the per-unit royalty scheme grow faster (with the patent strength) than those resulting from the use of a fixed fee scheme, which yields the result.

Interpretation. Condition (3) means that the *strategic* effect of an identical increase in the marginal cost of all (symmetric) firms on the firms' profits is positive. This interpretation results from the following decomposition (where we use the generic variable c instead of $\bar{c} - \epsilon$ as the second argument of the considered functions) :

$$\frac{\partial \pi^e}{\partial c}(n, c) = \underbrace{-q^e(n, c)}_{direct\ effect} + \underbrace{q^e(n, c)\frac{\partial p^e}{\partial c}(n, c) + (p^e(n, c) - c)\frac{\partial q^e}{\partial c}(n, c)}_{strategic\ effect}$$

An increase in all firms' marginal cost c affects their equilibrium profits $\pi^e(n, c)$ through two channels. First, it yields an increase in each firm's production costs (for a given output). Second, it entails an adjustment of their outputs and/or prices. The first effect, captured by the term $-q^e(n, c)$, can be interpreted as a *direct* effect of a marginal cost variation on equilibrium profits while the second effect, captured by the term $q^e(n, c)\frac{\partial p^e}{\partial c}(n, c) + (p^e(n, c) - c)\frac{\partial q^e}{\partial c}(n, c)$, can be interpreted as the *strategic* effect of cost variation on profits.[8]

To the best of our knowledge, condition (3) has not been studied in the literature on the effects of cost variations on oligopolists' profits which has mainly focused on the *overall* effect of cost changes on profits (see e.g. Kimmel, 1992; Février and Linnemer 2004). In section 6 we show the mildness of this condition by establishing that it holds under weak assumptions

[8] This strategic effect can further be split into a *price effect* captured by the term $q^e(n, c)\frac{\partial p^e}{\partial c}(n, c)$ and an *output effect* captured by the term $(p^e(n, c) - c)\frac{\partial q^e}{\partial c}(n, c)$.

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in two of the most usual competition models, namely Cournot competition with homogenous goods and Bertrand competition with differentiated goods.

5 Extensions

5.1 Litigation costs

Let us assume in this section that a firm that challenges the patent's validity before a court has to incur some legal costs $C \geq 0$.[9] It is straightforward that the higher those costs the higher the licensing revenues the patent holder can extract from the licensees without triggering litigation. This qualitative observation holds under both schemes. However, we show in what follows that on the quantitative side, the marginal effect of litigation costs on the patent holder's licensing revenues is higher under the per-unit royalty scheme than under the fixed fee scheme if condition (3) holds. This implies that the result in Proposition 1 remains true - and is actually strenghtened - if the model is extended to include (small) legal costs that have to be incurred by a challenger.

Suppose first that the patent holder makes a license offer involving the payment of a per-unit royalty $r \in [0, \epsilon[$. Note that the inclusion of legal costs in our setting does not affect the fact that the strategy "not buy a license *and* not challenge the patent's validity" is always dominated by the strategy "buy a license". Therefore, the only way a patent holder can deter litigation is to make a license offer that is accepted by all firms. This will be the case if and only if:

$$\pi^e(n, \bar{c} - \epsilon + r) \geq \theta \pi^i(n-1, \bar{c} - \epsilon + r) + (1-\theta)\, \pi^e(n, \bar{c} - \epsilon) - C$$

It is easily shown the latter constraint is met if and only if $r \leq r(\theta, C)$ where $r(\theta, C)$ is the solution in r to the equation:

$$\pi^e(n, \bar{c} - \epsilon + r) = \theta \pi^i(n-1, \bar{c} - \epsilon + r) + (1-\theta)\, \pi^e(n, \bar{c} - \epsilon) - C$$

and that, for θ and C sufficiently small, the optimal per-unit royalty license deterring litigation involves the payment of the royalty $r(\theta, C)$ (i.e. the constraint is binding). Note also that $r(\theta, C)$ is strictly increasing in both its arguments.

Suppose now that the patent holder makes a license offer involving the payment of a fixed fee F. Such a license offer is accepted by all firms if and only if:

$$\pi^e(n, \bar{c} - \epsilon) - F \geq \theta \pi^i(n-1, \bar{c} - \epsilon) + (1-\theta)\, \pi^e(n, \bar{c} - \epsilon) - C$$

[9] If we consider opposition before the European Patent Office (EPO) instead of litigation before a court, the cost C can be interpreted as the administrative fee an opponent to a patent has to pay to the EPO for the patent to be reexamined.

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and, therefore, the optimal fixed fee license deterring litigation involves the payment of the fee:

$$F(\theta, C) = \theta \left[\pi^e(n, \bar{c} - \epsilon) - \pi^i(n-1, \bar{c} - \epsilon)\right] + C$$

Let us now compare the licensing revenues derived by the patent holder under the two schemes. Under the optimal per-unit royalty scheme, they are given by

$$\tilde{P}_r(\theta, C) = nr(\theta, C)\, q^e(n, \bar{c} - \epsilon + r(\theta, C))$$

and under the optimal fixed fee licensing, they are given by:

$$\tilde{P}_F(\theta, C) = nF_n(\theta, C) = n\theta \left[\pi^e(n, \bar{c} - \epsilon) - \pi^i(n-1, \bar{c} - \epsilon)\right] + nC$$

Since $\tilde{P}_r(0,0) = \tilde{P}_F(0,0)$, a sufficient condition for the existence of $\tilde{\theta} > 0$ and $\tilde{C} > 0$ such that the inequality $\tilde{P}_r(\theta, C) > \tilde{P}_F(\theta, C)$ hold any $\theta < \tilde{\theta}$ and $C < \tilde{C}$ is that:

$$\frac{\partial \tilde{P}_r}{\partial \theta}(0,0) > \frac{\partial \tilde{P}_F}{\partial \theta}(0,0)$$

and

$$\frac{\partial \tilde{P}_r}{\partial C}(0,0) > \frac{\partial \tilde{P}_F}{\partial C}(0,0)$$

The former inequality has already been shown to be equivalent to condition (3). Surprisingly, the latter inequality is equivalent to condition (3) too. Indeed,

$$\frac{\partial \tilde{P}_F}{\partial C}(0,0) = n$$

and

$$\frac{\partial \tilde{P}_r}{\partial C}(0,0) = n\frac{\partial r}{\partial C}(0,0)\, q^e(n, \bar{c} - \epsilon)$$

Differentiating with respect to C the equation definition $r(\theta, C)$ at point $(\theta, C) = (0,0)$, we get: $\frac{\partial r}{\partial C}(0,0) = -\frac{1}{\frac{\partial \pi^e}{\partial c}(n, \bar{c} - \epsilon)}$. Thus,

$$\frac{\partial \tilde{P}_r}{\partial C}(0,0) = -\frac{nq^e(n, \bar{c} - \epsilon)}{\frac{\partial \pi^e}{\partial c}(n, \bar{c} - \epsilon)}$$

Hence

$$\frac{\partial \tilde{P}_r}{\partial C}(0,0) > \frac{\partial \tilde{P}_F}{\partial C}(0,0) \Longleftrightarrow \frac{\partial \pi^e}{\partial c}(n, \bar{c} - \epsilon) > -q^e(n, \bar{c} - \epsilon)$$

Therefore, the result in Proposition 1 is robust - and actually strengthened - in the presence of small legal costs (borne by the challenger).

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5.2 Internal patentee

Let us consider the case where the patent holder is active in the (downstream) market**.** More specifically, we assume that one of the n firms operating in the market, say firm 1, gets a patent on a technology that lowers the unit production cost from $\bar{c}$ to $\bar{c}-\epsilon$. We assume that $n \geq 3$ (as we want to have at least two potential licensees beside the patent holder).

