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Abstract

Suppose a decision maker (DM) has partial information about certain events of a σ-algebra $\mathcal{A}$ belonging to a set $\mathcal{E}$ and assesses their likelihood through a capacity v. When is this information probabilistic, i.e. compatible with a probability? We consider three notions of compatibility with a probability in increasing degree of preciseness. The weakest requires the existence of a probability P on $\mathcal{A}$ such that $P(E) \geq v(E)$ for all $E \in \mathcal{E}$, we then say that v is a probability minorant. A stronger one is to ask that v be a lower probability, that is the infimum of a family of probabilities on $\mathcal{A}$. The strongest notion of compatibility is for v to be an extendable probability, i.e. there exists a probability P on $\mathcal{A}$ which coincides with v on $\mathcal{A}$.

We give necessary and sufficient conditions on v in each case and, when $\mathcal{E}$ is finite, we provide effective algorithms that check them in a finite number of steps.

Keywords: Partial probabilistic information, exact capacity, core, extensions of set functions.

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1 Introduction

Following the pioneering work of von Neumann-Morgenstern [11] and Savage [5], numerous authors have considered the problem of defining a mathematical framework allowing to model situations of uncertainty. Notably, Dempster [1] and Shafer [8, 9] have proposed a representation of uncertain environments requiring a "lower probability" function (Dempster) or "degree of belief" (Shafer). This belief function is the lower envelope of a family of probabilities which are compatible with the given data. Though it is not additive in general, it is nevertheless a capacity.

These works have been at the source of the study of the properties of the core of a capacity. In particular, Shapley [10] has shown that the core of a convex capacity is not empty and has studied its geometry in detail, giving economic interpretations of his results.

In this paper, we are interested in the situation of a decision maker (DM) who considers a set of states of nature Ω and has partial subjective or objective information about certain events in a subset $\mathcal{E}$ of $\mathcal{A}$ a σ-algebra of subsets of Ω. $\mathcal{E}$ will be naturally assumed to contain Ω and $\emptyset$ and his assessment of likelihood v to take its values in $[0, 1]$ and satisfy $v(\Omega) = 1$ and $v(\emptyset) = 0$.

The aim of this paper is to answer the three following questions:

1) When can v be interpreted as a minorant for some probability P on $\mathcal{A}$, that is when does there exist a probability P on $\mathcal{A}$ such that $P(E) \geq v(E)$ $\forall E \in \mathcal{E}$ (i.e. $C(v) \neq \emptyset$)?

2) When can v be interpreted as a lower probability i.e. when is v the infimum of family of probabilities on $\mathcal{A}$? When this is the case v, according to the usual denomination, will be called exact (i.e. $\forall E \in \mathcal{E}$, $\exists P$ on $\mathcal{A}$ such that $P(E) = v(E)$ and $P \in C(v)$).

3) When can v be interpreted as the restriction to $\mathcal{E}$ of a given probability on $\mathcal{A}$? When this is the case, v will be called an extendable probability on $\mathcal{E}$. By definition, this means that there exists a probability P on $\mathcal{A}$ such that
$P(E) = v(E)$ for all $E \in \mathcal{E}$.

One notices that, indeed, these three notions correspond to more and more precise probabilistic "information" (objective or subjective).

In all the paper, we will assume the natural requirement that v is furthermore monotone i.e. a capacity. This condition is indeed needed for v to be a lower probability or else an extendable probability.

We show that Ky Fan's theorem [4] allows to derive such criteria but, unfortunately, the application of these criteria requires the checking of an infinite number of conditions, even when the set of states of nature Ω is finite. Using a trick due to Wolf (see Huber [2]), we then show that it is possible to modify these criteria in such a way that when $\mathcal{E}$ is finite (even if Ω is not) there remains only a finite number of conditions that can be checked through an effective algorithm in a finite number of steps.

In section 2, we give the definitions of the main notions needed.

In section 3, we give preliminary results which will be useful in the sequel.
In section 4, we state the main results of the paper.

2   Definitions

We first give some definitions that will be used in the sequel.

Definition 2.1 *Let Ω be a set. A collection $\mathcal{A} \subset \mathcal{P}(\Omega)$ is a λ-system if it satisfies the following three properties:*
1) $\emptyset, \Omega \in \mathcal{A}$.
2) $A, B \in \mathcal{A}$, $A \cap B = \emptyset \Rightarrow A \cup B \in \mathcal{A}$.
3) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$.
$\mathcal{A}$ is an algebra if it satisfies in addition:
4) $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$.
5) An algebra $\mathcal{A}$ is a σ-algebra if it is closed under denumerable union i.e. if $(A_n)_{n \in \mathbb{N}}$ is a sequence of elements of $\mathcal{A}$, then $\cup_{n \in \mathbb{N}} A_n \in \mathcal{A}$.

Definition 2.2 *Let $\mathcal{A}$ be an algebra, $\mathcal{M}_\mathcal{A}$ will denote the set of finitely additive set-functions on $\mathcal{A}$, i.e. the set of set-functions μ such that*
$\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \emptyset$ and $P_\mathcal{A} \subset \mathcal{M}_\mathcal{A}$ the set of probabilities on $\mathcal{A}$.

