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sigma-algebras for non-additive measures

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Abstract

Since the seminal paper of Ghirardato, it is known that Fubini Theorem for non-additive measures can be available only for functions defined as "slice-comonotonic". We give different assumptions that provide such Fubini theorems in the framework of product σ -algebras.

1 Introduction

Non-additive measure theory has risen up with the seminal work of Choquet. It has been widely popularised with the works of Schmeidler [15] for its use in decision theory. One of the most important and natural questions that has arisen is the following: how robust are the results of expected utility when considering Choquet capacities instead of probability measures? The Fubini theorem is a useful tool in expected utility theory, so it would be useful in Choquet expected utility theory, e.g. in game theory or portfolio problems. Since the pioneer paper of P. Ghirardato [9] (see also R. Dyckerhoff [8]) it is known that Fubini theorem for non-additive measures can hold only considering a special class of functions, namely slice-comonotonic functions. With this restriction to slice-comonotonic functions, Ghiradato [9] has got a Fubini Theorem with a product algebra. However in the infinite case, product σ -algebras are more usual than product algebras. As far as we know, the only Fubini Theorem with a product σ -algebra is the one of Brüning [2], who takes as marginals totally monotone capacities on compact spaces. In our work, we show that, not surprisingly, in case of product σ -algebras,

some continuity assumptions of the capacities are also required.

The paper begins by revisiting the notion of independent product of two belief functions v_1 and v_2 in the countable case. Building upon Chateauneuf and Rébillé [5], where it is proved that σ -continuous belief functions defined on $\mathcal{P}(\mathbb{N})$ are characterised through Möbius inverses with non null masses only on finite sets, we show that on $\mathcal{P}(\mathbb{N}) \otimes \mathcal{P}(\mathbb{N})$ the independent product of σ -continuous belief functions can be readily defined. Then building upon a natural method for deriving Fubini theorems on product σ -algebras for capacities, through the obtention of σ -additive probability measures coinciding with given capacities on suitable chains, we propose some inversion order version of Fubini Theorem. We consider a pair of capacities $v_i, i = 1, 2$ (v_i is either convex or concave) defined on σ -algebras Σ_i of sets Ω_i . We show that Fubini holds true for a bounded $\Sigma = \Sigma_1 \otimes \Sigma_2$ measurable mapping $f : \Omega_1 \times \Omega_2 \mapsto \mathbb{R}$ when assuming continuity from below at Ω_i for v_i . Then assuming Ω_i metric spaces and $f : \Omega_1 \times \Omega_2 \mapsto \mathbb{R}$ is bounded and continuous, we show that σ -continuity of v_i can be relaxed to inner continuity on open sets and outer continuity on closed sets. For capacities $v_i, i = 1, 2$ defined on the σ -algebra of the Borel sets \mathcal{B}_i of compact metric spaces Ω_i , Fubini Theorem applied to a continuous slice-comonotone mapping $f : \Omega_1 \times \Omega_2 \mapsto \mathbb{R}$ holds by merely assuming that each v_i is either continuous from below on open sets or continuous from above for closed sets. In the last part we deal with the product of capacities. In Ghiradato's Theorem [9] on product algebras, it is proved that a product capacity which has a property defined by Ghiradato [9] as the Fubini independent property, allows to compute a product integral which equals the iterated integrals. In our paper, we show that, for convex or concave σ -continuous capacities, there always exists a product capacity whose Choquet integral equals the iterated integrals when they invert. In general, it is not unique. Next we take care of finding out capacities product which have the same properties as their marginals (convexity, total monotonicity). With two totally monotone σ -continuous marginals on countable spaces we prove there exists a product which fulfils the good conditions, moreover it is unique.

2 Notations and existing results

Ω is a set. In this paper we need to distinguish clearly an algebra from a σ -algebra. \mathcal{A} will be used for an algebra and Σ for a σ -algebra of subsets of Ω . Let us note that all the definitions (definitions 3, 4, 6, 7, 10) below, stated for algebras, are also available for σ -algebras. Further when we consider topological spaces, we call \mathcal{B} the σ -algebra of Borel sets.

A *capacity* v is a set function from Σ (or \mathcal{A}) to \mathbb{R} with $v(\emptyset) = 0$ and with for all A and B in Σ , $A \subset B$, $v(A) \leq v(B)$.

We consider only normalised capacities, i.e. $v(\Omega) = 1$.

Let us recall the definition of the Choquet integral:

$$\int_{\Omega} f(\omega) dv = \int_{-\infty}^0 (v(f(\omega) \geq t) - 1) dt + \int_0^{+\infty} v(f(\omega) \geq t) dt$$

Let $E \in \Sigma$, E^* denotes the indicator function of E ($\forall x \in \Omega$, $E^*(x) = 1$ if $x \in E$ and 0 else).

Let us recall the classical Fubini theorem on iterated integrals.

Fubini's theorem

Let $P_i : \Sigma_i \rightarrow [0, 1]$, $i = 1, 2$, be two σ -additive probability measures, where Σ_i are σ -algebras of subsets of Ω_i , $i = 1, 2$. Let $\Sigma = \Sigma_1 \otimes \Sigma_2$ be the product σ -algebra generated by the set of rectangles $A_1 \times A_2$, $A_1 \in \Sigma_1$, $A_2 \in \Sigma_2$. Let $f : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$ be a bounded Σ -measurable mapping, then:

1. $f(\cdot, \omega_2)$ is Σ_1 -measurable and $\omega_2 \in \Omega_2 \rightarrow \int_{\Omega_1} f(\cdot, \omega_2) dP_1$ is Σ_2 -measurable

$f(\omega_1, \cdot)$ is Σ_2 -measurable and $\omega_1 \in \Omega_1 \rightarrow \int_{\Omega_2} f(\omega_1, \cdot) dP_2$ is Σ_1 -measurable

2. The iterated integrals $\int \int f dP_1 dP_2$ $\int \int f dP_2 dP_1$ exist and are equal:

$$\int_{\Omega_2} \left(\int_{\Omega_1} f(\omega_1, \omega_2) dP_1 \right) dP_2 = \int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2) dP_2 \right) dP_1$$

Moreover

3. There exists a unique σ -additive probability P on $(\Omega_1 \times \Omega_2, \Sigma_1 \otimes \Sigma_2)$

such that for any bounded Σ -measurable mapping $f : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$

$$\int f dP = \int \int f dP_1 dP_2 \left(= \int \int f dP_2 dP_1 \right)$$

P is uniquely defined by its values on rectangles:

$$P(A_1 \times A_2) = P(A_1) \cdot P(A_2) \quad (A_1 \in \Sigma_1, A_2 \in \Sigma_2)$$

Remark 1 *There exist more general versions of the Fubini Theorem (see e.g. Aliprantis and Border [1]). In one of these more general versions f can be only almost everywhere P -measurable. However almost everywhere is much more delicate to define with capacities.*

2.1 Fubini theorem of Ghirardato [9] on product algebras for non-additive measures

Let $v_i : \mathcal{A}_i \rightarrow [0, 1]$, $i = 1, 2$, be two capacities, where \mathcal{A}_i are algebras of subsets of Ω_i , $i = 1, 2$.

Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be the product algebra generated by the set of rectangles $A_1 \times A_2$, $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$.

Remark 2 *The algebra \mathcal{A} consists of all finite disjoint unions of rectangles.*

Definition 1 *Mappings $f, g : \Omega \rightarrow \mathbb{R}$ are comonotonic if for every $x, y \in \Omega$ $[f(x) - f(y)][g(x) - g(y)] \geq 0$*

The following definitions and lemmas are due to Ghirardato [9].

Definition 2 *$f : \Omega_1 \times \Omega_2 \mapsto \mathbb{R}$ is slice comonotonic if $\forall (x_1, x_2) \in \Omega_1 \times \Omega_1$, $f(x_1, \cdot)$ and $f(x_2, \cdot)$ are comonotonic and $\forall (y_1, y_2) \in \Omega_2 \times \Omega_2$, $f(\cdot, y_1)$ and $f(\cdot, y_2)$ are comonotonic.*

Definition 3 *$A \in \mathcal{A}$ is slice-comonotonic if its characteristic function is slice-comonotonic.*

The next lemma enlightens why slice-comonotonicity can be considered as a necessary condition to obtain Fubini Theorems for non-additive measures.

Lemma 1 *Let $f : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$ be a bounded Σ -measurable mapping. In order that iterated integrals $\int \int f dv_1 dv_2$, $\int \int f dv_2 dv_1$ exist and are equal for any pair of capacities, f must be slice-comonotonic.*

For a discussion about slice-comonotonic functions, see Ghirardato [9] p.278, 279. He concludes that "the class of slice-comonotonic functions is quite large", including functions that are monotone in each argument.

Theorem 1 (Ghirardato[9]) *Let $v_i, i = 1, 2$ be capacities on \mathcal{A}_i algebras of $\Omega_i, i = 1, 2$. Let $\Omega = \Omega_1 \times \Omega_2$ be endowed with the product algebra $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$. Let $f : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$ be a slice-comonotonic bounded \mathcal{A} -measurable mapping, then:*

1. $f(\cdot, \omega_2)$ is \mathcal{A}_1 -measurable and $\omega_2 \in \Omega_2 \rightarrow \int_{\Omega_1} f(\cdot, \omega_2) dv_1$ is \mathcal{A}_2 -measurable

$f(\omega_1, \cdot)$ is \mathcal{A}_2 -measurable and $\omega_1 \in \Omega_1 \rightarrow \int_{\Omega_2} f(\omega_1, \cdot) dv_2$ is \mathcal{A}_1 -measurable

2. The iterated integrals $\int \int f dv_1 dv_2, \int \int f dv_2 dv_1$ exist and are equal:

$$\int_{\Omega_2} \left(\int_{\Omega_1} f(\omega_1, \omega_2) dv_1 \right) dv_2 = \int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2) dv_2 \right) dv_1$$

Moreover

3. A capacity v on $(\Omega_1 \times \Omega_2, \mathcal{A})$ satisfies:

for any slice-comonotonic bounded \mathcal{A} -measurable mapping

$$f : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}, \int f dv = \int \int f dv_1 dv_2 \left(= \int \int f dv_2 dv_1 \right).$$

if and only if v satisfies $v(A) = \int \int A^* dv_1 dv_2$ for any slice comonotonic A belonging to \mathcal{A} .

2.2 Continuity assumptions of the measures for Fubini theorems on product σ -algebra

Capacities behave symmetrically as measures: Fubini theorem is available for finitely additive measures (see Marinacci [12]) with algebras but no more with σ -algebras. We give two examples that show the need for some continuity assumptions of the measures for Fubini theorems on product σ -algebra.

Let $\Omega = \mathbb{N}^*$, $\Sigma = \mathcal{P}(\mathbb{N}^*)$, $\mathcal{B} = \Sigma \otimes \Sigma = \mathcal{P}(\mathbb{N} \times \mathbb{N})$ the σ -algebra product on $\Omega \times \Omega$. P is a finitely additive and non σ -additive probability defined on Σ by $P(A) = 0$ if A finite $\in \Sigma$. Let us note that P can be defined on all $A \in \Sigma$ with $P(A) = \lim_{x \nearrow 1} \frac{\sum_{n \in A} x^n}{\sum_{n \in \mathbb{N}} x^n}$.

Example 1 $f : (m, n) \in \Omega \times \Omega \mapsto f(m, n) = \begin{cases} 1 & \text{if } m \geq n \\ 0 & \text{if } m < n \end{cases}$
f is Σ -measurable, slice-comonotonic since f is monotone in each argument.
But $\int_{\Omega} \int_{\Omega} f(m, n) dP(n) dP(m) = 0$ and $\int_{\Omega} \int_{\Omega} f(m, n) dP(m) dP(n) = 1$

Example 2 $f : (m, n) \in \Omega \times \Omega \mapsto f(m, n) = \begin{cases} \frac{1}{n} + 1 & \text{if } m > n \\ 0 & \text{if } m \leq n \end{cases}$
f is Σ -measurable, slice-comonotonic since f is monotone in each argument.
But $\int_{\Omega} \int_{\Omega} f(m, n) dP(n) dP(m) = 0$ and $\int_{\Omega} \int_{\Omega} f(m, n) dP(m) dP(n) = 1$

3 A fubini theorem on $\mathcal{P}(\mathbb{N}) \otimes \mathcal{P}(\mathbb{N})$ for belief functions

In this part we will deal with a special class of capacities: the belief functions. Belief functions have been studied first by Dempster and Shafer [16]. We place them in the begining of our work because we can obtain a complete Fubini theorem.

