Pricing the Weather Derivatives in the Presence of Long Memory in Temperatures

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ABSTRACT

Weather derivatives are financial contracts which allow companies to protect themselves against weather fluctuations. Since their underlying is not a traded asset, they cannot be priced by the traditional financial theory based on the hedging portfolio and on the arbitrage-free argument. Some authors suggest to use the actuarial pricing approach to value the weather derivatives. Although this method is simple to implement and does not repose on so restrictive assumptions as the financial one, it suffers from the fact that it is only based on the modelling of the temperature since the market information is not necessary to value the weather derivatives by this approach. On the contrary, the financial method needs to infer the market price of weather risk since the weather index is not a traded asset. Since 1999, a weather futures market exists in the United-States and information from the quotations of these contracts can be extracted to price the weather derivatives by the financial method. The purpose of this paper is to compute and to compare the prices stemming from the both approaches in a fractional framework since tests reveal the presence of a long memory phenomenon in the mean and in the volatility of the New York daily average temperatures.

Keywords : weather derivatives, incomplete market, long memory, ARFIMA process, FIGARCH process, LMSV process, fractional Brownian motion, PDE, Monte-Carlo simulations

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1. Introduction

Since the weather options lie upon a weather index which is not a traded asset, that is to say there is no spot market for it, no self-financing portfolio composed of the weather index and the riskless asset can be constructed. The market is said to be incomplete for the weather options and therefore, we cannot use the non-arbitrage principle to price them. In the following, we will refer to this approach as the 'financial method'. Very few financial asset prices are correlated with the weather index to construct such a hedging portfolio. Geman (1999) suggests a hedging strategy based upon the power or gas derivatives to price the weather options based on the temperature index. According to Brix, Jewson and Ziehmann (2002), the temperature index is highly correlated with the gas demand but not with the gas prices. However, since 1999, weather futures contracts are traded on the standardized markets such as the Chicago Mercantile Exchange (CME). The price of these contracts is very correlated with the underlying of the weather options. Then, a weather option can be hedged by a self-financing portfolio constituted by the weather futures and the riskless asset. But these markets are not liquid enough, large transaction costs prevent from creating such a portfolio. For all these reasons, an actuarial method was proposed to price the weather derivatives (Augros and Moréno (2002) and Brix, Jewson and Ziehmann (2002)). Their value corresponds to the expected outcome (under the historical probability) plus a charge depending on a risk measure which is usually the standard deviation. The popularity of this method among practitioners comes from the fact that it is very easy to implement it. The expected outcome is calculated by using the Monte-Carlo simulations. No equivalent martingale measure has to be considered with this approach on the contrary to the financial one. But this pricing method is not satisfactory since it does not necessitate to account for the information conveyed by the market whereas the financial pricing method requires the calibration of the model price to the market conditions. In the case of the weather options, a market price of weather risk has to be inferred from the observed prices to value them since the market is incomplete. But no quotation exists for the weather options since they are negociated only over-the-counter. Hamisultane (2006) suggests to extract this information from the frequently traded weather futures prices which are quoted on the CME. It is recovered by solving numerically a partial differential equation (PDE). She extracts also the risk-neutral density from these prices by using the Monte-Carlo simulations. We propose in this paper to infer the both information to compute the financial prices and to compare them to the actuarial ones. All the prices will be calculated by using the long memory processes since we show the presence of persistence in the mean and in the volatility of the New York daily average temperatures. This phenomenon in the temperature serie was already observed by Caballero, Jewson and Brix (2002) and Moréno (2003). Since the financial pricing method usually appeals to the continuous time processes while the actuarial pricing approach lies generally on the discrete time models, we propose to estimate the discrete time ARFIMA-FIGARCH and ARFIMA-LMSV processes and the continuous time fractional mean-reverting-FIGARCH process for modelling the long memory in the New York daily average temperatures. The estimation method used for these models is the spectral likelihood method which is simple to implement and not computationally demanding. Throughout this study we will compare the results brought by the long memory processes to those given by the simple AR-GARCH process so as to gauge the contribution of these processes. The originalities of this paper are the comparison of the prices provided by the financial and actuarial approaches and the estimation of the processes with a long memory in the mean and in the variance in both discrete and continuous time by the frequency-domain likelihood method. The plan of this paper is as follows : in section 2, we describe the weather derivatives, in section 3 we present the discrete time long memory processes which are the ARFIMA (AutoRegressive