Definition 2.3 *The characteristic function of a set A will be noted A^*.*

Definition 2.4 *Let Ω be a set and $\mathcal{E}$ a subset of $\mathcal{P}(\Omega)$ (in this paper, we will always assume that $\mathcal{E}$ contains $\emptyset$ and Ω) the algebra (repectively σ-algebra) generated by $\mathcal{E}$ is denoted by $\mathcal{A}(\mathcal{E})$ (respectively $\mathcal{A}^\sigma(\mathcal{E})$).*

Definition 2.5 *Let Ω be a set and $\mathcal{E}$ a subset of $\mathcal{P}(\Omega)$ containing $\emptyset$ and Ω. A set function v on $\mathcal{E}$ is called a (generalized) capacity if it is monotone i.e. $A, B \in \mathcal{E}$ $A \subset B \Rightarrow v(A) \leq v(B)$, $v(\emptyset) = 0$ and $v(\Omega) = 1$.*

Definition 2.6 *Given a set Ω, an algebra $\mathcal{A}$ of subsets of Ω, $\mathcal{E} \subset \mathcal{A}$ and v a capacity on $\mathcal{E}$, the inner set-function v_* associated with v is defined on $\mathcal{A}$ by:*

$$v_*(A) = sup\left\{v(E) \mid E \in \mathcal{E}, \ E \subset A\right\}.$$

Definition 2.7 *Let v a capacity defined on a subset $\mathcal{E}$ of an algebra $\mathcal{A}$. The core of v is defined by:*

$$C(v) = \{P \in \mathcal{P}_\mathcal{A} \mid P(E) \geq v(E) \ \forall E \in \mathcal{E}\}$$

(in case there is more than one algebra, we will note $C_\mathcal{A}(v)$ the core of v with respect to $\mathcal{A}$, in order to avoid any risk of confusion).

Definition 2.8 *Let $\mathcal{E}$ be a subset of an algebra $\mathcal{A}$. A capacity v is said to be a probability minorant if $C(v) \neq \emptyset$ i.e. if there exists a probability P on $\mathcal{A}$ such that $P \in C(v)$ i.e. such that $P(E) \geq v(E)$ for all $E \in \mathcal{E}$.*

Definition 2.9 *Let $\mathcal{E}$ be a subset of an algebra $\mathcal{A}$. A capacity v is said to be a lower probability if v is exact i.e. if for all $E \in \mathcal{E}$, there exists P on $\mathcal{A}$ such that $P(E) = v(E)$ and $P \in C(v)$.*

Definition 2.10 *Let $\mathcal{E}$ be a subset of an algebra $\mathcal{A}$. A capacity v is said to be an extendable probability if there exists a probability P on $\mathcal{A}$ such that $P(E) = v(E)$ for all $E \in \mathcal{E}$.*

Definition 2.11 *A capacity v on an algebra $\mathcal{A}$ is said to be convex if whenever $A, B \in \mathcal{A}$,*

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B).$$

3 Preliminary results

First we introduce some preliminary results that will be useful in the rest of the paper.

Lemma 3.1 *Let $\mathcal{A}$ be an algebra of subsets of a set Ω, $\mathcal{E} \subset \mathcal{A}$ and v a capacity on $\mathcal{E}$. Then v_* is a capacity on $\mathcal{A}$ such that $v_*(E) = v(E)$ for all $E \in \mathcal{E}$.*

Proof : $\star v_*(\emptyset) = sup\{v(E) \mid E \in \mathcal{E}, \ E \subset \emptyset\} = v(\emptyset) = 0$ by hypothesis.
$\star v_*(\Omega) = sup\{v(E) \mid E \in \mathcal{E}, \ E \subset \Omega\} = v(\Omega) = 1$ by hypothesis.
$\star$ Let $A_1, A_2 \in \mathcal{A}$ such that $A_1 \subset A_2$. Since $\{E \in \mathcal{E}, \ E \subset A_1\} \subset \{E \in \mathcal{E}, \ E \subset A_2\}$,
$v_*(A_1) = sup\{v(E) \mid E \in \mathcal{E}, \ E \subset A_1\} \leq sup\{v(E) \mid E \in \mathcal{E}, \ E \subset A_2\} = v_*(A_2)$.
$\star$ Finally, it is obvious that v_* extends v since v is monotone.
$\square$

Lemma 3.2 *Let $\mathcal{E}$ be a subset of an algebra $\mathcal{A}$ on a set Ω and v a capacity defined on $\mathcal{E}$. Then $C(v) = C(v_*)$.*

Proof : It is obvious that $C(v_*) \subset C(v)$, since if $P \in C(v_*)$ and $E \in \mathcal{E}$,

$$P(E) \geq v_*(E) \geq v(E).$$

Conversely, if $P \in C(v)$ and $A \in \mathcal{A}$. For all $E \in \mathcal{E}$ such that $E \subset A$,
$P(A) \geq P(E) \geq v(E)$, which clearly implies that $P(A) \geq v_*(A)$.
$\square$

Proposition 3.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras of subsets of a set Ω such that $\mathcal{B} \subset \mathcal{A}$, then every probability P on $\mathcal{B}$ can be extended into a probability on $\mathcal{A}$ and the set of extensions of P to a probability on $\mathcal{A}$ is equal to $C_{\mathcal{A}}(P_*)$.*

Proof : Let P be a probability on $\mathcal{B}$ and $\mathcal{Q} := \left\{ Q \in \mathcal{P}_{\mathcal{A}} \mid Q_{|\mathcal{B}} = P \right\}$ be the set of extensions of P to $\mathcal{A}$. We want to show that $\mathcal{Q}$ is non-empty and is equal to $C_{\mathcal{A}}(P_*)$. We first note that P_* is a convex capacity on $\mathcal{A}$. Indeed, let A_1, $A_2 \in \mathcal{A}$, we must show that $P_*(A_1) + P_*(A_2) \leq P_*(A_1 \cup A_2) + P_*(A_1 \cap A_2)$.