Definition 4 *Let \mathcal{A} be an algebra of subsets of Ω then $v : \Sigma \rightarrow \mathbb{R}$ is a belief function if v is a capacity and v is totally monotone: $\forall k \geq 2, A_1, \dots, A_k \in \Sigma :$*

$$v\left(\bigcup_{i=1}^k A_i\right) \geq \sum_{\substack{I \subset \{1 \dots k\} \\ I \neq \emptyset}} (-1)^{|I|+1} v\left(\bigcap_{i \in I} A_i\right)$$

We define the usual notions of continuity for capacities.

Definition 5 *A capacity v on Σ is σ -continuous if v is (everywhere) continuous from below and from above i.e. if for all $E \in \Sigma$, for all increasing and decreasing sequence $(E_n)_n$ to E : $\lim_{n \rightarrow \infty} v(E_n) = v(E)$.*

Totally monotone capacities are special cases of convex capacities.

Definition 6 A capacity v on \mathcal{A} is convex (resp. concave) if for all A and B in Σ :

$$v(A \cup B) + v(A \cap B) \geq (\text{resp. } \leq) v(A) + v(B).$$

The following property simplifies the verifications to see if a convex capacity is continuous.

Property 1 (Rosenmüller [14]): A convex capacity on (Ω, Σ) is σ -continuous if it is continuous from below at Ω .

In $\mathcal{P}(\mathbb{N})$, totally monotone capacities have this property which we use to prove the unicity of a totally monotone σ -continuous product capacity. $B_\infty(\mathbb{N})$ is the set of bounded real valued functions on \mathbb{N} .

Definition 7 $C(v) = \{m ; m \text{ is a measure such that } m(\Omega) = v(\Omega) \text{ and such that, for all } A \text{ in } \mathcal{A}, m(A) \geq v(A)\}$. $C(v)$ is called the core of v .

Property 2 (Chateauneuf Rébillé [5]) Let v be a σ -continuous belief function on $\mathcal{P}(\mathbb{N})$ then there exists a unique $m : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$ with $m(A) = 0, \forall A \text{ infinite} \in \mathcal{P}(\mathbb{N})$ and $\sum_{A \in \mathcal{P}(\mathbb{N})} m(A) = 1$.

$$\forall A \in \mathcal{P}(\mathbb{N}), v(A) = \sum_{B \subset A} m(B).$$

Futhermore for any finite $A \in \mathcal{P}(\mathbb{N}), m(A) = \sum_{B \subset A} (-1)^{|A \setminus B|} v(B)$.

Conversely, let $m : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$ be null on the empty set and the infinite sets, with $\sum_{A \in \mathcal{P}(\mathbb{N})} m(A) = 1$. Let us define v such that:

$$v : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1], \forall A \in \mathcal{P}(\mathbb{N}), v(A) = \sum_{B \subset A} m(B).$$

Then v is a σ -continuous belief function on $\mathcal{P}(\mathbb{N})$.

Moreover $\forall f \in B_\infty(\mathbb{N}), \int_{\mathbb{N}} f dv = \sum_{A, A \text{ finite}} m(A) \cdot \min f(A)$.

In other words a capacity v on $\mathcal{P}(\mathbb{N})$ is a σ -continuous belief function if and only if its usual Möbius inverse on finite subsets is non-negative and the capacity $v(A)$ of any subset A of \mathbb{N} is the sum of the masses $m(B)$ of all the finite subsets of A . In what follows the unique $m : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$ defined as in property 2, for a σ -continuous belief function v on $\mathcal{P}(\mathbb{N})$ will be called the Möbius inverse of v . Take $\Omega_1 = \Omega_2 = \mathbb{N}$, $\Sigma_1 = \Sigma_2 = \mathcal{P}(\mathbb{N})$, $\mathcal{B} = \Sigma \otimes \Sigma = \mathcal{P}(\mathbb{N} \times \mathbb{N})$ the σ -algebra product on $\Omega \times \Omega$.

Theorem 2 *Let $v_i : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$, $i = 1, 2$ be two σ -continuous belief functions and denote by m_i , $i = 1, 2$ their respective Möbius inverse. Let $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ be a slice-comonotonic bounded Σ -measurable mapping, then: the iterated integrals $\int \int f dv_1 dv_2$ and $\int \int f dv_2 dv_1$ exist and are equal:*

$$\int \int f dv_1 dv_2 = \sum_{A_1 \times A_2, A_i \text{ finite}} m_1(A_1) \cdot m_2(A_2) \cdot \min_{A_1 \times A_2} f(\omega_1, \omega_2) = \int \int f dv_2 dv_1$$

Moreover there exists a unique v , σ -continuous belief function on $(\mathbb{N} \times \mathbb{N}, \mathcal{P}(\mathbb{N}) \otimes \mathcal{P}(\mathbb{N}))$ such that for any $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ slice-comonotonic bounded \mathcal{B} -measurable mapping $\int f dv = \int \int f dv_1 dv_2 (= \int \int f dv_2 dv_1)$. v is uniquely defined by the values of its Möbius inverse m on rectangles of finite subsets of \mathbb{N} : $m(A_1 \times A_2) = m_1(A_1) \cdot m_2(A_2)$ (A_i finite set of \mathbb{N}) and $m(B) = 0$ otherwise.

That theorem is a complete Fubini Theorem: not only we can interchange the order of integration but also there exists a unique product capacity whose integral equals the iterated integrals.

Proof.

Part 1

Let us recall this property from Chateauneuf-Rébillé [5]:

$$\int_{\mathbb{N}} f dv = \sum_{A, A \text{ finite}} m(A) \cdot \min f(A).$$

So

$$\begin{aligned}
\int \int f dv_1 dv_2 &= \int_{\mathbb{N}} \left[\int_{\mathbb{N}} f(\cdot, \omega_2) dv_2 \right] dv_1 \\
&= \int_{\mathbb{N}} \left[\sum_{A_1, A_1 \text{ finite}} m_1(A_1) \cdot \min f(A_1, \omega_2) \right] dv_2 \\
&= \sum_{A_2, A_2 \text{ finite}} \sum_{A_1, A_1 \text{ finite}} m_2(A_2) \cdot m_1(A_1) \cdot \min f(A_1, A_2)
\end{aligned}$$

which proves part 1.

Part 2

Let $E \in \mathcal{P}(\Omega) \times \mathcal{P}(\Omega)$, E finite. Let us consider $v_{/E}$. For every comonotonic set F of $\Sigma(E) = \mathcal{P}\{(i, j), (i, j) \in E\}$, as v is a Fubini independent product, we have: $v(F) = \int_{E_1} \int_{E_2} F dv_2 dv_1$. Applying Ghirardato's theorem 3 [9] we can compute $v_{/E}$ Möbius transform: it equals $m_1(A_1) \cdot m_2(A_2)$ for the rectangles $A_1 \times A_2$, it is null for every non rectangle. From Chateauneuf-Rébillé [5] we can get the Möbius transform of v , it has exactly the same values on $\mathbb{N} \times \mathbb{N}$ for the finite sets.