Fractionally Integrated Moving Average), the FIGARCH (Fractionally Integrated Generalized AutoRegressive Conditional Heteroskedasticity), the FIEGARCH (Fractionally Integrated Exponential GARCH) and the LMSV (Long Memory Stochastic Volatility) processes, in section 4 we deal with the continous time long memory process represented by the fractional mean-reverting diffusion process, in section 5 we describe the estimation of the processes with long memory in the mean and in the variance, in discrete and continuous time by the spectral likelihood approach, section 6 is dedicated to the application of the long memory tests to the New York daily average temperatures, we conclude the presence of a persistent phenomenon in the mean and in the volatility of the serie, we then estimate an ARFIMA-FIGARCH, an ARFIMA-LMSV and a fractional mean-reverting-FIGARCH process, we compare their performances to those of the AR-GARCH process for reproducing and forecasting the temperatures, section 7 discusses the financial and actuarial pricing methods, computes the weather futures prices and compares them to the observations and section 8 concludes the paper.

2. Brief description of the weather derivatives

Weather derivatives are financial intruments based on a weather index which can be rain, snow, frost or temperature index. But the most traded contracts are the temperature-based contracts because the temperature is a more manageable parameter than the rain or the snow which is subject to discontinuities (Dischel (1999)). Temperature-based contracts are mainly based on the degree-day index which is the accumulation of the Heating Degree Days (HDDs) during the winter period from November to March and of the Cooling Degree Days (CDDs) during the summer period from May to September. April and October are often referred to as the 'shoulder months'[(1)]. These contracts can be monthly[(2)] or seasonal ones. The HDD and CDD indexes for n days in the contract period are respectively defined as :

$$I_n^H = \sum_{j=1}^{n} HDD_j \quad \text{and} \quad I_n^C = \sum_{j=1}^{n} CDD_j \qquad (1)$$

where $HDD_j = \max(65°F - T_j , 0)$, $CDD_j = \max(T_j - 65°F , 0)$, $T_j = \frac{T_j^{max} + T_j^{min}}{2}$ represents the average temperature for day j and 65° F (≈18°Celcius) is the reference temperature above which people start turning their air conditioners on for cooling and under which people start heating their homes.

The weather derivatives are structured as options, futures and forwards. There are two types of options, calls and puts.

A CDD call option delivers at the end of a period to the buyer of the contract the following payoff[(3)] if I_n^C is greater than the predetermined strike price level K :

$$C = \delta.\max(I_n^C - K , 0) \qquad (2)$$

(1) They are usually excluded from contracts because fluctuations during these months are greater than during the other months of the year.

(2) The accumulation period for a monthly contract corresponds to the calendar period of the month. Monthly and seasonal contracts are typical contracts on the CME.

(3) Weather options are European contracts.

where δ is the tick size which represents the value of one degree-day.

A CDD put option delivers at the end of a period to the buyer the following payoff if I_n^C is lower than the predetermined strike level :

$$C = \delta.\max(K- I_n^C , 0). \qquad (3)$$

For HDD call and put options, I_n^C is replaced by I_n^H in Eq.(2) and (3).

The CDD/HDD weather futures are agreements to buy or sell the value of the I_n^C/I_n^H index at a specific future date. They are traded on the standardized markets. Unlike the CDD/HDD weather options, the buyer of the contract does not pay any premium to write the contract but he is obliged to buy or sell the I_n^C/I_n^H index at the end of the contract.

The CDD/HDD weather forwards are capped contracts which are traded over-the-counter.