By the very definition of P_*, for all $\epsilon > 0$ we can find B_1, $B_2 \in \mathcal{B}$ such that $B_1 \subset A_1$, $B_2 \subset A_2$ and $P_*(A_1) - \epsilon < P(B_1)$, $P_*(A_2) - \epsilon < P(B_2)$.

Adding these two inequalities, we obtain

$$P_*(A_1) + P_*(A_2) - 2\epsilon \;\; < P(B_1) + P(B_2)$$

$$= P(B_1 \cup B_2) + P(B_1 \cap B_2)$$

$$\leq P_*(A_1 \cup A_2) + P_*(A_1 \cap A_2)$$

Which gives the result since ϵ is arbitrary. Now, since P_* is convex, its core is non-empty (see Schmeidler (1986) [7]), so that the only thing that remains to be proved is that $\mathcal{Q} = C_{\mathcal{A}}(P_*)$.

Let $Q \in \mathcal{Q}$ and $A \in \mathcal{A}$. Since, for every $B \in \mathcal{B}$ such that $B \subset A$, $P(B) = Q(B)$ and $Q(B) \leq Q(A)$, we see that $P_*(A) \leq Q(A)$ and therefore $\mathcal{Q} \subset C_{\mathcal{A}}(P_*)$.

Conversely, let $Q \in C_{\mathcal{A}}(P_*)$. Then $Q_{|\mathcal{B}} \in C_{\mathcal{B}}(P)$, since, for all $B \in \mathcal{B}$, $Q(B) \geq P_*(B) = P(B)$. Now, since P is a probability, $C_{\mathcal{B}}(P) = \{P\}$[1] so that $Q_{|\mathcal{B}} = P$ i.e. $Q \in \mathcal{Q}$.

$\square$

4 Partial probabilistic information

Let $\mathcal{A}$ be a σ-algebra of subsets of a set Ω and $\mathcal{E}$ a subset of $\mathcal{A}$ containing $\emptyset$ and Ω. Let also v be a capacity on $\mathcal{E}$ assumed to represent the likelihood over $\mathcal{E}$ obtained through subjective or objective information by a decision-maker.

We first give a characterization of "probability minorants".

Theorem 4.1 *The following assertions are equivalent:*

1) v is a probability minorant.

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[1] Indeed, let $B \in \mathcal{B}$, then $P(B) + P(B^c) = 1 = Q(B) + Q(B^c)$. Therefore, $P(B) - Q(B) = Q(B^c) - P(B^c)$ and since $P(B) - Q(B) \leq 0$ and $Q(B^c) - P(B^c) \geq 0$, this shows that $P(B) = Q(B)$

2) *For all $n \in \mathbb{N}^*$, $a_i > 0$ and $A_i \in \mathcal{A}^\sigma(\mathcal{E})$ such that the functions A_i^* are linearly independent,*

$$\sum_{i=1}^{n} a_i A_i^* = 1 \Rightarrow \sum_{i=1}^{n} a_i v_*(A_i) \leq 1.$$

Proof : $\star$ 1) implies 2) :

Let $P \in C(v)$, a_i, A_i be given as in 2) such that $\sum_{i=1}^{n} a_i A_i^* = 1$. By Lemma 3.2, $P(A) \geq v_*(A)$ for all $A \in \mathcal{A}^\sigma(\mathcal{E})$. Therefore,

$$\sum_{i=1}^{n} a_i v_*(A_i) \leq \sum_{i=1}^{n} a_i P(A_i)$$

$$= \int_\Omega \left(\sum_{i=1}^{n} a_i A_i^* \right) dP$$

$$= \int_\Omega dP$$

$$= 1$$

$\star$ 2) implies 1) :

Since by Proposition 3.3, any probability on $\mathcal{A}^\sigma(\mathcal{E})$ can be extended to a probability on $\mathcal{A}$ and by Lemma 3.2 $C(v) = C(v_*)$, it is enough to show that $C(v_*)$ is non-empty.

$\star$ In a first step, we show that $C(v_*) \neq \emptyset$ if and only if for all $n \in \mathbb{N}^*$, $a_i > 0$, $A_i \in \mathcal{A}^\sigma(\mathcal{E})$,

$$\sum_{i=1}^{n} a_i A_i^* \leq 1 \Rightarrow \sum_{i=1}^{n} a_i v_*(A_i) \leq 1 \quad (*)$$

It is clear that this condition is necessary. In order to show that it is sufficient it is enough to prove that it implies the existence of a functional $f \in \mathcal{B}'_\infty(\mathcal{A}^\sigma(\mathcal{E}))$ (the topological dual of the space of bounded functions on $\mathcal{A}^\sigma(\mathcal{E})$, $\mathcal{B}_\infty(\mathcal{A}^\sigma(\mathcal{E}))$) which satisfies the following conditions:

i) $f(A^*) \geq v_*(A)$ $\forall A \in \mathcal{A}^\sigma(\mathcal{E})$.
ii) $f(\Omega^*) = 1$.