Here again, we assume there exists a unique equilibrium of the competition game for any cost structure (with identical profits for firms producing at the same unit cost) and we set some general assumptions on the equilibrium profit functions. We focus on industry cost structures that can emerge following the licensing game, that is, situations in which: one firm - the patent holder - produces at unit cost $\bar{c}-\epsilon$, a number $k \leq n-1$ of firms - the licensees - produce at a unit cost $c \in [\bar{c}-\epsilon, \bar{c}]$ and the remaining $n-k$ firms - the non-licensees - produce at unit cost $\bar{c}$. We denote by $\pi^p(k,c)$, $\pi^l(k,c)$ and $\pi^n(k,c)$ the equilibrium market profits of the patent holder, a licensee producing at an effective unit cost c and a non-licensee respectively.

We make the following general assumptions for any given n and $k = 1, ..., n$.

A1'. The equilibrium profits $\pi^p(k,c)$, $\pi^l(k,c)$ and $\pi^n(k,c)$ are continuously differentiable in c over $[0,\bar{c}]$ over the subset of $[0,\bar{c}]$ in which $\pi^n(c,k) > 0$. Furthermore, the function $c \to q^l(n,c)$ is continuously differentiable over the subset of $[0,\bar{c}]$ in which it is strictly positive.

A2'. An identical increase in the costs of all firms but the patent holder decreases each one of those firms' equilibrium profit: $\frac{\partial \pi^l}{\partial c}(n-1,c) < 0$.

A3'. A non-licensee's equilibrium profit is increasing in the licensees' unit cost: If $\pi^n(k,c) > 0$ then $\frac{\partial \pi^n}{\partial c}(k,c) > 0$ and if $\pi^n(k,c) = 0$ then $\pi^n(k,c') = 0$ for any $c' < c$.

A4'. A firm's market profit is decreasing in the number of licensees in the industry: for any $c < \bar{c}$ and any $k < n-1$ it holds that $\pi^p(k,c) > \pi^p(k+1,c)$; $\pi^l(k,c) > \pi^l(k+1,c)$ and $\pi^n(k,c) \geq \pi^n(k+1,c)$.

A5'. A firm's market profit increases as it moves from the subgroup of non-licensees to the subgroup of licensees: for any $c < \bar{c}$ and any $k < n-1$ it holds that $\pi^n(k,c) < \pi^l(k+1,c)$.

Let us compare the innovator's overall profit, i.e. *the sum of its market profit and licensing revenues*, under the two licensing schemes.

Under the per-unit royalty scheme, the optimal royalty $r(\theta)$ is the solution in r to the following equation:

$$\pi^l(n-1,\bar{c}-\epsilon+r) = \theta\pi^n(n-2,\bar{c}-\epsilon+r) + (1-\theta)\,\pi^l(n-1,\bar{c}-\epsilon)$$

and the patent holder's overall profit is:

$$\tilde{\Pi}_r(\theta) = \pi^p(n-1,\bar{c}-\epsilon+r(\theta)) + (n-1)\,r(\theta)\,q^l(n-1,\bar{c}-\epsilon+r(\theta))$$

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Under the fixed fee scheme, the optimal fee is given by:

$$F(\theta) = \theta \left[\pi^l(n-1, \bar{c}-\epsilon) - \pi^n(n-2, \bar{c}-\epsilon)\right]$$

and the patent holder's overall profit is then:

$$\tilde{\Pi}_F(\theta) = \pi^p(n-1, \bar{c}-\epsilon) + (n-1)\,\theta \left[\pi^l(n-1, \bar{c}-\epsilon) - \pi^n(n-2, \bar{c}-\epsilon)\right]$$

Since $\tilde{\Pi}_r(0) = \tilde{\Pi}_F(0)$ then $\tilde{\Pi}_r(\theta) > \tilde{\Pi}_F(\theta)$ for θ sufficiently small if:

$$\frac{d\tilde{\Pi}_r(\theta)}{d\theta} \mid_{\theta=0} > \frac{d\tilde{\Pi}_F(\theta)}{d\theta} \mid_{\theta=0} \tag{4}$$

which can be rewritten as:

$$r'(0)\left[\frac{\partial \pi^p}{\partial c}(n-1, \bar{c}-\epsilon) + (n-1)\,q^l(n, \bar{c}-\epsilon)\right] > (n-1)\left[\pi^l(n-1, \bar{c}-\epsilon) - \pi^n(n-2, \bar{c}-\epsilon)\right]$$

because $r(0) = 0$. Moreover, differentiating at $\theta = 0$ the equation defining $r(\theta)$, we get:

$$r'(0)\frac{\partial \pi^l}{\partial c}(n-1, \bar{c}-\epsilon) = \pi^n(n-2, \bar{c}-\epsilon) - \pi^l(n-1, \bar{c}-\epsilon)$$

which yields:

$$r'(0) = \frac{\pi^n(n-2, \bar{c}-\epsilon) - \pi^l(n-1, \bar{c}-\epsilon)}{\frac{\partial \pi^l}{\partial c}(n-1, \bar{c}-\epsilon)}$$

Hence, inequality (1) is equivalent to:

$$\frac{\pi^n(n-2,\bar{c}-\epsilon)-\pi^l(n-1,\bar{c}-\epsilon)}{\frac{\partial \pi^l}{\partial c}(n-1,\bar{c}-\epsilon)}\left[\frac{\partial \pi^p}{\partial c}(n-1, \bar{c}-\epsilon) + (n-1)\,q^l(n, \bar{c}-\epsilon)\right] > (n-1)\left[\pi^e(n-1, \bar{c}-\epsilon) - \pi^i(n-2, \bar{c}-\epsilon)\right]$$

which can be rewritten as:

$$\frac{\partial \pi^p}{\partial c}(n-1, \bar{c}-\epsilon) + (n-1)\,q^l(n, \bar{c}-\epsilon) > -(n-1)\frac{\partial \pi^l}{\partial c}(n-1, \bar{c}-\epsilon)$$

that is,

$$\frac{\partial \pi^p}{\partial c}(n-1, \bar{c}-\epsilon) + (n-1)\frac{\partial \pi^l}{\partial c}(n-1, \bar{c}-\epsilon) > -(n-1)\,q^l(n, \bar{c}-\epsilon)$$

Thus, we get the following result:

Proposition 4 *The optimal per-unit royalty scheme deterring litigation generates higher overall profit for the patent holder than the optimal fixed fee scheme deterring litigation for sufficiently weak patents if:*

$$\frac{\partial \pi^p}{\partial c}(n-1, \bar{c}-\epsilon) + (n-1)\frac{\partial \pi^l}{\partial c}(n-1, \bar{c}-\epsilon) > -(n-1)\,q^l(n, \bar{c}-\epsilon) \tag{5}$$

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The reverse holds if the reverse strict inequality is satisfied.

To see how Condition (5) compares to its counterpart when the patent holder is not active on the market, i.e. Condition (3), let us rewrite both of them with the same notations. For that purpose, let us denote by $\Pi^*(c_1, c_2, ..., c_n)$ the sum of all firms' equilibrium profits, i.e. the equilibrium aggregate profit, and $q_i^*(c_1, c_2, ..., c_n)$ firm i's output when each firm $j = 1, 2, ..., n$ produces at unit cost c_j.

The sufficient condition for the patent holder to prefer the per-unit royalty scheme for sufficiently weak patents when *it is not active in the market* can be rewritten as (here, $\bar{c} - \epsilon$ is replaced by the generic variable c):

$$\sum_{i=1}^{n} \frac{\partial \Pi^*}{\partial c_i}(c, c, ..., c) > -\sum_{i=1}^{n} q_i^*(c, c, ..., c) \tag{6}$$

The sufficient condition for the patent holder, denoted as firm k, to prefer the per-unit royalty scheme for sufficiently weak patents when *it is active in the market* can be rewritten as (here again, $\bar{c} - \epsilon$ is replaced by the generic variable c):

$$\sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{\partial \Pi^*}{\partial c_i}(c, c, ..., c) > -\sum_{\substack{i=1 \\ i \neq k}}^{n} q_i^*(c, c, ..., c) \tag{7}$$

Both inequalities have almost the same interpertation: Condition (6) means that the *strategic* effect of an identical increase in all firms' (common) unit cost on the aggregate profit is positive and Condition (7) means that the strategic effect of an increase in the costs of all firms but one on the aggregate profit is positive (firms being equally efficient initially). Note also that both conditions are implied by the following inequality when it holds for any $i = 1, 2, ..., n$:

$$\frac{\partial \Pi^*}{\partial c_i}(c, c, ..., c) > -q_i^*(c, c, ..., c) \tag{8}$$

This condition means that when firms are equally efficient initially, the strategic effect of an increase in one firm's unit cost on the aggregate profit is positive.