The most common approach for computing the weather derivative prices is to model the temperature T_j and not the index because some information is lost when the CDD or the HDD is calculated. Indeed for the HDD, it is equal to zero if T_j is above 65°F and for the CDD, it is equal to zero for T_j below 65°F. Therefore, a part of the CDD/HDD historical data is useless for estimating the model.

Some authors like Caballero, Jewson and Brix (2002), Moréno (2003), Brody, Syroka and Zervos (2002) and Benth (2003) point out the presence of a long memory in the daily average temperatures. Caballero, Jewson and Brix (2002) propose to use an ARFIMA process to model it in discrete time while Moréno (2003) recommends an ARFIMA-FIGARCH process. Brody, Syroka and Zervos (2002) and Benth (2003) deal with the continuous time process based on the fractional Brownian motion. We show in the following that a long memory is indeed present in the New York daily average temperature as well as in its volatility. But first, we define and describe the long memory processes treated in literature.

3. Modelling the long memory in discrete time

In the time domain, a long memory process is characterized by an autocorrelation function which decays very slowly at an hyperbolic rate while a short memory process has an autocorrelation function decaying at an exponential rate. Short memory processes are represented by the ARIMA (Autoregressive Integrated Moving Average) processes. In the frequency domain, the long memory process has a spectral density which is concentrated at low frequencies.

The presence of a long memory in a serie is modelled by the ARFIMA process in discrete time and in continuous time by the diffusion process with a fractional Brownian motion.

These latters allow the modelling of the long memory in the mean. For long memory in the variance, FIGARCH, FIEGARCH and LMSV processes are used.

Let us recall their definition :

Definition 1 : an ARFIMA(p,d,q) process for the serie $\{y_j\}$ is stated as follows

$$[1-\Phi(L)]\,(1-L)^d\, y_j = [1-\Theta(L)]\,\varepsilon_j\,, \qquad (4)$$

where d is the fractional differencing parameter taking any non-integer values, $\{\varepsilon_j\}$ is the white noise process with $\varepsilon_j = \sigma_\varepsilon \tilde{\varepsilon}_j$, $\tilde{\varepsilon}_j$ ~>iid(0,1) ,

L is the backward-shift operator, i.e $Ly_j = y_{j-1}$, $L^n y_j = y_{j-n}$,

$$\Phi(L) = \sum_{l=1}^{p} \Phi_l L^l \;,\quad \Theta(L) = \sum_{l=1}^{q} \Theta_l L^l \,, \qquad (5)$$

all the roots of 1-Φ(L) and 1-Θ(L) lie outside the unit circle ($|\Phi_l|<1$ and $|\Theta_l|<1$) ,

$$(1-L)^d = 1 - dL - \frac{d(1-d)}{2!}L^2 - \frac{d(1-d)(2-d)}{3!}L^3 - \ldots = \sum_{l=0}^{\infty} \pi_l L^l \,, \qquad (6)$$

$$\pi_l = \frac{\Gamma(l-d)}{\Gamma(l+1)\Gamma(-d)} = \prod_{0<k\le l} \left(\frac{k-1-d}{k}\right) \,,\ l=0,1,\ldots \text{ and } \Gamma \text{ is a gamma function.} \qquad (7)$$

For $0 < d < \frac{1}{2}$, the process $\{y_j\}$ has a long memory. Its autocorrelations are all positive and decay at a hyperbolic rate.

For $-\frac{1}{2} < d < 0$, the process is said to be “antipersistent” or to have an “intermediate memory” which can be considered as a long memory since its autocorrelations are negative and decay hyperbolically to zero (see Baillie (1996)).

For d = 0, the process has a short memory : $\{y_j\}$ is an ARMA process. For d = 1, it turns to be an ARIMA process.

For $-\frac{1}{2} < d < \frac{1}{2}$, the process $\{y_j\}$ is stationary and invertible.