This will give the result, indeed P defined on $\mathcal{A}^\sigma(\mathcal{E})$ by $P(A) := f(A^*)$ is then clearly a probability and by condition $i)$ it belongs to the core of v_*.

In order to show the existence of such a functional, we will use the following theorem of Ky Fan (see Theorem 13 p. 126 of "On systems of linear inequalities" [4]).

Theorem (Ky Fan) *Let $(x_\nu)_{\nu \in J}$ be a family of elements, not all 0, in a real normed linear space X, and let $(\alpha_\nu)_{\nu \in J}$ be a corresponding family of real numbers. Let*

$$\sigma := Sup \left\{ \sum_{j=1}^{n} \beta_j \alpha_{\nu_j} \text{ such that } n \in \mathbb{N}^*, \; \beta_1, ..., \beta_n \in \mathbb{R}_*^+ \; \| \sum_{j=1}^{n} \beta_j x_{\nu_j} \| = 1 \right\}.$$

Then:
1) The system $f(x_\nu) \geq \alpha_\nu$ ($\nu \in J$) has a solution $f \in X'$ if and only if σ is finite.
2) If the system $f(x_\nu) \geq \alpha_\nu$ ($\nu \in J$) has a solution $f \in X'$ and if the zero-functional is not a solution, then σ is equal to the minimum of the norms of all solutions f of this system.

We apply the theorem to the normed vector space $\mathcal{B}_\infty(\mathcal{A}^\sigma(\mathcal{E}))$ and the family of vectors A^* ($A \in \mathcal{A}^\sigma(\mathcal{E})$) and the corresponding family of real numbers $v_*(A)$.

In order to prove that σ is finite, we need to find an upper bound for $\sum_{i=1}^n a_i v_*(A_i)$ over all families $(a_i)_{1 \leq i \leq n}$, $(A_i)_{1 \leq i \leq n}$ such that $a_i > 0$, $A_i \in \mathcal{A}^\sigma(\mathcal{E})$ and

$$\| \sum_{i=1}^n a_i A_i^* \|_\infty = 1. \quad (**)$$

By $(**)$,

$$\sum_{i=1}^n a_i A_i^* \leq 1$$

and therefore by $(*)$:

$$\sum_{i=1}^n a_i v_*(A_i) \leq 1.$$

so that $\sigma \leq 1$.

Therefore by Ky Fan's theorem there exists $g \in \mathcal{B}'_\infty(\mathcal{A}^\sigma(\mathcal{E}))$ of norm σ satisfying $i)$.

Now, either $v(E) = 0$ $\forall E \in \mathcal{E}$, in which case the core of v is obviously non-empty or there exists $E \in \mathcal{E}$ such that $v(E) > 0$, in which case the zero-functional is not a solution and therefore by Ky Fan's theorem, since $0 < \sigma \leq 1$, we obtain the desired functional by setting $f = \frac{1}{\sigma}g$.

$\star$ We now show that 2) is equivalent to $(*)$, which will complete the proof.

First, we show that we can assume that equality holds in the premise of $(*)$ i.e. that $(*)$ is equivalent to

$$\sum_{i=1}^n a_i A_i^* = 1 \Rightarrow \sum_{i=1}^n a_i v_*(A_i) \leq 1 \quad (***)$$

where $n \in \mathbb{N}^*$, $a_i > 0$ and $A_i \in \mathcal{A}^\sigma(\mathcal{E})$.

Indeed $(*)$ obviously implies $(***)$. Now, in order to show that $(***)$ implies $(*)$, suppose that $\sum_{i=1}^n a_i A_i^* \leq 1$ and set

$$\phi = 1 - \sum_{i=1}^n a_i A_i^*.$$

8

Since ϕ is a positive $\mathcal{A}^\sigma(\mathcal{E})$-measurable function which takes only a finite number of values there exist $b_j > 0$, $B_j \in \mathcal{A}^\sigma(\mathcal{E})$ such that

$$\phi = \sum_{j=1}^{m} b_j B_j^*.$$

So that,

$$\sum_{i=1}^{n} a_i A_i^* + \sum_{j=1}^{m} b_j B_j^* = 1$$

Therefore by $(***)$,

$$\sum_{i=1}^{n} a_i v_*(A_i) + \sum_{j=1}^{m} b_j v_*(B_j) \leq 1$$

and since $b_j > 0$ and $v_*(B_j) \geq 0$, we conclude that

$$\sum_{i=1}^{n} a_i v_*(A_i) \leq 1.$$

$\star$ We now show that $(***)$ is equivalent to 2), which will conclude the proof.

It is obvious that $(***)$ implies 2).