We show in what follows that Condition (8) holds under mild conditions for a Cournot oligopoly (with homogenous products) and a Bertrand oligopoly (with differentiated products). It then follows that both Condition (6) and Condition (7) hold since they are implied by Condition (8).

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6 Two standard oligopoly applications

In this section, we provide sufficient conditions of a general nature on the primitives of the two most widely used models of imperfect competition, which lead to Assumptions **A1-A5** and **A1'-A5'** and Condition (8) being verified. Since some of the results below are new to the oligopoly literature, and of some independent interest, we derive them for fully asymmetric versions of the Cournot and Bertrand oligopolies with linear costs. Accordingly, we also change the notation as needed, relative to the other parts of the paper.

6.1 Cournot competition with homogenous products

Consider an industry consisting of n firms competing in Cournot fashion. Firm i's marginal cost is denoted c_i (fixed production costs are assumed to be zero or otherwise sunk). Suppose the firms face an inverse demand function $P(\cdot)$ satisfying the following minimal conditions:

C1 $P(\cdot)$ is twice continuously differentiable and $P'(\cdot) < 0$ whenever $P(\cdot) > 0$.
C2 $P(0) > c_i > P(Q)$ for Q sufficiently high, $i = 1, 2, ..., n$.
C3 $P'(Q) + QP''(Q) < 0$ for all $Q \geq 0$ with $P(\cdot) > 0$.

These assumptions are quite standard and minimal. **C3** is the familiar condition used by Novshek (1985) to guarantee downward-sloping reaction curves (for any cost function). It is equivalent to the statement that each firm's marginal revenue is decreasing in rivals' output (see Amir, 1996 for an alternative condition).

Firm i's profit function and reaction correspondence are (here, $Q_{-i} = \sum_{j \neq i} q_j$)

$$\pi_i(q_i, Q_{-i}) = q_i \left[P(q_i + Q_{-i}) - c_i\right] \text{ and } r_i(Q_{-i}) = \arg\max_{q_i \geq 0} \pi_i(q_i, Q_{-i})$$

The next proposition provides general conditions under which Assumptions **A1-A5** and **A1'-A5'** hold for a Cournot oligopoly.

Proposition 5 *Under Assumptions **C1-C3**, the following holds:*
(a) There exists a unique Cournot equilibrium.
(b) Firm i's equilibrium profit π_i^* is differentiable in c_i and in c_j for any $j \neq i$.
(c) Firm i's equilibrium profit π_i^* is decreasing in c_i and increasing in c_j for any $j \neq i$.
If in addition, the game is symmetric (with c denoting the unit cost), then
(d) The unique Cournot equilibrium is symmetric.
(e) The equilibrium output q^* strictly decreases in c.
(f) Per-firm equilibrium profit π^* decreases in c.

Proof. See Appendix. ■

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It is straightforward to relate the different parts of Proposition (5) to Assumptions **A1-A5** and **A1'-A5'.** Part (a) is needed to avoid vacuous statements. Assumptions **A1** and **A1'** are implied by part (b) and the proof of part (e). Assumption **A2** follows from part (f) and Assumption **A2'** follows from combining part (f) with part (c). Assumptions **A3** and **A3'** are implied by part (c). Assumptions **A4** and **A4'** follow from repeated applications of part (c), with one rival firm's cost decreasing at a time. Assumptions **A5** and **A5'** follow from part (c).

While Parts (c) and (f) appear quite intuitive, they actually both have a less universal scope that one might think. Indeed, there is an extensive literature dealing with taxation in oligopolistic industries and one of its key insights is that a common cost increase can lead to some firms benefiting at the expense of others (Seade, 1985, Kimmel, 1992, and Février and Linnemer, 2004). More surprisingly, in a symmetric setting, a cost increase may be beneficial to all firms, when the inverse demand function is sufficiently convex. In light of this result, part (f) may be viewed as giving sufficient conditions for this counter-intuitive effect of uniform taxation not to arise.

On the other hand, while the questions addressed in Part (c) do not have any direct counterparts in the taxation literature, the results from the latter do suggest that the intuitive conclusions of Part (c) will not hold for sufficiently convex inverse demand functions.[10]

In fact, in the present otherwise quite general framework, it is worth noting that the most restrictive assumptions made are uniqueness of pure-strategy equilibrium, **A2** and **A5**. In order to substantiate this claim via an instructive illustration, we provide a well-known class of demand functions that leads to a violation of each of the three assumptions.

Example. Consider a duopoly industry with inverse demand

$$P(Q) = Q^{-\frac{1}{\beta}}, \text{ with } \frac{1}{2} < \beta < 1, \tag{9}$$

which clearly fails Assumption **C3**.The profit functions are

$$\pi_1(x_1, x_2) = (q_1 + q_2)^{-\frac{1}{\beta}} x_1 - c_1 q_1 \text{ and } \pi_2(q_1, q_2) = (q_1 + q_2)^{-\frac{1}{\beta}} q_2 - c_2 q_2.$$

The equilibrium outputs are

$$q_i^* = \frac{(2\beta - 1)^\beta}{\beta^\beta (c_1 + c_2)^{\beta+1}} (c_i (1 - \beta) + c_j \beta), \text{ with } i, j \in \{1, 2\}, i \neq j.$$

[10] To the best of our knowledge, this comparative statics issue, which might be thought of as single-firm taxation, has not been considered in the literature.

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The equilibrium profits are

$$\pi_i^*(c_1, c_2) = \frac{(2\beta - 1)^{\beta-1}}{\beta^\beta} \frac{(c_i(1-\beta) + c_j\beta)^2}{(c_i + c_j)^{\beta+1}}.$$

It is easily verified that

(i) $\frac{\partial \pi_i^*(c_1,c_2)}{\partial c_i}$ can be positive under some robust parameter conditions. In particular, $\left.\frac{\partial \pi_i^*(c_1,c_2)}{\partial c_i}\right|_{c_1=c_2} > 0$ if $\frac{1}{2} < \beta < \frac{3}{5}$.

(ii) In the $n-$firm symmetric version of this example with $\beta > 0$, $\frac{\partial \pi^*}{\partial c} > 0$ (see Kimmel, 1992, and Février and Linnemer, 2004).

(iii) this example gives rise to two Cournot equilibria, one of which has each firm producing zero output.

In light of this discussion and example, an insightful perspective on Proposition (5) is that, by imposing one of the commonly used conditions to guarantee existence and uniqueness of Cournot equilibrium, **C3**, one also obtains as a byproduct that the counter-intuitive results on the effects of uniform or unilateral cost increases (or uniform or individual taxation) do not hold.

Proposition 6 *Under Assumptions **C1-C3**, Condition (8) is verified.*

Proof. See Appendix. ■

We can then state that under the general assumptions **C1-C3**, the holder of a weak patent prefers to license it out using a per-unit royalty licensing contract rather than a fixed fee contract. This result holds whether the patent holder is active in the (downstream) market or not.