As v is σ -continuous its Möbius transform is null on the infinite set. So it is unique and follows the definition 8. ■

Definition 8 Let $v_i, i = 1, 2$ be σ -continuous belief functions on $P(\mathbb{N})$, with Möbius inverses $m_i, i = 1, 2$. The independent product of v_1 and v_2 denoted by $v_1 \times v_2$ is the unique σ -continuous belief function on $P(\mathbb{N}) \times P(\mathbb{N})$ defined by $m(A_1 \times A_2) = m_1(A_1) \cdot m_2(A_2)$ (A_i finite sets of \mathbb{N}) and $m(B) = 0$ otherwise.

4 Some Fubini theorems on iterated integrals for non-additive measures on product σ -algebras

4.1 Abstract space

With conditions on the capacities (namely concave or convex capacities) we obtain a Fubini theorem of the type interchange the order of integration:

Theorem 3 Let $v_i, i = 1, 2$ be convex or concave σ -continuous capacities on Σ_i σ -algebras of $\Omega_i, i = 1, 2$. Let $\Omega = \Omega_1 \times \Omega_2$ be endowed with the product σ -algebra $\Sigma = \Sigma_1 \otimes \Sigma_2$. Let $f : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$ be a slice-comonotonic bounded Σ -measurable mapping, then:

1. $f(., \omega_2)$ is Σ_1 -measurable and $\omega_2 \in \Omega_2 \rightarrow \int_{\Omega_1} f(., \omega_2) dv_1$ is bounded and Σ_2 -measurable.
 $f(\omega_1, .)$ is Σ_2 -measurable and $\omega_1 \in \Omega_1 \rightarrow \int_{\Omega_2} f(\omega_1, .) dv_2$ is bounded and Σ_1 -measurable.
2. The iterated integrals $\int \int f dv_1 dv_2, \int \int f dv_2 dv_1$ exist and are equal:

$$\int_{\Omega_2} \left(\int_{\Omega_1} f(\omega_1, \omega_2) dv_1 \right) dv_2 = \int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2) dv_2 \right) dv_1$$

Roughly speaking the general principles of the proofs are the following: show that there exist P_i σ -additive probabilities on (Ω_i, Σ_i) such that

$$\int_{\Omega_2} \left(\int_{\Omega_1} f(\omega_1, \omega_2) dv_1 \right) dv_2 = \iint f dP_1 dP_2$$

and

$$\int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2) dv_2 \right) dv_1 = \iint f dP_2 dP_1$$

and then apply the classical Fubini's theorem.

Proof. Slice-comonotonicity of f implies that $\mathcal{C} = \{f(., \omega_2), \omega_2 \in \Omega_2\}$ is a class of comonotonic mappings, hence from Denneberg [6] proposition 10.1, there exists P'_1 additive probability in the core of v_1 (resp. in the anti-core of v_1) if v_1 is convex (resp. v_1 is concave) such that

$\int_{\Omega_1} f(., \omega_2) dv_1 = \int_{\Omega_1} f(., \omega_2) dP'_1 \forall \omega_2 \in \Omega_2$. σ -continuity of v_1 implies σ -additivity of P'_1 hence

$f_2 : \omega_2 \in \Omega_2 \mapsto f_2(\omega_2) = \int_{\Omega_1} f(., \omega_2) dv_1$ is Σ_2 -measurable (and bounded).

From this, $\int \int f dv_1 dv_2$ and $\int \int f dv_2 dv_1$ exist. Slice-comonotonicity of f implies that, if $f(\omega_1, \omega_2) \leq f(\omega_1, \omega'_2)$ then for all ω'_1 in Ω_1 , $f(\omega'_1, \omega_2) \leq f(\omega'_1, \omega'_2)$.

Hence by monotonicity of the Choquet integral we have: $\int_{\Omega_1} f(., \omega_2) dv_1 \leq \int_{\Omega_1} f(., \omega'_2) dv_1$. We can conclude that $\mathcal{C} = \{f_2(.), f(\omega_1, .), \omega_1 \in \Omega_1\}$ is a class of comonotonic mappings, hence as above, there exists P_2 σ -additive on (Ω_2, Σ_2) such that $\int \int f dv_1 dv_2 = \int \int f dv_1 dP_2$

and $\int \int f dv_2 dv_1 = \int \int f dP_2 dv_1$. A similar reasoning allows to replace v_1 by a σ -additive probability P_1 on (Ω_1, Σ_1) . ■

Remark 3 For this Theorem, an "almost everywhere" version would be available. A continuous convex capacity v has a measure P in its core such that all the other measures of its core are absolutely continuous with respect to P (see e.g. Chateauneuf et al. [4]). Hence we can define v -almost everywhere as P -almost everywhere. Thanks to dualities properties, we can do the same for a concave continuous capacity. We can thus obtain a more general version of Theorem 3.

4.1.1 Introducing some topological assumptions

In the following, we consider topological spaces in order to get less strong conditions on the capacity. Now let Ω_1, Ω_2 be metric spaces, and \mathcal{B}_i be the Borel σ -algebras of $\Omega_i, i = 1, 2$.

Definition 9 \mathcal{G} is the set of open sets, \mathcal{F} the set of closed sets.

- v is continuous from below on open sets if $G_n, G \in \mathcal{G}, G_n$ an increasing sequence to G : $\lim_{n \rightarrow \infty} v(G_n) = v(G)$.
- v is continuous from above on closed sets if $F_n, F \in \mathcal{F}, F_n$ an increasing sequence to F : $\lim_{n \rightarrow \infty} v(F_n) = v(F)$.