To show the converse implication, suppose that 2) holds and that $(***)$ is not satisfied. Then, for a certain choice of a_i and A_i, $\sum_{i=1}^{n} a_i A_i^* = 1$ and $\sum_{i=1}^{n} a_i v_*(A_i) > 1$, where we can suppose that n is the minimum integer for which these two relations hold simultaneously. Obviously, since 2) is supposed to hold, the functions A_i^* are linearly dependent and therefore there exist $b_i \in \mathbb{R}$ not all equal to zero such that

$$\sum_{i=1}^{n} b_i A_i^* = 0.$$

From which it follows that for all $t \in \mathbb{R}$,

$$\sum_{i=1}^{n} (a_i + t b_i) A_i^* = 1.$$

Set $I = \{t \in \mathbb{R} \mid a_i + t b_i \geq 0 \quad \forall i \in \{1, ..., n\}\}.$

Since $\sum_{i=1}^{n} b_i A_i^* = 0$ and the coefficients are not all equal to zero, there exist i, j such that $b_i > 0$ and $b_j < 0$, so that for all $t \in I$,

$$-\frac{a_i}{b_i} \leq t \leq -\frac{a_j}{b_j}.$$

Hence I is a closed interval which contains 0 in its interior and it is bounded. Therefore the function $t \mapsto \sum_{i=1}^{n} (a_i + t b_i) v_*(A_i)$ reaches its maximum at an

endpoint t_0 of I. At this point we still have

$$\sum_{i=1}^{n}(a_i + t_0 b_i)A_i^* = 1$$

while

$$\sum_{i=1}^{n}(a_i + t_0 b_i)v_*(A_i) \geq \sum_{i=1}^{n} a_i v_*(A_i) > 1.$$

But since t_0 is an endpoint of I, one of the coefficient must be equal to zero contradicting the minimality of n.

$\square$

We now come to a generalization of well-known characterizations of exact games previously performed in case of capacities defined on σ-algebras (see e.g. Kannai (1969) [3], Schmeidler (1972) [6]) or else finite algebras (see Huber (1981) [2]).

Theorem 4.2 *The following assertions are equivalent:*

1) v is a lower probability.

2) For all $E \in \mathcal{E}$, $n \in \mathbb{N}^*$, $a_i \in \mathbb{R}$, $A_i \in \mathcal{A}^\sigma(\mathcal{E}) \setminus \{E\}$ such that $a_i > 0$ if $A_i \neq \Omega$ and the functions A_i^* are linearly independent.

$$\sum_{i=1}^{n} a_i A_i^* = E^* \Rightarrow \sum_{i=1}^{n} a_i v_*(A_i) \leq v(E).$$

Proof : $\star$ 1) implies 2) :

Let $\sum_{i=1}^{n} a_i A_i^* = E^*$ as in 2). Since v is exact, there exists $P \in C(v)$ such that $P(E) = v(E)$. By Lemma 3.2, $P(A) \geq v_*(A)$ for all $A \in \mathcal{A}^\sigma(\mathcal{E})$ therefore, since $a_i > 0$ if $A_i \neq \Omega$, we have:

$$
\begin{aligned}
\sum_{i=1}^{n} a_i v_*(A_i) \quad &\leq \sum_{i=1}^{n} a_i P(A_i) \\
&= \int_\Omega \left(\sum_{i=1}^{n} a_i A_i^*\right) dP \\
&= \int_\Omega E^* \, dP \\
&= P(E) \\
&= v(E).
\end{aligned}
$$

$\star$ 2) implies 1) :

- In a first step, we show that 2) is sufficient, without restricting the A_i^* to form a linear independent system.

We must show that for all $E \in \mathcal{E}$, there exists $P \in C(v)$ such that $P(E) = v(E)$. We need only consider $E \neq \Omega$ since otherwise condition 2) of Theorem 4.2 is exactly condition 2) of Theorem 4.1 and the existence of such a P follows from that theorem. Proceeding as in Theorem 4.1, given $E \in \mathcal{E} \setminus \{\Omega\}$, we have to find $f \in \mathcal{B}'_\infty(\mathcal{A}^\sigma(\mathcal{E}))$ satisfying the following conditions:

i) $f(\Omega^*) \geq 1$
ii) $f(-\Omega^*) \geq -1$
iii) $f(E^*) \geq v(E)$
iv) $f(-E^*) \geq -v(E)$
v) $f(A^*) \geq v_*(A) \ \ \forall A \in \mathcal{A}^\sigma(\mathcal{E}) \setminus \{\Omega, E\}$.

Again as in Theorem 4.1, we derive the existence of such an f from Ky Fan's theorem.

Here we have to show that the upper bound σ of the quantities

$$\sum_{i=1}^n a_i v_*(A_i) + av(E)$$

is finite, where $E \in \mathcal{E}$, $A_i \in \mathcal{A}^\sigma(\mathcal{E}) \setminus \{E\}$, $n \in \mathbb{N}^*, a, \ a_i \in \mathbb{R} \ (a_i > 0$ if $A_i \neq \Omega)$ are such that

$$\| \sum_{i=1}^n a_i A_i^* + aE^* \|_\infty = 1. \quad (*)$$

Since $(*)$ implies

$$\sum_{i=1}^n a_i A_i^* - \Omega^* + aE^* \leq 0,$$

it is clear that if we assume that for $E \in \mathcal{E}$, $A_i \in \mathcal{A}^\sigma(\mathcal{E}) \setminus \{E\}$, $n \in \mathbb{N}^*, a, \ a_i \in \mathbb{R}$ $(a_i > 0$ if $A_i \neq \Omega)$

$$\sum_{i=1}^n a_i A_i^* + aE^* \leq 0 \Rightarrow \sum_{i=1}^n a_i v_*(A_i) + av(E) \leq 0 \quad (2')$$

then $\sigma \leq 1$ and that, by applying Ky Fan's theorem, we can therefore find the desired functional, thereby showing that v is exact.