6.2 Bertrand competition with differentiated products

Consider an industry consisting of n single-product firms, with constant unit costs $c_1, c_2, ..., c_n$. Assume that the goods are imperfect substitutes. Denoting $D_i(p_1, p_2, ..., p_n)$ the demand for the good produced by firm i, its profit function and reaction correspondence are defined by

$$\pi_i(p_i, p_{-i}) = (p_i - c_i)D_i(p_i, p_{-i}) \text{ and } r_i(p_{-i}) = \arg\max_{p_i} \pi_i(p_i, p_{-i})$$

We will say that the Bertrand oligopoly is symmetric if the demand functions are symmetric and $c_1 = c_2 = ... = c_n \triangleq c$.

Let $S_i \triangleq \{(p_1, p_2, ..., p_n) \in R_+^n \mid D_i(p_1, p_2, ..., p_n) > 0\}$. We assume that for every firm i

B1 D_i is twice continuously differentiable on S_i.

B2 (i)$\frac{\partial D_i}{\partial p_i} < 0$, (ii)$\frac{\partial D_i}{\partial p_j} > 0$ and (iii) $\sum_{k=1}^n \frac{\partial D_i}{\partial p_k} < 0$ over the set S_i.

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B3 $D_i \frac{\partial D_i}{\partial p_j \partial p_i} - \frac{\partial D_i}{\partial p_j} \frac{\partial D_i}{\partial p_i} > 0$ over the set S_i, for $j \neq i$.
B4 $\sum_{k=1}^{n} \frac{\partial^2 \log D_i}{\partial p_i \partial p_j} < 0$ over the set S_i.

These general conditions are commonly invoked for differentiated-good demand systems (e.g., Vives, 1999). They have the following meanings and economic interpretations. For **B2,** part (i) is just the ordinary law of demand; Part (ii) says that goods i and j are substitutes; and Part (iii) is a dominant diagonal condition for the Jacobian of the demand system. It says that own price effect on demand exceeds the total cross-price effects. **B3** says that each demand has (differentiably) strict log-increasing differences in own price and any rival's price. The exact economic interpretation is that the price elasticity of demand strictly increases in any rival's price, which is a very natural assumption for substitute products (Milgrom and Roberts, 1990). **B4** says that the Hessian of the log-demand system has a dominant diagonal, which is a standard assumption invoked to guarantee uniqueness of Bertrand equilibrium (Milgrom and Roberts, 1990 or Vives, 1999). **B2(iii)** and **B4** hold that own effects of price changes dominate cross effects, for the level and the Jacobian of demand, respectively.

The following proposition provides sufficient conditions for Assumptions **A1-A5** and **A1'-A5'** to hold in this framework of price competition.

Proposition 7 *Under Assumptions **B1-B4**,*

(a) The Bertrand game is of strict strategic complements and has a unique equilibrium.
(b) Firm i's equilibrium price p_i^* is increasing in c_j for any j.
(c) Firm i's equilibrium profit π_i^* is differentiable in c_j for any j.
(d) Firm i's equilibrium profit π_i^* is increasing in c_j for any $j \neq i$.
(e) Firm i's equilibrium profit π_i^* is decreasing in c_i.
If, in addition, the game is symmetric, then
(f) the unique Bertrand equilibrium is symmetric.
(g) the equilibrium price p^* increases in c and $\frac{p^*(c')-p^*(c)}{c'-c} \leq 1$ for all $c' > c$.
(h) per-firm equilibrium profit π^* is differentiable in c, and decreasing in c.

Proof. See Appendix. ■

We leave to the reader the task of matching the different parts of Proposition (7) to Assumptions **A1-A5** and **A1'-A5'**, as this step is quite similar to the Cournot case.

Anderson, DePalma and Kreider (2001) extends the analysis of the effects of taxation to Bertrand competition with differentiated products, and report analogous findings as in the Cournot case. Since Proposition (7) contains only intuitive results on the effects of cost changes on profits, one concludes that the standard assumptions for existence and uniqueness of Bertrand equilibrium preclude any counter-intuitive effects of taxation.

Proposition 8 *Under Assumptions **B1-B4**, Condition (8) is verified.*

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Proof. See Appendix. ■

We can then state that under the general assumptions **B1-B4**, the holder of a weak patent prefers to license it out using a per-unit royalty licensing contract rather than a fixed fee contract (be it active or not in the market).

7 Conclusion

This paper shows that the "weakness" of a patent is an alternative justification for the use of a per-unit royalty instead of a fixed fee in a licensing scheme. A sufficient condition under which the holder of a weak patent prefers to license out through a per-unit royalty rather than a fixed fee is provided and shown to be mild in the sense that it holds under weak conditions for a Cournot oligopoly with homogenous goods and a Bertrand oligopoly with heterogenous goods. A significant difference with respect to the literature on the licensing of ironclad patents is that we get a clear-cut result on the comparison of a patent holder's profits under the two schemes, *independent of* the type of downstream competition, the degree of differentiation between products and whether the patent holder is active or not in the downstream market, while varying any of these three features can overturn the outcome of the comparison when ironclad patents are considered. Furthermore, our model generates some testable predictions that might be worth investigating: First, our results suggest that per-unit royalty licenses should be more prevalent in industries with a significant proportion of firms holding questionable patents, e.g., industries relying on some new patentable subject matter (biotechnology, software, business methods,...). Second, if our predictions are correct then under the presumption that the EPO is more stringent in checking the patentability standards than the USPTO, the use of per-unit royalties should be more prevalent in the US than in the EU.

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9 Appendix

Equilibria of the second stage subgame

a. Per-unit royalty scheme

Assume the patent holder makes a license offer (in the first stage) involving the payment of a per-unit royalty $r \in [0, \epsilon]$.

Assume first that $r < \epsilon$. Let us show that in this case, any outcome with $k \leq n-2$ licensees cannot be an equilibrium. We have already shown that a situation where not all firms buy a license and no firm challenges the patent cannot be an equilibrium so we can focus on situations with $k \leq n-2$ licensees and at least one non-licensee, which we denote by K, challenging the patent. Any of the $n-k-1 \geq 1$ non-licensees different from K has an incentive to unilaterally deviate since it would get an expected profit of $\theta\pi^e(k+1, \bar{c}-\epsilon+r)+(1-\theta)\,\pi^e(n, \bar{c}-\epsilon)$ instead of the strictly lower expected profit $\theta\pi^i(k, \bar{c}-\epsilon+r)+(1-\theta)\,\pi^e(n, \bar{c}-\epsilon)$ if it remains a non-licensee. Thus, any equilibrium of the second stage subgame involves at least $n-1$ firms if $r < \epsilon$. The latter result extends to the case $r = \epsilon$ if it is assumed that a firm which is indifferent between getting a license at this royalty rate and not buying a license does purchase a license. Let us now use this result to determine the equilibria for each value of $r \leq \epsilon$:

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- If $\pi^e(n, \bar{c} - \epsilon + r) \geq \theta\pi^i(n-1, \bar{c} - \epsilon + r) + (1-\theta)\,\pi^e(n, \bar{c} - \epsilon)$ (which is shown later on to be equivalent to $r \leq r(\theta)$ where $r(\theta)$ is defined in Proposition 1) then all firms buying a license is an equilibrium. Moreover, a situation where all firms but one buy a license is not an equilibrium. Thus, in this case, all firms purchasing a license is the unique equilibrium of the second stage subgame.
- If $\pi^e(n, \bar{c} - \epsilon + r) < \theta\pi^i(n-1, \bar{c} - \epsilon + r) + (1-\theta)\,\pi^e(n, \bar{c} - \epsilon)$ ((which is shown later on to be equivalent to $r(\theta) < r \leq \epsilon$ where $r(\theta)$ is defined in Proposition 1) then all firms but one buying a license (and the remaining firm challenging the patent's validity) are equilibria. Moreover, these are the only equilibria since all firms purchasing a license is not an equilibrium.