Definition 2 : a FIGARCH(p_0,d_0,q_0) process is expressed as

$$[1-\Psi(L)]\,(1-L)^{d_0}\varepsilon_j^2 = \alpha_0 + [1-\beta(L)]\,v_j \qquad (8)$$

where $0 \le d_0 < 1$,

$$\Psi(L) = \sum_{l=1}^{q_0} \Psi_l L^l \,, \qquad (9)$$

$$\beta(L) = \sum_{l=1}^{p_0} \beta_l L^l \qquad (10)$$

and all the roots of $1-\Psi(L)$ and $1-\beta(L)$ lie outside the unit circle.

By noting that $v_j = \varepsilon_j^2 - \sigma_{\varepsilon,j}^2$, this process can be restated as

$$[1-\beta(L)]\,\sigma_{\varepsilon,j}^2 = \alpha_0 + \left[1-\beta(L) - [1-\Psi(L)](1-L)^{d_0}\right] \varepsilon_j^2 . \qquad (11)$$

Equivalently, we have

$$\sigma_{\varepsilon,j}^2 = \alpha_0[1-\beta(1)]^{-1} + \left[1 - [1-\Psi(L)]\,[1-\beta(L)]^{-1}(1-L)^{d_0}\right] \varepsilon_j^2 \ , \qquad (12)$$

$$\sigma_{\varepsilon,j}^2 = \alpha_0[1-\beta(1)]^{-1} + \lambda(L)\, \varepsilon_j^2 \ , \qquad (13)$$

where

$$\lambda(L) = \left[1 - [1-\Psi(L)]\,[1-\beta(L)]^{-1}(1-L)^{d_0}\right] = \lambda_1 L + \lambda_2 L^2 + \ldots \qquad (14)$$

and $\lambda_k \geq 0$, $k = 1,2,\ldots$

In the case of the FIGARCH process, conditions are required to ensure the positiveness of the conditional variance while no condition is needed for the FIEGARCH and LMSV processes. These conditions will be presented in part 6.3.

Definition 3 : a FIEGARCH(p_0,d_0,q_0) process is defined as

$$\ln \sigma_{\varepsilon,j}^2 = \omega + \varnothing(L)^{-1}(1-L)^{-d_0}[1+\alpha(L)]\, g(e_{j-1}) \qquad (15)$$

where

$$\alpha(L) = \sum_{l=1}^{q_0} \alpha_l L^l , \qquad (16)$$

$$\varnothing(L) = 1- \sum_{l=1}^{p_0} \varnothing_l L^l , \qquad (17)$$

$$g(e_j) = \gamma_1 e_j + \gamma_2 [\, |e_j| - E|e_j|\,] \ , \qquad (18)$$

$$e_j = \frac{\varepsilon_j}{\sigma_{\varepsilon,j}} \ \sim> \text{iid}(0,1) \qquad (19)$$

and all the roots of $\varnothing(L) = 0$ lie outside the unit circle.

Definition 4 : a LMSV(p_0,d_0,q_0) process is given by

$$x_j = \sigma_{\varepsilon,j}\, \tilde{\varepsilon}_j \quad \text{where} \quad \tilde{\varepsilon}_j \sim> \text{iid}(0,1) , \qquad (20)$$

$$\sigma_{\varepsilon,j} = \sigma \exp(v_j/2) \qquad (21)$$

which can be written as

$$\ln \sigma^2_{\varepsilon,j} = \ln\sigma^2 + v_j \tag{22}$$

where

$$[1-\theta(L)]\,(1-L)^{d_0} v_j = [1-\varphi(L)]\,\eta_j \text{ with } \eta_j \sim> iid(0,\sigma^2_\eta)\,, \tag{23}$$

$$\varphi(L) = \sum_{l=1}^{q_0} \varphi_l L^l\,, \tag{24}$$

and

$$\theta(L) = \sum_{l=1}^{p_0} \theta_l L^l. \tag{25}$$

4. Modelling the long memory in continuous time

Brody, Syroka and Zervos (2002) are the first to suggest a mean-reverting fractional Brownian motion process (or a fractional Ornstein-Uhlenbeck process) for the daily average temperature which is written as