Theorem 4 Let $v_i, i = 1, 2$ be convex or concave capacities on \mathcal{B}_i Borel σ -algebras of the metric spaces $\Omega_i, i = 1, 2$, continuous from below on open sets, continuous from above on closed sets. Let $\Omega = \Omega_1 \times \Omega_2$ endowed with the σ -algebra generated by $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$. Let $f : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$ be a slice-comonotonic bounded continuous mapping, then:

1. $f(\cdot, \omega_2)$ is continuous and $\omega_2 \in \Omega_2 \rightarrow \int_{\Omega_1} f(\cdot, \omega_2) dv_1$ is bounded and continuous.
 $f(\omega_1, \cdot)$ is continuous and $\omega_1 \in \Omega_1 \rightarrow \int_{\Omega_2} f(\omega_1, \cdot) dv_2$ is bounded and continuous.
2. The iterated integrals $\int \int f dv_1 dv_2, \int \int f dv_2 dv_1$ exist and are equal:

$$\int_{\Omega_2} \left(\int_{\Omega_1} f(\omega_1, \omega_2) dv_1 \right) dv_2 = \int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2) dv_2 \right) dv_1$$

Proof. Suppose that v_1 is convex then let us define v_{1*} on \mathcal{B}_1 by:

$$A \in \mathcal{B}_1 \quad v_{1*} = \sup\{v_1(F), F \subset A, F \in \mathcal{F}_1\}.$$

\mathcal{F}_1 is closed under finite union and intersection and closed from above (i.e.

for any decreasing sequence $F_n \in \mathcal{F}_1$, $\bigcap_{n=1}^{+\infty} F_n \in \mathcal{F}_1$).

Since v_1 is convex and continuous from above on \mathcal{F}_1 , it comes (see proposition 2.4 in Denneberg [6]) that v_{1*} is convex and continuous from above on \mathcal{B}_1 .

Ω_1 metric space, v_1 continuous from below on \mathcal{G}_1 implies that

for any $G \in \mathcal{G}_1$, $v_1(G) = \sup\{v_1(F), F \subset G, F \in \mathcal{F}_1\}$, therefore:

$v_{1*}(G) = v_1(G)$ for any $G \in \mathcal{G}_1$, futhermore v_{1*} is continuous from below on \mathcal{G}_1 .

To summarize there exists v_{1*} convex continuous from above, continuous from below on open sets, coinciding with v_1 on closed and open sets.

Clearly $f(\cdot, \omega_2)$ is continuous hence \mathcal{B}_1 measurable and therefore $\int_{\Omega_1} f(\cdot, \omega_2) dv_1$ exists for any $\omega_2 \in \Omega_2$. Recall that

$$\int_{\Omega_1} f(\cdot, \omega_2) dv_1 = \int_{-\infty}^0 (v_1(f(\cdot, \omega_2) \geq t) - 1) dt + \int_0^{+\infty} v_1(f(\cdot, \omega_2) \geq t) dt$$

or else $\int_{\Omega_1} f(\cdot, \omega_2) dv_{1*} = \int_{-\infty}^0 (v_{1*}(f(\cdot, \omega_2) \geq t) - 1) dt + \int_0^{+\infty} v_{1*}(f(\cdot, \omega_2) \geq t) dt$

hence $\int_{\Omega_1} f(\cdot, \omega_2) dv_1 = \int_{\Omega_1} f(\cdot, \omega_2) dv_{1*}$.

$\mathcal{C} = \{f(\cdot, \omega_2), \omega_2 \in \Omega_2\}$ is a class of comonotonic mappings, hence from Denneberg [6] proposition 10.1 there exists ν_1 additive probability on \mathcal{B}_1 such that $\nu_1(f(\cdot, \omega_2) \geq t) = v_{1*}(f(\cdot, \omega_2) \geq t)$, $\nu_1(f(\cdot, \omega_2) > t) = v_{1*}(f(\cdot, \omega_2) > t)$ for any $(t, \omega_2) \in \mathbb{R} \times \Omega_2$ and ν_1 belongs to the core of v_{1*} .

Therefore from theorem 6 of Parker [13] there exists P'_1 σ -additive on \mathcal{B}_1 such that P'_1 agrees with ν_1 on $\mathcal{D} = \{\{f(\cdot, \omega_2) > t\}, t \in \mathbb{R}, \omega_2 \in \Omega_2\}$ hence agrees with v_1 on \mathcal{D} . Therefore $\int_{\Omega_1} f(\cdot, \omega_2) dv_1 = \int_{\Omega_1} f(\cdot, \omega_2) dP'_1 \forall \omega_2 \in \Omega_2$. This entails that: $f_2 : \omega_2 \in \Omega_2 \rightarrow f_2(\omega_2) = \int_{\Omega_1} f(\cdot, \omega_2) dv_1$ is continuous and indeed bounded.

A similar reasoning would apply if v_1 is concave, which completes the proof of part 1 of theorem 5.

Let us come to part 2 of theorem 5. Suppose again that v_1 is convex (similar reasoning if v_1 is concave). Let denote $f_1 : \omega_1 \in \Omega_1 \mapsto f_1(\omega_1) = \int_{\Omega_2} f(\omega_1, \cdot) dv_2$. As in the proof of theorem 3 slice comonotonicity of f implies that $\mathcal{C}_1 = \{f_1(\cdot), f(\cdot, \omega_2), \omega_2 \in \Omega_2\}$ is a class of comonotonic mappings, hence the same reasoning as above shows that there exists P_1 σ -additive on \mathcal{B}_1 such that P_1 agrees with v_1 on \mathcal{D} where \mathcal{D} is now defined by $\mathcal{D} = \{\{f_1 > t\}, \{f(\cdot, \omega_2) > t\}, t \in \mathbb{R}, \omega_2 \in \Omega_2\}$. This implies $\int \int f dv_1 dv_2 = \int \int f dP_1 dv_2$ and $\int \int f dv_2 dv_1 = \int \int f dv_2 dP_1$. Applying a similar reasoning to v_2 now al-

lows to replace v_2 by a σ -additive probability P_2 on $(\Omega_2, \mathcal{B}_2)$ hence equality of the iterated integrals. ■

Remark 4 *Theorem 4 and 5 allow to take a convex and a concave capacity reflecting on one space a kind of uncertainty aversion and on the other one a kind of uncertainty loving.*

Proofs of theorem 5 and 6 are more delicate than theorem 4's. Our proofs use tricks similar to those of Brüning and Denneberg [3] for their theorem 1 and make intensive use of Parker's theorem [13] which follows:

Theorem 5 (Parker [13]) *Let Ω be a metric space and v be a capacity on $\mathcal{B}(\Omega)$ everywhere continuous from above and continuous from below on open sets.*

Then for every finitely additive probability μ in the core of v and every non-empty sub-system D of open sets such that μ agrees with v on D , there exists a σ -additive probability P in the core of v with $P(G) = v(G), \forall g \in D$.

Let assume finally that the Ω_i 's, $i = 1, 2$ are compact metric spaces. That strong restriction for the state space allows to consider general capacities (not necessarily convex or concave) with continuity only for closed or compact sets.