b. Fixed fee scheme

Denoting F the fixed fee each firm has to pay to get a license, all firms deciding to purchase a license is an equilibrium of the second stage subgame if and only if:

$$\pi^e(n, \bar{c} - \epsilon) - F \geq \theta\pi^i(n-1, \bar{c} - \epsilon) + (1-\theta)\,\pi^e(n, \bar{c} - \epsilon)$$

which can be rewritten as:

$$F \leq F(\theta) = \theta\left[\pi^e(n, \bar{c} - \epsilon) - \pi^i(n-1, \bar{c} - \epsilon)\right]$$

All firms but one deciding to purchase a license is an equilibrium of the second stage subgame if the following two conditions hold:

$$\theta\pi^i(n-1, \bar{c} - \epsilon) + (1-\theta)\,\pi^e(n, \bar{c} - \epsilon) \geq \pi^e(n, \bar{c} - \epsilon)$$

and

$$\theta\left[\pi^e(n-1, \bar{c} - \epsilon) - F\right] + (1-\theta)\,\pi^e(n, \bar{c} - \epsilon) \geq \theta\pi^i(n-2, \bar{c} - \epsilon) + (1-\theta)\,\pi^e(n, \bar{c} - \epsilon)$$

Thus, all firms but one deciding to purchase a license is an equilibrium if and only if:

$$F(\theta) \leq F \leq F_{n-1} \equiv \pi^e(n-1, \bar{c} - \epsilon) - \pi^i(n-2, \bar{c} - \epsilon)$$

Denoting $F_k = \pi^e(k, \bar{c} - \epsilon) - \pi^i(k-1, \bar{c} - \epsilon)$ for each $k = 1, 2, .., n$, we can further show that for any $k = 1, 2, ..., n-2$, a number k of firms accepting the license offer and the other $n-k$ firms refusing to do so is an equilibrium if and only if:

$$F_{k+1} \leq F \leq F_k$$

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Moreover, all firms deciding not to buy a license is an equilibrium if and only if:

$$F \geq F_1$$

If we assume that the sequence $(F_k)_{1\leq k\leq n}$ is decreasing, i.e. a firm's willingness to pay for a license (under ironclad patent protection) decreases with the number of licensees, and that a firm which is indifferent between accepting and refusing the license offer buys a license, then for any $F \geq 0$, there is a unique equilibrium to the second stage subgame up to a permutation of firms: all the equilibria of the second stage subgame involve the same number of licensees (which allows to define a "demand function" for licenses which is decreasing in the fixed fee F). However, if $(F_k)_{1\leq k\leq n}$ is not decreasing then there might exist some values of F for which there is either no (pure-strategy) equilibrium or multiple equilibria with different number of licensees.

However, if we focus on small values of θ and do not care about whether pure-strategy equilibria exist - and which one arises in case they do - if all firms accepting the license offer is not an equilibrium (as in the present paper), then the problem of multiplicity or inexistence of equilibria depicted does not affect our analysis. The reason is that, to be sure that all firms accepting a license is the unique equilibrium whenever it is an equilibrium, i.e. whenever $F \leq F(\theta)$, we only need the inequality $F(\theta) \leq F_k$ to hold for any $k = 1, 2, ..., n-1$, which, given that $F(\theta) = \theta F_n$, is true if θ is small enough, and more specifically if

$$\theta \leq \bar{\theta} = \frac{\min\limits_{1\leq k\leq n-1} F_k}{F_n}$$

Proof of Proposition 1

All firms accepting the payment of a per-unit royalty r is an equilibrium if and only if:

$$\pi^e(n, \bar{c} - \epsilon + r) \geq \theta\pi^i(n-1, \bar{c} - \epsilon + r) + (1-\theta)\,\pi^e(n, \bar{c} - \epsilon)$$

which can be rewritten as:

$$g(r, \theta) \equiv \pi^e(n, \bar{c} - \epsilon + r) - \theta\pi^i(n-1, \bar{c} - \epsilon + r) - (1-\theta)\,\pi^e(n, \bar{c} - \epsilon) \geq 0$$

We have $g(0, \theta) = \theta\left[\pi^e(n, \bar{c} - \epsilon) - \pi^i(n-1, \bar{c} - \epsilon)\right] \geq \theta\left[\pi^e(n, \bar{c}) - \pi^i(n-1, \bar{c} - \epsilon)\right]$ $\geq \theta\left[\pi^i(n-1, \bar{c}) - \pi^i(n-1, \bar{c} - \epsilon)\right] > 0$ and $g(\epsilon, \theta) = \pi^e(n, \bar{c}) - \theta\pi^i(n-1, \bar{c}) - (1-\theta)\,\pi^e(n, \bar{c} - \epsilon) = (1-\theta)\left(\pi^e(n, \bar{c}) - \pi^e(n, \bar{c} - \epsilon)\right) < 0$. Combining this with g being continuous and strictly decreasing in r yields: i/ the existence and uniqueness of a solution in r to the equation $g(r, \theta) = 0$ (within the interval $[0, \epsilon[$), which we denote by $r(\theta)$; ii/ the equivalence between $g(r, \theta) \geq 0$ and $r \leq r(\theta)$. Moreover $g(r, \theta)$ is strictly increasing in θ (for any $r \in [0, \epsilon]$) because $\pi^e(n, \bar{c} - \epsilon) - \pi^i(n-1, \bar{c} - \epsilon + r) > 0$ (for any $r \in [0, \epsilon]$), which entails that $r(\theta)$

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is strictly increasing in θ. Furthermore, A5 ensures that the licensing revenue function $r \rightarrow nrq^e(n, \bar{c} - \epsilon + r)$ is strictly increasing in the neighborhood of 0 (its derivative at $r = 0$ being $q^e(n, \bar{c} - \epsilon) > 0$). Hence, Since $r(\theta)$ is continuous, increasing and has $r(0) = 0$, we can conclude that, for θ sufficiently small, the optimal per-unit royalty license accepted by all firms involves the payment of the royalty rate $r(\theta)$.

Proof of Proposition 2

All firms accepting the payment of a pure fixed fee F is an equilibrium if and only if:

$$\pi^e(n, \bar{c} - \epsilon + r) \geq \theta \pi^i(n-1, \bar{c} - \epsilon + r) + (1 - \theta)\, \pi^e(n, \bar{c} - \epsilon)$$

which yields:

$$F \leq F(\theta) \equiv \theta \left[\pi^e(n, \bar{c} - \epsilon) - \pi^i(n-1, \bar{c} - \epsilon)\right]$$

The optimal fixed fee accepted by all firms is the solution to the maximization program $\max nF$ subject to the constraint $F \leq F(\theta)$, which is obviously $F(\theta)$.

Proof of Proposition 3

The licensing revenues from the optimal per-unit royalty scheme deterring litigation are given by:

$$\tilde{P}_r(\theta) = nr(\theta)\, q^e(n, \bar{c} - \epsilon + r(\theta))$$

and the licensing revenues from the optimal fixed fee licensing scheme deterring litigation are:

$$\tilde{P}_F(\theta) = nF_n(\theta) = n\theta \left[\pi^e(n, \bar{c} - \epsilon) - \pi^i(n-1, \bar{c} - \epsilon)\right]$$

Since $\tilde{P}_r(0) = \tilde{P}_F(0)$ then $\tilde{P}_r(\theta) > \tilde{P}_F(\theta)$ for θ sufficiently small if:

$$\frac{d\tilde{P}_r(\theta)}{d\theta} \mid_{\theta=0} > \frac{d\tilde{P}_F(\theta)}{d\theta} \mid_{\theta=0} \tag{10}$$

which can be rewritten as:

$$nr'(0)\, q^e(n, \bar{c} - \epsilon) > n \left[\pi^e(n, \bar{c} - \epsilon) - \pi^i(n-1, \bar{c} - \epsilon)\right]$$

because $r(0) = 0$. Moreover differentiating at $\theta = 0$ the equation defining $r(\theta)$, that is,

$$\pi^e(n, \bar{c} - \epsilon + r(\theta)) = \theta \pi^i(n-1, \bar{c} - \epsilon + r(\theta)) + (1 - \theta)\, \pi^e(n, \bar{c} - \epsilon)$$

we get:

$$r'(0) \frac{\partial \pi^e}{\partial c}(n, \bar{c} - \epsilon) = \pi^i(n-1, \bar{c} - \epsilon) - \pi^e(n, \bar{c} - \epsilon)$$