The fact that $(2')$ implies 2) without restricting the A_i^*'s to be linearly independent can be straightforwardly obtained in a similar way as in Theorem 4.1. This ends the first step of the proof.

- In a second step we intend to show that assuming as in 2) that the A_i^*'s are linearly independent is sufficient.

Let us reason ad absurdum.

Assume that $E \in \mathcal{E}$, $n \in \mathbb{N}^*$, $a_i \in \mathbb{R}$, $A_i \in \mathcal{A}^\sigma(\mathcal{E}) \setminus \{E\}$ such that $a_i > 0$ if $A_i \neq \Omega$, the functions A_i^* are linearly dependent and $\sum_{i=1}^n a_i A_i^* = E^*$, $\sum_{i=1}^n a_i v_*(A_i) > v(E)$.

Let n be the minimum integer such that these relations are both satisfied and assume that $A_i \in \mathcal{A}^\sigma(\mathcal{E}) \setminus \{E\}$ for $1 \leq i \leq n-1$ and $A_n = \Omega$.

Since the A_i^*'s are linearly dependent, there exist c_i's $\in \mathbb{R}$ such that $\sum_{i=1}^n c_i A_i^* = 0$ and indeed there exist $i, \, j \in \{1, ..., n\}$ such that $c_i \times c_j < 0$.

Several cases must be consedered:

Case 1: There exist $i, \, j \in \{1, ..., n-1\}$ such that $c_i \times c_j < 0$.

In such a case, it is easy to see that $I := \{t \in \mathbb{R} \mid a_i + tc_i \geq 0 \; \forall i \in \{1, ..., n-1\}\}$ is a compact interval containing 0 in its interior.

For a t belonging to I, $\sum_{i=1}^n (a_i + tc_i) A_i^* = E^*$ and indeed $a_i + tc_i \geq 0$ $\forall i \in \{1, ..., n-1\}$.

Furthermore, $g(t) := \sum_{i=1}^n (a_i + tc_i) v_*(A_i)$ is linear, therefore reaches its maximum at an endpoint $t_0 \in I$, so since $0 \in I$, one gets $g(t_0) > v(E)$ and n is decreased by at least one. But n was minimal, which leads to a contradiction.

Case 2: For $i \in \{1, ..., n-1\}$ all the c_i's are non-negative (a similar proof applies if the c_i's are non-positive). So indeed there exists $c_{i_0} > 0$ for $i \in \{1, ..., n-1\}$ and necessarily $c_n < 0$.

In such a case, it is easy to see that $I := \{t \in \mathbb{R} \mid a_i + tc_i \geq 0 \; \forall i \in \{1, ..., n-1\}\}$ writes $I = [t_0, +\infty)$ with $t_0 < 0$.

Furthermore, $g(t) := \sum_{i=1}^{n-1} (a_i + tc_i) v_*(A_i) + (a_n + tc_n)$, so since $c_n < 0$ and $0 \in I$, g reaches its maximum on I at t_0, so since $0 \in I$, one gets $g(t_0) > v(E)$, and n is decreased by at least one. But n was minimal, which leads to a contradiction. $\square$

We are now interested in finding necessary and sufficient conditions for v to be an extendable probability.

A necessary condition for v to be extendable is that, when E and E^c belong to $\mathcal{E}$, then $v(E) + v(E^c) = 1$.

Now, suppose that this condition is satisfied and set:

$$\widetilde{\mathcal{E}} = \{A \mid A \in \mathcal{E} \; or \; A^c \in \mathcal{E}\}.$$

Then, we can unambiguously extend v to $\widetilde{\mathcal{E}}$ by setting:

$$\widetilde{v}(E) = \begin{cases} v(E) & \text{if } E \in \mathcal{E} \\ 1 - v(E^c) & \text{if } E^c \in \mathcal{E} \end{cases}$$

With this definition, we can state the following theorem:

Theorem 4.3 *The following assertions are equivalent:*

1) v is an extendable probability.

2) $\begin{cases} \text{(a) For all } E \in \mathcal{E} \text{ such that } E^c \in \mathcal{E}, \ v(E) + v(E^c) = 1 \text{ and } \widetilde{v} \text{ is a capacity on } \widetilde{\mathcal{E}}. \\ \\ \text{(b) For all } n \in \mathbb{N}^*, \ a_i > 0 \text{ and } A_i \in \mathcal{A}^\sigma(\mathcal{E}) \text{ such that the functions } A_i^* \text{ are} \\ \text{linearly independent, } \sum_{i=1}^{n} a_i A_i^* = 1 \Rightarrow \sum_{i=1}^{n} a_i \widetilde{v}_*(A_i) \leq 1. \end{cases}$

Furthermore, if we let $\mathcal{P}_{\mathcal{A}}(v)$ (respectively $\mathcal{P}_{\mathcal{A}}(\widetilde{v})$) denote the set of probabilities on $\mathcal{A}$ which extend v (respectively $\widetilde{v}$). Then, $\mathcal{P}_{\mathcal{A}}(v) = \mathcal{P}_{\mathcal{A}}(\widetilde{v}) = C_{\mathcal{A}}(\widetilde{v})$.