$$dT_t = \left[\frac{dT^m_t}{dt} + a(T^m_t - T_t)\right] dt + \sigma_t\, dW^H_t \tag{26}$$

where a is the speed of the mean-reversion, σ_t is the volatility of the temperature, W^H_t corresponds to the fractional Brownian motion, H refers to the Hurst exponent, for ½ <H<1 the process has a long memory, for 0<H<½, it is said to be "antipersistent" and for H = ½, it has a short memory. T^m_t reflects the mean temperature defined by

$$T^m_t = \alpha + \beta t + \zeta \sin(\omega t + \varphi)\,, \quad \omega = \frac{2\pi}{365}\,, \tag{27}$$

where α, β, ζ, φ are constants ,

$$\frac{dT^m_t}{dt} = \beta + \omega\, \zeta \cos(\omega t + \varphi)\,, \tag{28}$$

this term is required in Eq.(26) to allow the temperature to revert to the mean temperature in the long run (see Dornier and Queruel (2000)).

Eq.(26) can be reformulated with the differencing parameter d substituted for H as follows

$$dT_t = \left[\frac{dT^m_t}{dt} + a(T^m_t - T_t)\right] dt + \sigma_t\, dW_{d,t} \tag{29}$$

which solution is given by

$$T_t = e^{-a(t-s)}(T_s - T^m_s) + T^m_t + \int_s^t \sigma_\tau\, e^{-a(t-\tau)} dW_{d,\tau}\,, \quad 0 \le s \le t \tag{30}$$

where $d = H - \frac{1}{2}$ and $-\frac{1}{2} < d < \frac{1}{2}$.

Stating the fractional Ornstein-Uhlenbeck process with the parameter d allows us to use the following calculation rules. Let

$$z_t = \int_s^t \sigma_\tau \, e^{-a(t-\tau)} dW_{d,\tau} \ , \qquad (31)$$

the derivative of order d of z_t is given by

$$z_t^{(d)} = \int_s^t \sigma_\tau \, e^{-a(t-\tau)} dW_\tau \qquad (32)$$

where $dW_t = \tilde{\varepsilon}_t \sqrt{dt}$ is the Wiener process with $\tilde{\varepsilon}_t \sim> N(0,1)$.

The derivative of order d can also be calculated as follows (see Comte (1996))

$$z_t^{(d)} = \int_s^t \frac{(t-\tau)^{-d}}{\Gamma(1-d)} dz_\tau \qquad (33)$$

where Γ is the gamma function and $z_t = T_t - e^{-a(t-s)}(T_s - T_s^m) - T_t^m$.

The discretization of Eq.(33) gives

$$z_t^{(d)} = \sum_{ph<t} \frac{(t - ph)^{-d}}{\Gamma(1 - d)} \Delta z_{ph} \qquad (34)$$

where $z_1, z_2, \ldots, z_n$ correspond to the discrete sample of n observations with step h , t = jh, j = 1,…,n , p = 0, 1,…, j and h is assumed here to be equal to 1.

These formulas will be useful in the following section to estimate the fractional Ornstein-Uhlenbeck process.

By discretizing Eq.(29), we show that the fractional Ornstein-Uhlenbeck process corresponds to the AR(1) process in discrete time :

$$\Delta T_t = \left[\frac{\Delta T_t^m}{\Delta t} + a(T_{t-1}^m - T_{t-1}) \right] \Delta t + \sigma_{t-1} \Delta W_{d,t} \qquad (35)$$

and then

$$\left(T_t - T_t^m\right) = (1-a) \left(T_{t-1} - T_{t-1}^m\right) + \sigma_{t-1} \Delta W_{d,t} \qquad (36)$$

where $\Delta t = 1$.