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which yields:

$$r'(0) = \frac{\pi^i(n-1, \bar{c}-\epsilon) - \pi^e(n, \bar{c}-\epsilon)}{\frac{\partial \pi^e}{\partial c}(n, \bar{c}-\epsilon)}$$

Hence, (2) is equivalent to:

$$n\frac{\pi^i(n-1, \bar{c}-\epsilon) - \pi^e(n, \bar{c}-\epsilon)}{\frac{\partial \pi^e}{\partial c}(n, \bar{c}-\epsilon)} q^e(n, \bar{c}-\epsilon) > n\left[\pi^e(n, \bar{c}-\epsilon) - \pi^i(n-1, \bar{c}-\epsilon)\right]$$

which can be rewritten as:

$$\frac{\partial \pi^e}{\partial c}(n, \bar{c}-\epsilon) > -q^e(n, \bar{c}-\epsilon)$$

because $\pi^i(n-1, \bar{c}-\epsilon) - \pi^e(n, \bar{c}-\epsilon) < 0$.

Proof of Proposition 5

(a) The existence of a unique Cournot equilibrium follows from the contraction-like property that every selection of r_i satisfies (See Amir, 1996 or Amir and Lambson, 2000 for details)

$$-1 < \frac{r_i(Q'_{-i}) - r_i(Q_{-i})}{Q'_{-i} - Q_{-i}} < 0 \text{ for all } Q'_{-i} > Q_{-i}. \tag{11}$$

(b) We first show that q_i^* is continuously differentiable in c_i. Viewed as a correspondence in the parameter c_i, q_i^* is upper hemi-continuous (or u.h.c.), as a direct consequence of the well-known property of u.h.c. of the equilibrium correspondence for games with continuous payoff functions (jointly in own and rivals' actions), see e.g., Fudenberg and Tirole, 1990. Since q_i^* is also single-valued in c (from part (b)), q_i^* must be a continuous function. Then the fact that q_i^* is continuously differentiable in c_i follows from the Implicit Function Theorem applied to the first order conditions, and the smoothness of $P(\cdot)$.

The fact that π_i^* is also continuously differentiable in c_i follows directly from the fact that q_i^* has the same property for all i.

The proof for the parameter $c_j, j \neq i$, follows along the same lines.

(c) Denote industry output, firm i's output, profit and its rivals' total outputs at equilibrium by Q^*, q_i^*, π_i^* and Q_{-i}^* respectively when the cost vector is $(c_1, c_2, ..., c_n)$. Denote the same four variables by $\widehat{Q}$, $\widehat{q}_i, \widehat{\pi}_i$ and $\widehat{Q}_{-i}$ after firm i's cost alone increases to $\widehat{c}_i > c_i$.

We first show that $\widehat{Q} < Q^*$. Adding the n first order conditions at the Cournot equilibrium yields

$$nP(Q^*) + Q^*P'(Q^*) = \sum_{k=1}^{n} c_k \tag{12}$$

Since the LHS of (12) is strictly decreasing in Q^*, the increase in firm i's cost from c_i to $\widehat{c}_i$ increases the RHS of (12), which causes the solution to (12) to decrease. In other words, $\widehat{Q} < Q^*$.

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We now show that for any firm $j \neq i$, we must have $\widehat{Q}_{-j} < Q^*_{-j}$. To this end, first observe that $\widehat{Q}_{-j} + r_j(\widehat{Q}_{-j}) = \widehat{Q} < Q^* = Q^*_{-j} + r_j(Q^*_{-j})$. Since (11) holds that $Q_{-j} + r_j(Q_{-j})$ is increasing in Q_{-j}, we must have $\widehat{Q}_{-j} < Q^*_{-j}$.

To show that $\widehat{\pi}_j > \pi^*_j$, consider for firm j,

$$\begin{aligned}
\widehat{\pi}_j &= \widehat{q}_j[P(\widehat{q}_j + \widehat{Q}_{-j}) - c_j] \\
&\geq q^*_j[P(q^*_j + \widehat{Q}_{-j}) - c_j] \text{ by the Cournot property} \\
&> q^*_j[P(q^*_j + Q^*_{-j}) - c_j] \text{ since } \widehat{Q}_{-j} < Q^*_{-j} \\
&= \pi^*_j
\end{aligned}$$

We now show that for firm i, we must have $\widehat{Q}_{-i} > Q^*_{-i}$. To this end, first observe that since r_j is strictly decreasing (cf. (11)) and $\widehat{Q}_{-j} < Q^*_{-j}$, we have $\widehat{q}_j = r_j(\widehat{Q}_{-j}) > r_j(Q^*_{-j}) = q^*_j$, for every firm $j \neq i$. Then since $\widehat{Q}_{-i} = \sum_{j \neq i} \widehat{q}_j$ and $Q^*_{-i} = \sum_{j \neq i} q^*_j$, we have $\widehat{Q}_{-i} > Q^*_{-i}$.

To show that $\pi^*_i > \widehat{\pi}_i$, consider

$$\begin{aligned}
\pi^*_i &= q^*_i \left[P(q^*_i + Q^*_{-i}) - c_i\right] \\
&\geq \widehat{q}_i \left[P(\widehat{q}_i + Q^*_{-i}) - c_i\right] \text{ by the Cournot property} \\
&> \widehat{q}_i[P(\widehat{q}_i + \widehat{Q}_{-i}) - c_i] \text{ since } \widehat{Q}_{-i} > Q^*_{-i}. \\
&> \widehat{q}_i[P(\widehat{q}_i + \widehat{Q}_{-i}) - \widehat{c}_i] = \widehat{\pi}_i.
\end{aligned}$$

The remaining part of the proof deals with the case of a symmetric Cournot oligopoly.

(d) Due to the symmetry of the game, asymmetric equilibria, if any, would come in n-tuples. Hence, the conclusion follows from part (a) directly.

(e) Let q^* denote each firm's equilibrium output. Differentiating the first order condition with respect to c, we get:

$$\frac{\partial q^*}{\partial c}\left[(n+1)\,P'\,(nq^*) + nq^*P"\,(nq^*)\right] = 1 \tag{13}$$

Using the first order condition and **C3**, it is easy to see that the term in brackets is strictly negative, it follows that $\frac{\partial q^*}{\partial c} < 0$.

We now show that per-firm profit decreases in c. Denote the equilibrium variables by q^*_i, π^*_i and Q^*_{-i} when the unit cost is c, and by q'_i, π'_i and Q'_{-i} the same variables when the unit cost is $c' > c$.

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Differentiating $\pi_i^* = q^* \left[P(nq^*) - c\right]$ with respect to c yields

$$\begin{aligned}
\frac{\partial \pi_i^*}{\partial c} &= \frac{\partial q^*}{\partial c}\left[P(nq^*) - c\right] + q^*\left[P'(nq^*)n\frac{\partial q^*}{\partial c} - 1\right] \qquad (14)\\
&= \frac{\partial q^*}{\partial c}(n-1)q^*P'(nq^*) - q^* \text{ by } (\mathbf{??})\\
&= -q^*\frac{2P'(Q^*) + Q^*P"\left(Q^*\right)}{(n+1)\,P'\left(Q^*\right) + Q^*P"\left(Q^*\right)} \text{ by } (13)
\end{aligned}$$

Clearly, **C3** implies that $2P'(Q) + QP"\left(Q\right) < 0$ for all Q, so the numerator in the above fraction is < 0. It is then easy to see that the denominator is then also < 0. Hence $\frac{\partial \pi_i^*}{\partial c} < 0$.