Proof : $\star$ We first prove that $\mathcal{P}_{\mathcal{A}}(v) = \mathcal{P}_{\mathcal{A}}(\widetilde{v}) = C_{\mathcal{A}}(\widetilde{v})$.

This is obvisous. Indeed:

$\star$ $\mathcal{P}_{\mathcal{A}}(v) = \mathcal{P}_{\mathcal{A}}(\widetilde{v})$:

Clearly $\mathcal{P}_{\mathcal{A}}(\widetilde{v}) \subset \mathcal{P}_{\mathcal{A}}(v)$.

For the converse inclusion, let $P \in \mathcal{P}_{\mathcal{A}}(v)$ and $A \in \widetilde{\mathcal{E}}$.

- If $A \in \mathcal{E}$, then $P(A) = v(A) = \widetilde{v}(A)$.

- If $A^c \in \mathcal{E}$, then $P(A) = 1 - P(A^c) = 1 - v(A^c) = \widetilde{v}(A)$.

$\star$ $\mathcal{P}_{\mathcal{A}}(\widetilde{v}) = C_{\mathcal{A}}(\widetilde{v})$:

It is obvious that $\mathcal{P}_{\mathcal{A}}(\widetilde{v}) \subset C_{\mathcal{A}}(\widetilde{v})$.

On the other hand, if $P \in C_{\mathcal{A}}(\widetilde{v})$ and $A \in \widetilde{\mathcal{E}}$ then, since $1 = P(A) + P(A^c) \geq \widetilde{v}(A) + \widetilde{v}(A^c) = 1$, it is clear that $P(A) = \widetilde{v}(A)$.

$\star$ 1) $\Rightarrow$ 2) :

Let P be a probability on $\mathcal{A}$ which extends v and $E \in \mathcal{E}$ such that $E^c \in \mathcal{E}$. Then, clearly $v(E) + v(E^c) = P(E) + P(E^c) = 1$.

Furthermore, since v is an extendable probability, it is obvious that $\widetilde{v}$ is a capacity on $\widetilde{\mathcal{E}}$, so that $a)$ is satisfied.

Now, from what we have just shown $P \in C_{\mathcal{A}}(\widetilde{v}) = C_{\mathcal{A}}(\widetilde{v}_*)$. Therefore, since $a_i > 0$

$$\sum_{i=1}^{n} a_i A_i^* = 1 \Rightarrow \sum_{i=1}^{n} a_i \widetilde{v}_*(A_i) \leq \sum_{i=1}^{n} a_i P(A_i) = \int_{\Omega} \left(\sum_{i=1}^{n} a_i A_i^* \right) dP = 1,$$

so that $b)$ is satisfied.

$\star$ 2) $\Rightarrow$ 1) :

$a)$ implies that $\widetilde{v}$ is a well-defined capacity on $\widetilde{\mathcal{E}}$ and by Theorem 4.1, $b)$ implies that $C_{\mathcal{A}}(\widetilde{v}_*)$ is non-empty. Since we have already shown that $C_{\mathcal{A}}(\widetilde{v}_*) = \mathcal{P}_{\mathcal{A}}(v)$, we see that v is an extendable probability.

$\square$

13

Remark 4.4 *Note that Theorems 4.1 to 4.3 are valid even when Ω is not finite. When Ω is finite, there is clearly only a finite number of conditions to check in order to see if the core of a (generalized) capacity is non-empty and if it is a lower or an extendable probability. Furthermore we note that, since the characteristic functions appearing in the left member of the relations in Theorems 4.1 to 4.3 are linearly independent, the coefficients are fixed by these relations. Therefore when $\mathcal{E}$ is finite (so that $\mathcal{A}^\sigma(\mathcal{E})$ is also finite), one has only to check a finite number of relations even if Ω is not assumed to be finite.*

However it is necessary to take into consideration all A_i in $\mathcal{A}^\sigma(\mathcal{E})$ and not only those belonging to $\mathcal{E}$. This cannot be improved by replacing $\mathcal{A}^\sigma(\mathcal{E})$ by $\mathcal{E}$ in condition 2) of Theorem 4.1. Consider for instance the following example:

$\Omega := \{1,2,3\}$, $\mathcal{E} := \{\emptyset, \Omega, \{1\}, \{2\}\}$ and define v on $\mathcal{E}$ by:
$v(\emptyset) = 0$, $v(\Omega) = 1$, $v(\{1\}) = v(\{2\}) = \frac{3}{4}$.

It is obvious that v is a generalized capacity and that the only way to obtain 1 as a linear combination with positive coefficients of linearly independent characteristic functions of elements of $\mathcal{E}$ is $\Omega^* = 1$.

Since $v(\Omega) = 1$, condition 2) of Theorem 4.1 where $\mathcal{A}^\sigma(\mathcal{E})$ is replaced by $\mathcal{E}$ would be satisfied. However it is clear that $C(v)$ is empty since $v(\{1\}) + v(\{2\}) = \frac{3}{2} > 1$.