Theorem 6 *Let $v_i, i = 1, 2$ be capacities on \mathcal{B}_i Borel σ -algebras of the compact space $\Omega_i, i = 1, 2$ continuous from below on open sets or continuous from above on closed sets. Let $\Omega = \Omega_1 \times \Omega_2$ be endowed with the σ -algebra generated by $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$. Let $f : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$ be a slice-comonotonic bounded continuous mapping, then:*

1. *$f(\cdot, \omega_2)$ is continuous and $\omega_2 \in \Omega_2 \rightarrow \int_{\Omega_1} f(\cdot, \omega_2) dv_1$ is bounded and continuous*

$f(\omega_1, \cdot)$ is continuous and $\omega_1 \in \Omega_1 \rightarrow \int_{\Omega_2} f(\omega_1, \cdot) dv_2$ is bounded and continuous.

2. *The iterated integrals $\int \int f dv_1 dv_2, \int \int f dv_2 dv_1$ exist and are equal:*

$$\int_{\Omega_2} \left(\int_{\Omega_1} f(\omega_1, \omega_2) dv_1 \right) dv_2 = \int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2) dv_2 \right) dv_1$$

Proof.

Suppose that v_1 is continuous from above on closed sets.

For any $\omega_2 \in \Omega_2$, $f(\cdot, \omega_2)$ is continuous hence \mathcal{B}_1 measurable and therefore $\int f(\cdot, \omega_2) dv_1$ exists for any ω_2 . Slice-comonotonicity of f and continuity of $f(\cdot, \omega_2)$ implies that $\mathcal{C} = \{f(\cdot, \omega_2) \geq t, \omega_2 \in \Omega_2, t \in \mathbb{R}\}$ is a chain of closed sets of Ω_1 . Applying the dual form of lemma 2 of Brüning and Denneberg [3], we show that the capacity v_{1*} defined by $v_{1*} = \sup\{v_1(F), F \in \underline{\mathcal{C}}, F \subset A\}$ is continuous from above on \mathcal{B}_1 , continuous from below on open sets and convex.

$\underline{\mathcal{C}}$ denotes the closure from above of \mathcal{C} , i.e. $\underline{\mathcal{C}}$ consists of all sets in Ω_1 which are the intersection of decreasing sequences of sets in \mathcal{C} . Clearly $\underline{\mathcal{C}}$ is a chain of closed sets.

$\mathcal{E} = \{f(\cdot, \omega_2), \omega_2 \in \Omega_2\}$ is a class of comonotonic mappings, hence from Denneberg [6] proposition 10.1 there exists ν_1 additive probability on \mathcal{B}_1 belonging to the core of v_{1*} and such that $\nu_1(f(\cdot, \omega_2) > t) = \nu_{1*}(f(\cdot, \omega_2) > t)$ for any $(t, \omega_2) \in \mathbb{R} \times \Omega_2$. Therefore from theorem 6 of Parker [13] there exists P'_1 σ -additive on \mathcal{B}_1 such that P'_1 agrees with v_{1*} on \mathcal{D} . Let ω_2 be fixed since $t \rightarrow \nu_{1*}(f(\cdot, \omega_2) \geq t)$ is monotone with compact support $[a, b]$, one obtains:

$\nu_{1*}(f(\cdot, \omega_2) \geq t) = \nu_{1*}(f(\cdot, \omega_2) > t) = P_1(f(\cdot, \omega_2) > t) = P'_1(f(\cdot, \omega_2) \geq t)$.
 Since $\{f(\cdot, \omega_2) > t\} \in \underline{\mathcal{C}}$ one gets $\nu_1(f(\cdot, \omega_2) \geq t) = \nu_{1*}(f(\cdot, \omega_2) \geq t)$ hence $\nu_1(f(\cdot, \omega_2) \geq t) = P'_1(f(\cdot, \omega_2) \geq t)$, therefore $\int_{\Omega_1} f(\cdot, \omega_2) dv_1 = \int_{\Omega_1} f(\cdot, \omega_2) dP'_1$, $\forall \omega_2 \in \Omega_2$, this entails that $f : \omega_2 \in \Omega_2 \rightarrow f_2(\omega_2) = \int_{\Omega_1} f(\cdot, \omega_2) dv_1$ is continuous and indeed bounded.

A similar reasoning would apply if v_1 is continuous from below on open sets, what completes the proof of part 1 of theorem 6.

Let us come to part 2 of theorem 6. Suppose again that v_1 is continuous from above on closed sets (similar reasoning if v_1 is continuous from below on open sets).

Let denote $f_1 : \omega_1 \in \Omega_1 \mapsto \int_{\Omega_2} f(\omega_1, \cdot) dv_2$. Slice comonotonicity of f , continuity of f_1 and of $f(\cdot, \omega_2)$ implies that $\mathcal{C} = \{f_1(\cdot) \geq t, \{f(\cdot, \omega_2) \geq t\}, \omega_2 \in \Omega_2, t \in \mathbb{R}\}$ is a chain of closed sets of Ω_1 , hence the same reasoning as above shows that there exists P_1 σ -additive on \mathcal{B}_1 such that $\int \int f dv_1 dv_2 = \int \int f dP_1 dv_2$ and $\int \int f dv_2 dv_1 = \int \int f dv_2 dP_1$.

Applying a similar reasoning to v_2 now allows to replace v_2 by a σ -additive probability P_2 on $(\Omega_2, \mathcal{B}_2)$, hence equality of the iterated integrals. ■

5 Product capacity

In the preceding section we have got theorems which allow to change the order of integration. It is then natural to wonder if there exists a capacity v that verifies

$\int f dv = \int \int f dv_1 dv_2 = \int \int f dv_2 dv_1$ for every slice comonotonic f . In order to answer in some cases, we need to adapt a definition of Ghirardato [9] (definition 5 p. 270) for a σ -algebra. Let us recall it.

Definition 10 *A capacity v on (Ω, \mathcal{A}) is said to have the Fubini property (or to be a Fubini independent product) if for any slice-comonotonic set $A \in \mathcal{A}$:*
 $\int A^* dv = \int \int A^* dv_1 dv_2$.

First let us note that a Fubini independent product capacity always exists when $\int \int f dv_1 dv_2 = \int \int f dv_2 dv_1$ for every slice-comonotonic f . We can define v_m as $\forall E \in \Sigma, v_m(E) = \sup \int \int A^* dv_1 dv_2$, for A slice comonotonic set of Σ and $A \subset E$ (we call it v_m because it is the minimal Fubini independent product capacity). The next theorem shows Fubini independent product capacities allow us to state a "full" Fubini theorem when the assumptions of theorem 3 are fulfilled.