Proof of Proposition 6

Let us show that (8) holds (which will imply that both Condition (6) and Condition (7) are satisfied).

We have:

$$\Pi^* = \left(P(Q^*) - c_i\right)q_i^* + \sum_{j\neq i}\left(P(Q^*) - c_j\right)q_j^*$$

then:

$$\begin{aligned}
\frac{\partial \Pi^*}{\partial c_i} &= \left(P'(Q^*)\frac{\partial Q^*}{\partial c_i} - 1\right)q_i^* + \left(P(Q^*) - c_i\right)\frac{\partial q_i^*}{\partial c_i} + \\
&\quad \sum_{j\neq i}\left[P'(Q^*)\frac{\partial Q^*}{\partial c_i}q_j^* + \left(P(Q^*) - c_j\right)\frac{\partial q_j^*}{\partial c_i}\right]
\end{aligned}$$

which can be rewritten as:

$$\frac{\partial \Pi^*}{\partial c_i} = -q_i^* + \sum_j\left[P'(Q^*)\frac{\partial Q^*}{\partial c_i}q_j^* + \left(P(Q^*) - c_j\right)\frac{\partial q_j^*}{\partial c_i}\right]$$

When $c_i = c_j = c$, the latter becomes:

$$\begin{aligned}
\frac{\partial \Pi^*}{\partial c_i} &= -q_i^* + \sum_j\left[P'(Q^*)\frac{\partial Q^*}{\partial c_i}q_j^* + \left(P(Q^*) - c\right)\frac{\partial q_j^*}{\partial c_i}\right]\\
&= -q_i^* + P'(Q^*)\frac{\partial Q^*}{\partial c_i}\sum_j q_j^* + \left(P(Q^*) - c\right)\sum_j\frac{\partial q_j^*}{\partial c_i}\\
&= -q_i^* + P'(Q^*)\frac{\partial Q^*}{\partial c_i}.Q^* + \left(P(Q^*) - c\right)\frac{\partial Q^*}{\partial c_i}\\
&= \underbrace{-q_i^*}_{\text{direct effect}} + \underbrace{\frac{\partial Q^*}{\partial c_i}\left[P'(Q^*).Q^* + \left(P(Q^*) - c\right)\right]}_{\text{strategic effect}}
\end{aligned}$$

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Adding the n first order conditions at the Cournot equilibrium yields

$$nP\left(Q^*\right)+Q^*P'\left(Q^*\right)=\sum_{k=1}^{n}c_k=nc$$

Thus,

$$P'(Q^*).Q^*+(P(Q^*)-c)=\frac{n-1}{n}P'(Q^*).Q^*<0$$

Moreover, we have already shown (see the proof of Prop. 4) that $\frac{\partial Q^*}{\partial c_i}<0$. It then follows that:

$$\frac{\partial Q^*}{\partial c_i}\left[P'(Q^*).Q^*+(P(Q^*)-c)\right]>0$$

which yields:

$$\frac{\partial \Pi^*}{\partial c_i}>-q_i^*$$

Proof of Proposition 7

First note that for firm i, charging a price of c_i strictly dominates charging any price strictly below c_i. Hence, we restrict attention to the price space $[c_i,\infty)$ as the action set for firm $i, i=1,2,...,n$. Then the transfomed profit function $\log\pi_i(p_i,p_{-i})$ is well defined.

(a) For the proof that the game with log-profits as payoffs is of strict strategic complements, observe that, due to **B3**, each payoff $\pi_i(p_i,p_{-i})$ satisfies $\partial^2\log\pi_i(p_i,p_{-i})/\partial p_i\partial p_{-i}>0$. Hence, by the strong version of Topkis's Theorem (see Amir, 1996 or Topkis, 1998 p. 79), every selection of $r_i(p_{-i})$ is strictly increasing in p_{-i}. It follows directly from the property of strategic complements, via Tarski's fixed point theorem, that the Bertrand equilibrium set is nonempty. Uniqueness then follows from a well known argument from **B4** (for details, see Milgrom and Roberts 1990, pp. 1271-1272, or Vives, 1999 pp. 149-150).

(b) To show that for any firm j, the equilibrium price p_j^* is increasing in c_i, note first that the price game is log-supermodular (from part (a)). In addition, for player i, $\log\pi_i(p_i,p_{-i})=\log(p_i-c_i)+\log D_i(p_i,p_{-i})$ has increasing differences in (p_i,c_i) since $\partial^2\log\pi_i(p_i,p_{-i})/\partial p_i\partial c_i=(p_i-c_i)^{-2}>0$, and the constraint set $[c_i,\infty)$ is clearly ascending in c_i. For any player $k\neq i$, $\log\pi_k(p_k,p_{-k})$ is independent of c_i, so has increasing differences in (p_k,c_i) since $\partial^2\log\pi_k(p_k,p_{-k})/\partial p_k\partial c_i=0$. So the unique equilibrium price vector $(p_1^*,p_2^*,...,p_n^*)$ is increasing in c_i by [Milgrom and Roberts (1990), Theorem 6].

(c) We first show that every equilibrium price p_i^* is continuously differentiable in c_j, for all i and j. Viewed as a correspondence in the parameter c_j, p_i^* is u.h.c., by the u.h.c. property of the equilibrium correspondence for games with continuous payoff functions (jointly in own and rivals' actions), see e.g., Fudenberg and Tirole, 1990. Since p_i^* is also single-valued in c_j (from part (i)), p_i^* is a continuous function. Then the fact that p_i^* is continuously differentiable in c_j follows from the Implicit Function Theorem. Finally, continuous differentiability of π_i^*

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follows from that of all the $p_i^{*\prime}s$.

(d) Differentiating $\pi_i^* = (p_i^* - c_i)D_i(p_i^*, p_{-i}^*)$ with respect to c_j, for $i \neq j$, yields

$$\frac{\partial \pi_i^*}{\partial c_j} = \frac{\partial p_i^*}{\partial c_j} D_i(p_i^*, p_{-i}^*) + (p_i^* - c_i) \sum_k \frac{\partial D_i}{\partial p_k} \frac{\partial p_k^*}{\partial c_j} \tag{15}$$

Using the first order condition $D_i(p_i^*, p_{-i}^*) + (p_i^* - c_i)\frac{\partial D_i}{\partial p_i} = 0$, (15) reduces to

$$\frac{\partial \pi_i^*}{\partial c_j} = (p_i^* - c_i) \sum_{k \neq i} \frac{\partial D_i}{\partial p_k} \frac{\partial p_k^*}{\partial c_j} \geq 0$$

since $\frac{\partial D_i}{\partial p_k} > 0$ (goods are substitutes) and $\frac{\partial p_k^*}{\partial c_j} \geq 0$ from part (b).

(e) We first show that the slopes of p_j^* in c_i are all ≤ 1, for $j = 1, 2, ..., n$. With the change of variable $z_j = p_j - c_i$, $j = 1, 2, ..., n$, the log-profit functions of firm i and firm $k \neq i$ are $\log \widetilde{\pi}_i(z_i, z_{-i}) \triangleq \log z_i + \log D_i(z_i + c_i, z_{-i} + c_i)$ and $\log \widetilde{\pi}_k(z_k, z_{-k}) \triangleq \log(z_k + c_i) + \log D_k(z_k + c_i, z_{-k} + c_i)$. With these new action variables, the game is easily seen to remain supermodular. In addition, $\log \widetilde{\pi}_i(z_i, z_{-i})$ has decreasing differences in (z_i, c_i), since $\frac{\partial \log \widetilde{\pi}_i(z_i, z_{-i})}{\partial z_i \partial c_i} = \sum_k \frac{\partial^2 \log D_i}{\partial p_k \partial p_i} < 0$ by **B4**; and $\log \widetilde{\pi}_k(z_k, z_{-k})$ for $k \neq i$ has decreasing differences in (z_k, c_i), since $\frac{\partial \log \widetilde{\pi}_k(z_i, z_{-i})}{\partial z_k \partial c_i} = \frac{-1}{(z_k + c_i)^2} + \sum_j \frac{\partial^2 \log D_k}{\partial p_j \partial p_k} < 0$ by **B4**. Hence by [Milgrom and Roberts (1990), Theorem 6], the unique equilibrium $(z_1^*, z_2^*, ..., z_n^*)$ is decreasing in c_i, the action set $z_j \in [-c_j, \infty)$ being descending in c_j, for each j. This is equivalent to the slopes of p_j^* in c_i being all ≤ 1, since $z_j = p_j - c_i$, for all j. Combining with part (b), we get $0 \leq \frac{p^*(c') - p^*(c)}{c' - c} \leq 1$ for all $c' > c$.