We notice that it does not strictly correspond to an ARFIMA(1,d,0) process defined in our case as

$$(1-L)^d (T_j - T_j^m) = \Phi_1 (1-L)^d (T_{j-1} - T_{j-1}^m) + \varepsilon_j \ . \qquad (37)$$

After introducing the discrete and continuous time long memory processes, we now look after their estimation. In the following, we will focus on the estimation of the ARFIMA-FIGARCH and ARFIMA-LMSV processes since we show in section 6 with the help of tests the presence of a long memory both in the mean and in the variance. We will select the 'best' one based on the information criteria which will be described later. The results of this process will be confronted with the results provided by the AR(1)-GARCH(1,1) process to see whether there is an improvement when considering the long memory processes. Thereafter, we will use the chosen long memory process to compute the actuarial prices. We will also estimate the continuous time version of the 'best' long memory model in dicrete time so as to calculate the financial prices since the main findings of the financial theory lie on a continuous time framework. The actuarial prices will be also calculated with the estimated continuous time process to see whether there is a significant difference between the results provided by the both processes.

5. Estimating processes with long memory in the mean and in the variance by the spectral likelihood method

Chung (1999) is among the rare authors who have dealt with the estimation in one step of the processes presenting simultaneously a long memory in the mean and in the variance. In his paper, he suggests to use for estimating the ARFIMA-FIGARCH process the algorithm which consists in squaring the residuals $\hat{\varepsilon}_j$ of the ARFIMA process and thereafter in applying the filter $(1-L)^{d_0}$ to the squared residuals in order to compute the variance equation $\sigma^2_{\varepsilon,j}$ as defined in Eq.(12). More precisely, he applies the filter $(1-L)^{d_0}$ to the term $(\varepsilon_j^2 - \sigma_\varepsilon^2)$ where σ_ε^2 is the unconditional variance of ε_j. The estimates of $\delta = (\Phi_l, \Theta_l, d, \Psi_l, \beta_l, d_0, \alpha_0)$ are obtained by maximizing the following log-likelihood function :

$$L(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n ; \delta) = -\frac{n}{2} \ln(2\pi) - \frac{1}{2} \sum_{j=1}^{n} \ln \sigma^2_{\varepsilon,j} - \frac{1}{2} \sum_{j=1}^{n} \frac{\varepsilon_j^2}{\sigma^2_{\varepsilon,j}} \qquad (38)$$

where n is the number of observations.

This procedure is said in one step because it estimates all together the parameters $(\Phi_l, \Theta_l, d, \Psi_l, \beta_l, d_0, \alpha_0)$ with given initial values whereas the procedure in two steps consists in estimating separately the parameters. The parameters of the ARFIMA process are first calculated and they are thereafter employed to estimate the coefficients of the FIGARCH process.

Chung (1999) restates the Baillie, Bollerslev and Mikkelsen (1996)'s formulation of the FIGARCH given by Eq.(12) in order to simplify the computation of the variance equation. He expresses the FIGARCH process in the same form as the ARFIMA process without the constant term :

$$[1-\Psi(L)]\,(1-L)^{d_0}(\varepsilon_j^2 - \sigma_\varepsilon^2) = [1-\beta(L)]\, v_j \qquad (39)$$

where $v_j = \varepsilon_j^2 - \sigma^2_{\varepsilon,j}$ which yields

$$\sigma^2_{\varepsilon,j} = \beta(L)\sigma^2_{\varepsilon,j} + [1-\beta(L)]\,\varepsilon_j^2 - [1-\Psi(L)]\,(1-L)^{d_0}(\varepsilon_j^2 - \sigma^2). \qquad (40)$$