Theorems 4.1 to 4.3 provide effective algorithms in case $\mathcal{E}$ is finite. We give some details about this algorithm for the non-emptiness of the core, in the case of Theorem 4.1.

- From the data $\mathcal{E} := \{E_1, ..., E_p\}$, determine $\mathcal{A}^\sigma(\mathcal{E}) = \mathcal{A}(\mathcal{E})$ in a finite number of steps by taking finite unions, intersections and complements.

- From the data $v(E_i)$, $1 \leq i \leq p$, compute $v_*(A_i)$, $A_i \in \mathcal{A}(\mathcal{E})$, $1 \leq i \leq q = card(\mathcal{A}(\mathcal{E}))$. Let r be the number of atoms of $\mathcal{A}(\mathcal{E})$.

- For all $1 \leq s \leq r$, determine the free subsets of card s of $\mathcal{A}(\mathcal{E})$.

- Then, retain only the free subsets $\{A_{i_1}^*, ..., A_{i_s}^*\}$.

- Compute in a standard way the (uniquely determined) coefficients a_{i_k}, if they exist, such that $\sum_{k=1}^s a_{i_k} A_{i_k}^* = 1$.

- Retain only the linear combinations where all the coefficients a_{i_k} are positive.

- Finally, check for those combinations whether $\sum_{k=1}^s a_{i_k} v_*(A_{i_k}) \leq 1$.

This allows to decide in a finite number of steps whether $C(v)$ is empty or not.

Two simple examples:

1) An example of a set function whose core is non-empty.

Let $\Omega = \{1,2,3\}$, $\mathcal{E} = \{\{1\}, \{2,3\}\}$ and set $v(\{1\}) = \frac{1}{3}$ and $v(\{2,3\}) = \frac{1}{3}$.

It is obvious that $C(v) \neq \emptyset$ (take for instance $P(\{1\}) = P(\{2\}) = P(\{3\}) = \frac{1}{3}$).

One can also check this by applying Theorem 4.1:
$\mathcal{A}(\mathcal{E}) = \{\emptyset, \Omega, \{1\}, \{2,3\}\}, \ v_*(\emptyset) = 0, \ v_*(\Omega) = v_*(\{1\}) = v_*(\{2,3\}) = \frac{1}{3}$.

The only way to obtain $\sum_{i=1}^{4} a_i A_i^* = 1$ with $A_i \in \mathcal{A}(\mathcal{E})$ and A_i^* linearly independent is to set:

$$\Omega^* = 1 \ or \ \{1\}^* + \{2,3\}^* = 1.$$

In the first case, we have

$$v_*(\Omega) = \frac{1}{3} \leq 1$$

and in the second

$$v_*(\{1\}) + v_*(\{2,3\}) = \frac{2}{3} \leq 1.$$

2) An example of a set function whose core is empty.

Consider again the example of Remark 4.4. We have seen that $C(v)$ is empty, however considering linear combinations of characteristic functions of events in $\mathcal{E}$ did not yield this result. By contrast, the emptiness of the core follows immediately from Theorem 4.1, indeed:

$$\{1\}^* + \{2\}^* + \{3\}^* = 1$$

and

$$v_*(\{1\}) + v_*(\{2\}) + v_*(\{3\}) = \frac{3}{2} > 1.$$

5   Concluding comments

This paper is concerned with the situation of a DM who has partial information (objective or subjective) about certain events. We establish simple criteria that allow to decide whether his (her) information is probabilistic in increasing degree of preciseness. The lowest being merely the existence of a probability taking on each event considered by the DM a greater value than the likelihood attributed to that event by the DM. The highest being that there is a probability actually coinciding with the likelihood assessment of the DM. In the intermediary case, the DM is able to assess a lower probability. In each case, when the set of events considered by the DM is finite, we show that these criteria are effective in the sense that they can be checked in a finite number of steps.

15

References

[1] AP. Dempster. Upper and lower probabilities induced by a multivalued mapping. *The Annals of Mathematical Statistics*, 38, Number 2:325–339, 1967.

[2] PJ. Huber. *Upper and lower probabilities induced by a multivalued mapping.* Wiley Series in Probability and Statistics, 1981.

[3] Y. Kannai. Countably additive measures in core of games. *Journal of Mathematical Analysis and Applications*, 27:227–240, 1969.

[4] H.W. Kuhn and A.W. Tucker. *Linear inequalities and related systems.* Princeton University Press, Study 38, 1956.

[5] L. Savage. *The foundations of statistics.* John Wiley, New York,, 1954.

[6] D. Schmeidler. Cores of exact games, I. *Journal of Mathematical Analysis and Applications*, 40:214–225, 1972.

[7] D. Schmeidler. Integral representation without additivity. *Proceedings of the American Mathematical Society*, 97:255–261, 1986.

[8] G. Shafer. *A mathematical theory of evidence.* Princeton university Press, New Jersey,, 1976.

[9] G. Shafer. Constructive probability. *Synthese*, 48:1–59, 1981.

[10] LS. Shapley. Cores of exact games. *International journal of game theory*, 1:11–26, 1971.

[11] J. von Neumann and O. Morgenstern. *Theory of games an economic behavior.* Princeton university Press, New Jersey, 1947.