The FOC for maximizing firm i's (log-transformed) profit in the p variables is

$$\frac{1}{p_i^* - c_i} + \frac{\partial \log D_i(p_1, p_2, ..., p_n)}{\partial p_i} = 0 \tag{16}$$

Differentiating $\log \pi_i^*$ w.r.t. c_i yields

$$\begin{aligned}
\frac{\partial \log \pi_i^*}{\partial c_i} &= \frac{\frac{\partial p_i^*}{\partial c_i} - 1}{p_i^* - c_i} + \sum_k \frac{\partial \log D_i}{\partial p_k} . \frac{\partial p_k^*}{\partial c_i} \\
&= -\left(\frac{\partial p_i^*}{\partial c_i} - 1\right) \frac{\partial \log D_i}{\partial p_i} + \sum_k \frac{\partial \log D_i}{\partial p_k} . \frac{\partial p_k^*}{\partial c_i} \text{ by (16)} \\
&= \frac{\partial \log D_i}{\partial p_i} + \sum_{k \neq i} \frac{\partial \log D_i}{\partial p_k} . \frac{\partial p_k^*}{\partial c_i} \\
&\leq \frac{\partial \log D_i}{\partial p_i} + \sum_{k \neq i} \frac{\partial \log D_i}{\partial p_k} \text{ using } \mathbf{B2}\text{(ii) and } 0 \leq \frac{\partial p_k^*}{\partial c_i} \leq 1 \\
&< 0 \text{ by } \mathbf{B2}\text{(iii)}.
\end{aligned}$$

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(f) When the Bertrand game is symmetric, the unique Bertrand equilibrium must be symmetric, for otherwise equilibria would come in pairs.

(g) That the equilibrium price p^* increases in c follows from the same argument as for part (b) in view of the fact that $\log \pi_i$ has increasing differences in (p_i, c) since $\partial^2 \log \pi_i(p_i, p_{-i})/\partial p_i \partial c = (p_i - c)^{-2} > 0$. To show that p^* has all slopes in c less than 1, use the change of variable $z_i = p_i - c$ for each i, so that firm i's payoff becomes $\log z_i + \log D_i(z_i + c, z_{-i} + c)$, which is easily seen to have decreasing differences in (z_i, c) since $\sum_k \frac{\partial \log D_i}{\partial p_k \partial p_i} < 0$ by **B4**. By [Milgrom and Roberts (1990), Theorem 6], z^* is decreasing in c, which is equivalent to p^* having all slopes in c less than 1.

(h) From an argument similar to the proof of part (c), p^* and thus $\pi^* = (p^* - c) D_i(p^*, p^*, ..., p^*)$ are differentiable with respect to c. The FOC at a symmetric Bertrand equilibrium is

$$\frac{1}{p^* - c} + \frac{\partial \log D_i(p^*, ..., p^*)}{\partial p_i} = 0 \tag{17}$$

Differentiating $\log \pi^*$ w.r.t. c yields

$$\begin{aligned}
\frac{\partial \log \pi^*}{\partial c} &= \frac{\frac{\partial p^*}{\partial c} - 1}{p^* - c} + \sum_k \frac{\partial \log D_i}{\partial p_k} . \frac{\partial p^*}{\partial c} \\
&= -\left(\frac{\partial p^*}{\partial c} - 1\right) \frac{\partial \log D_i}{\partial p_i} + \sum_k \frac{\partial \log D_i}{\partial p_k} . \frac{\partial p^*}{\partial c} \text{ by (17)} \\
&= \frac{\partial \log D_i}{\partial p_i} + \sum_{k \neq i} \frac{\partial \log D_i}{\partial p_k} . \frac{\partial p^*}{\partial c} \\
&\leq \frac{\partial \log D_i}{\partial p_i} + \sum_{k \neq i} \frac{\partial \log D_i}{\partial p_k} \text{ by } \mathbf{B2}\text{(ii) and Part (g)} \\
&< 0 \text{ by } \mathbf{B2}\text{(iii)}.
\end{aligned}$$

Proof of Proposition 8

Let us show that Condition (8) holds (which will imply that both Condition (6) and Condition (7) are satisfied).

We have:

$$\Pi^* = (p_i^* - c_i) D_i^* + \sum_{j \neq i} \left(p_j^* - c_j\right) D_j^*$$

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then:

$$\begin{aligned}\frac{\partial \Pi^*}{\partial c_i} &= \left(\frac{\partial p_i^*}{\partial c_i} - 1\right) D_i^* + (p_i^* - c_i) \frac{\partial D_i^*}{\partial c_i} + \\ &\quad \sum_{j \neq i} \left[\frac{\partial p_j^*}{\partial c_i} D_j^* + (p_i^* - c_i) \frac{\partial D_j^*}{\partial c_i}\right]\end{aligned}$$

which can be rewritten as:

$$\begin{aligned}\frac{\partial \Pi^*}{\partial c_i} &= -D_i^* + \sum_j \left[\frac{\partial p_j^*}{\partial c_i} D_j^* + \left(p_j^* - c_j\right) \frac{\partial D_j^*}{\partial c_i}\right] \\ &= -D_i^* + \sum_j \left[\frac{\partial p_j^*}{\partial c_i} D_j^* + \left(p_j^* - c_j\right) \sum_k \frac{\partial D_j}{\partial p_k} . \frac{\partial p_k^*}{\partial c_i}\right] \\ &= -D_i^* + \sum_j \left[\frac{\partial p_j^*}{\partial c_i} D_j^* + \left(p_j^* - c_j\right) \frac{\partial D_j}{\partial p_j} . \frac{\partial p_j^*}{\partial c_i} + \left(p_j^* - c_j\right) \sum_{k \neq j} \frac{\partial D_j}{\partial p_k} . \frac{\partial p_k^*}{\partial c_i}\right] \\ &= -D_i^* + \sum_j \left[\frac{\partial p_j^*}{\partial c_i} \underbrace{\left(D_j^* + \left(p_j^* - c_j\right) \frac{\partial D_j}{\partial p_j}\right)}_{=0} + \left(p_j^* - c_j\right) \sum_{k \neq j} \frac{\partial D_j}{\partial p_k} . \frac{\partial p_k^*}{\partial c_i}\right] \\ &= -D_i^* + \sum_j \left[\left(p_j^* - c_j\right) \sum_{k \neq j} \frac{\partial D_j}{\partial p_k} . \frac{\partial p_k^*}{\partial c_i}\right]\end{aligned}$$

We have already shown that $\frac{\partial p_k^*}{\partial c_i} > 0$ for any k, i (see the proof for part (b) of Prop. 5). Moreover, we have $\frac{\partial D_j}{\partial p_k} > 0$ for any $j \neq k$ (from B2(ii)). It then follows that:

$$\frac{\partial \Pi^*}{\partial c_i} > -D_i^* = -q_i^*$$

This proof establishes a result which is actually more general than Condition (8). Indeed, it is shown that *for any* $(c_1, c_2, ..., c_n)$:

$$\frac{\partial \Pi^*}{\partial c_i} (c_1, c_2, ..., c_n) > -q_i^* (c_1, c_2, ..., c_n) \tag{18}$$