Theorem 7 *Let $v_i, i = 1, 2$ be convex or concave σ -continuous capacities on Σ_i σ -algebras of $\Omega_i, i = 1, 2$, let $\Omega = \Omega_1 \times \Omega_2$ be endowed with the product σ -algebra $\Sigma = \Sigma_1 \otimes \Sigma_2$, let $f : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$ be a slice-comonotonic bounded Σ -measurable mapping, let v be a product capacity on (Ω, Σ) which has the Fubini property,*
then we have $\int f dv = \int \int f dv_1 dv_2 = \int \int f dv_2 dv_1$

Proof. If f is slice-comonotonic and simple (i.e. finite ranged) then $f = \sum_i f_i A_i^*$ (actually $f = \alpha_1 \Omega^* + \sum_{i=2}^n (\alpha_i - \alpha_{i-1}) A_i^*$ with $\alpha_1 < \dots < \alpha_n$ and

$A_{i+1} \subset A_i$ for $i = 1, \dots, n-1$). The proof of Ghirardato [9] p.10 (i.e. if an A_i is not slice-comonotonic f cannot be slice-comonotonic) can be readily adapted for σ -algebras and so A_i are comonotonic sets with $A_i \subset A_{i+1}$. As v is Fubini independent we can then compute the integral of f (i.e. $\int f dv = \sum_i f_i v(A_i)$) and obtain the same result as the one with the iterated integral. So the equality is true for finite ranged functions.

Let us consider now a general slice-comonotonic f , let us define the sequence (f_n) , $f_n = \sum_{k=0}^{n2^n-1} [\frac{k}{2^n} \{ \frac{k}{2^n} \leq f < \frac{k+1}{2^n} \}^* - \frac{k+1}{2^n} \{ -\frac{k+1}{2^n} \leq f < -\frac{k}{2^n} \}^*]$. Each f_n is a finite ranged slice-comonotonic function: let us suppose f_n has not its ω_1 -sections comonotonic then: $\exists \omega_1, \omega'_1 \in \Omega_1, \omega_2, \omega'_2 \in \Omega_2$ such that $f_n(\omega_1, \omega_2) > f_n(\omega_1, \omega'_2)$ and $f_n(\omega'_1, \omega_2) < f_n(\omega'_1, \omega'_2)$ which implies $f(\omega_1, \omega_2) > f(\omega_1, \omega'_2)$ and $f(\omega'_1, \omega_2) < f(\omega'_1, \omega'_2)$. It is a contradiction of f comonotonic and so f_n has its ω_1 -sections comonotonic. A similar argument proves that f_n has its ω_2 -sections comonotonic and so f_n is slice-comonotonic. We are going to prove now that there exist σ -additive probability measures P_1 and P_2 such that $\int f_n dv = \int f_n dP_1 \times P_2$ and $\int f dv = \int f dP_1 \times P_2$. If $f_n(\omega_1, \omega_2) > f_n(\omega'_1, \omega_2)$ then $f(\omega_1, \omega_2) > f(\omega'_1, \omega_2)$ $\{f_n(\cdot, \omega_2), f(\cdot, \omega_2), \omega_2 \in \Omega_2, n \in \mathbb{N}\}$ is a class of comonotonic mappings so like in the proof of theorem 3 there exists a σ -additive probability measures P_1 such that $f_{n2}(\cdot) = \int_{\Omega_1} f_n(\cdot, \omega_2) dv_1 = \int_{\Omega_1} f_n(\cdot, \omega_2) dP_1$ and $f_2(\cdot) = \int_{\Omega_1} f(\cdot, \omega_2) dv_1 = \int_{\Omega_1} f(\cdot, \omega_2) dP_1$. In theorem 3 we proved $\{f_2(\cdot) = \int_{\Omega_1} f(\cdot, \omega_2) dv_1, f(\omega_1, \cdot), \omega_1 \in \Omega_1\}$ is a class of comonotonic mappings, so for all n in \mathbb{N} , $\{f_{n2}(\cdot) = \int_{\Omega_1} f_n(\cdot, \omega_2) dv_1, f_n(\omega_1, \cdot), \omega_1 \in \Omega_1\}$ is also a class of comonotonic mappings, then f_{n2} and f_2 are comonotonic and there exists a σ -additive probability measures P_2 such that $\int_{\Omega_2} f_2 dv_2 = \int_{\Omega_2} f_2 dP_2$ and $\int_{\Omega_2} f_{n2} dv_2 = \int_{\Omega_2} f_{n2} dP_2$. As we have proved $\int f_n dv = \int f_n dP_1 \times P_2$ and $\int f dv = \int f dP_1 \times P_2$ the monotone convergence theorem for measures allows us to conclude $\int f dv = \int \int f dv_1 dv_2$.

■

We are now looking for Fubini independent product capacities which inherit the properties of their marginals. It is not always true for the Fubini independent product capacity, v_m , defined above. The next example shows it with convex marginals. We set up two convex capacities v_1 and v_2 as follows,

let $\Omega = \{\omega_1, \omega_2, \omega_3\}$; v is defined on $\mathcal{P}(\Omega)$ by:

$$v(A) = \frac{6}{24} \text{ if } |A| = 1, v(A) = \frac{15}{24} \text{ if } |A| = 2, v(\Omega) = 1$$

v is convex but not totally monotone since $v(\Omega) - \sum_{\substack{E \subsetneq \Omega \\ I \neq \emptyset}} (-1)^{|E|+1} v(E) = \frac{-3}{24}$

We consider now the product space $\Omega \times \Omega$ and the sets

$$A_1 = \{(\omega_1, \omega_1), (\omega_2, \omega_1), (\omega_2, \omega_2)\}; A_2 = \{(\omega_1, \omega_2), (\omega_1, \omega_2), (\omega_2, \omega_2)\}.$$

$A_1, A_2, A_1 \cup A_2$ are slice-comotonic, $A_1 \cap A_2$ is not slice comonotonic. Simple computations show that $v_m(A_1 \cup A_2) = \frac{15 \cdot 15}{24 \cdot 24}$, $v_m(A_1 \cap A_2) = \frac{36}{24 \cdot 24}$,

$v_m(A_1) = v_m(A_2) = \frac{6}{24}$. Hence v_m is not convex since $v_m(A_1) + v_m(A_2) = \frac{288}{24 \cdot 24} > v_m(A_1 \cup A_2) + v_m(A_1 \cap A_2) = \frac{261}{24 \cdot 24}$

Another product capacity for convex marginals has been suggested in Walley and Fine [17] (see also Hendon et al. [11]; it has been called the "core-product"; for two convex capacities v_1 and v_2 , we define the core-product v_c as:

$$v_c = \inf_{\substack{P_1 \in C(v_1) \\ P_2 \in C(v_2)}} P_1 \times P_2$$

However it may be not convex as shown in the following example: we consider again $\Omega \times \Omega$, where v_1 and v_2 are respectively defined on $\mathcal{P}(\Omega)$ by:

$$\begin{aligned} v_1(A) &= \frac{4}{24} \text{ if } |A| = 1, v_1(A) = \frac{14}{24} \text{ if } |A| = 2, v_1(\Omega) = 1 \\ v_2(A) &= \frac{6}{24} \text{ if } |A| = 1, v_2(A) = \frac{15}{24} \text{ if } |A| = 2, v_2(\Omega) = 1 \end{aligned}$$

v_1 is convex and not totally monotone since $v(\Omega) - \sum_{\substack{E \subseteq \Omega \\ I \neq \emptyset}} (-1)^{|E|+1} v(E) = \frac{-6}{24}$

The three extreme points of the cores of v_i are respectively:

	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$		$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$
m_1	$\frac{4}{24}$	$\frac{10}{24}$	$\frac{10}{24}$	m_2	$\frac{6}{24}$	$\frac{9}{24}$	$\frac{9}{24}$
m'_1	$\frac{10}{24}$	$\frac{4}{24}$	$\frac{10}{24}$	m'_2	$\frac{9}{24}$	$\frac{6}{24}$	$\frac{9}{24}$
m''_1	$\frac{10}{24}$	$\frac{10}{24}$	$\frac{4}{24}$	m''_2	$\frac{9}{24}$	$\frac{9}{24}$	$\frac{6}{24}$

The extreme points of $C(v_1) \times C(v_2)$ are the products of the extreme points of $C(v_1)$ and $C(v_2)$. So the infimum is attained for one of those nine extreme points.

Let $A_1 = \{(\omega_1, \omega_2), (\omega_2, \omega_1)\}$ and $A_2 = \{(\omega_1, \omega_1), (\omega_2, \omega_1)\}$

Simple computations show that $v_C(A_1 \cup A_2) = \frac{120}{24 \cdot 24}$, $v_C(A_1 \cap A_2) = \frac{24}{24 \cdot 24}$, $v_C(A_1) = \frac{96}{24 \cdot 24}$ and $v_C(A_2) = \frac{84}{24 \cdot 24}$. Hence v_C is not convex since $v_C(A_1) + v_C(A_2) = \frac{180}{24 \cdot 24} > v_C(A_1 \cup A_2) + v_C(A_1 \cap A_2) = \frac{144}{24 \cdot 24}$

Now let us suppose that v_1 and v_2 are two totally monotone capacities. To the same question "can we get a totally monotone Fubini independent product?", some answers are available.

Ghirardato [9](p. 285-288) proposes a "Möbius product" which is totally monotone when the marginals are totally monotone, moreover it is the unique totally monotone Fubini independent product. In Ghirardato framework it is available only on finite spaces.

In section 3 we have generalised that result in the countable case. We have got a σ -continuous totally monotone product which is unique. Let us note in the

last example that "core product" and "Möbius product" do not necessarily coincide for two initial totally monotone marginals.

Consider the product space $\{\omega_1, \omega'_1\} \times \{\omega_2, \omega'_2\}$, two totally monotone capacities v_1 and v_2 , for which we site the two extreme points of their cores.

	$\{\omega_1\}$	$\{\omega'_1\}$	$\{\omega_1, \omega'_1\}$		$\{\omega_2\}$	$\{\omega'_2\}$	$\{\omega_2, \omega'_2\}$
v_1	0.2	0.3	1	v_2	0.4	0.1	1
m_1	0.2	0.8	1	m_2	0.4	0.6	1
m'_1	0.7	0.3	1	m'_2	0.9	0.1	1

$C(v_1) = \{\alpha m_1 + (1 - \alpha)m'_1; \alpha \in [0, 1]\}$; $C(v_2) = \{\beta m_2 + (1 - \beta)m'_2; \beta \in [0, 1]\}$.
The extreme points of $C(v_1) \times C(v_2)$ are the products of the extreme points of $C(v_1)$ and $C(v_2)$ because:

$$P_1 \times P_2 = \alpha\beta m_1 \times m_2 + \alpha(1 - \beta)m_1 \times m'_2 + (1 - \alpha)\beta m'_1 \times m_2 + (1 - \alpha)(1 - \beta)m'_1 \times m'_2$$

So the infimum is attained for one of those four extreme points.

Computations give us:

$$v_c(\{(\omega_1, \omega'_2), (\omega'_1, \omega_2)\}) + v_c(\{(\omega_1, \omega_2), (\omega'_1, \omega_2)\}) = \min\{0.2 \cdot 0.6 + 0.8 \cdot 0.4; 0.2 \cdot 0.1 + 0.8 \cdot 0.9; 0.7 \cdot 0.6 + 0.3 \cdot 0.4; 0.7 \cdot 0.1 + 0.3 \cdot 0.9\} + v_1(\{\omega_1, \omega'_1\}) \cdot v_2(\{\omega_2\}) = .0.34 + 0.4$$

$$v_c(\{(\omega_1, \omega'_2), (\omega'_1, \omega_2), (\omega_1, \omega_2)\}) + v_c(\{(\omega'_1, \omega_2)\}) = \min\{0.2 + 0.8 \cdot 0.4; 0.2 + 0.8 \cdot 0.9; 0.7 + 0.3 \cdot 0.4; 0.7 + 0.3 \cdot 0.9\} + v_1(\{\omega'_1\}) \cdot v_2(\{\omega_2\}) = 0.52 + 0.12$$

Thus v_c is neither totally monotone nor convex.

6

Concluding remarks

1. All our results can be naturally generalised for n spaces instead of two.
2. The usual Fubini Theorem has applications in game theory or portfolio selections. For non-additive measures one could use the Fubini theorem proposed but it requires to restrict only on slice-comonotonic functions.

3. Obtaining a convex (resp. totally monotone) Fubini independent product when marginals are convex (resp. totally monotone) will be the object of a future work. We think we could construct super additive or totally balanced product capacities when marginals are super additive or totally balanced. A convex product capacity with convex marginals seems more problematic.

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