In this formulation, the filter $(1-L)^{d_0}$ is applied to $(\varepsilon_j^2 - \sigma_\varepsilon^2)$ and not to ε_j^2 . The fact that we have $(\varepsilon_j^2 - \sigma_\varepsilon^2)$ instead of ε_j^2 allows us not to use N pre-sample terms to compute $(1-L)^{d_0}(\varepsilon_j^2 - \sigma_\varepsilon^2)$. Indeed, in the Baillie, Bollerslev and Mikkelsen (1996)'s representation, the variance equation lies on the calculation of $(1-L)^{d_0}\varepsilon_j^2$ as follows

$$(1-L)^{d_0}\varepsilon_j^2 = \sum_{l=0}^{\infty} \pi_l\, \varepsilon_{j-l}^2 \approx \sum_{l=0}^{N+j-1} \pi_l\, \varepsilon_{j-l}^2 \tag{41}$$

because the pre-sample values of ε_j^2 for j = 0, -1, -2, … cannot be set to zero. They cannot have a zero mean while in the Chung (1999)'s formulation of the FIGARCH, the pre-sample values of $(\varepsilon_j^2 - \sigma_\varepsilon^2)$ can have a zero mean. Therefore, the computation reduces to

$$(1-L)^{d_0}(\varepsilon_j^2 - \sigma_\varepsilon^2) \approx \sum_{l=0}^{j-1} \pi_l\, (\varepsilon_{j-l}^2 - \sigma_\varepsilon^2). \tag{42}$$

The pre-sample values of $(\varepsilon_j^2 - \sigma_\varepsilon^2)$ can be set to zero since they do not have to be all positive.

Futhermore, the author stresses on the fact that the estimated value of the constant term in Eq.(12) is sensitive to the choice of N.

Despite this improvement brought by Chung (1999), this estimation approach for the ARFIMA-FIGARCH process still has a major drawback : it is extremely slow because of the computation of the both terms $(1-L)^{d_0}(\varepsilon_j^2 - \sigma_\varepsilon^2)$[(4)] and $(1-L)^d\, y_j$.

Following Breidt, Crato and de Lima (1998) who have employed a spectral likelihood function to estimate the LMSV process, we suggest to estimate the FIGARCH process by this approach. Hence, when estimating the ARFIMA-FIGARCH process, we will only have to calculate the term $(1-L)^d\, y_j$ in the ARFIMA process to obtain the residuals which will be used in the following spectral likelihood function (also called the Whittle function) :

$$L(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n ; \gamma) = -\frac{1}{2}\, n \ln(2\pi) - n \ln(\sigma_\varepsilon) - \frac{1}{2}\sigma_\varepsilon^{-2} \sum_j \frac{I(\lambda_j)}{g(\lambda_j)} - \frac{1}{2}\sum_j \ln g(\lambda_j) \tag{43}$$

where $\gamma = (\Phi_l, \Theta_l, d, \Psi_l, \beta_l, d_0, \alpha_0, \sigma_\varepsilon)$, $\lambda_j = \frac{2\pi j}{n} \in (-\pi,\pi)$, j = 1,2,…,n , $I(\lambda_j)$ stands for the periodogram defined as

$$I(\lambda_j) = \frac{1}{2\pi\, n}\left|\sum_{k=1}^{n} (\varepsilon_k^2 - \bar{\varepsilon}^2)\, e^{ik\lambda_j}\right|^2, \quad \bar{\varepsilon}^2 = \frac{1}{n}\sum_{k=1}^{n} \varepsilon_k^2 \ , \tag{44}$$

$\sigma_\varepsilon^2\, g = f$ and f is the spectral density given by

(4) Baillie, Bollerslev and Mikkelsen (1996) truncate the infinite sum at N=1000 for a sample of 3000 observations.

$$f(\lambda) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{|\Theta(e^{-i\lambda})|^2}{|\Phi(e^{-i\lambda})|^2} \,|1-e^{-i\lambda}|^{-2d} \qquad (45)$$

$$= \frac{\sigma_\varepsilon^2}{2\pi} \frac{(1+\Theta_1 e^{-i\lambda}+\Theta_2 e^{-2i\lambda}+\ldots+\Theta_q e^{-qi\lambda})(1+\Theta_1 e^{i\lambda}+\Theta_2 e^{2i\lambda}+\ldots+\Theta_q e^{qi\lambda})}{(1-\Phi_1 e^{-i\lambda}-\Phi_2 e^{-2i\lambda}-\ldots-\Phi_p e^{-pi\lambda})(1-\Phi_1 e^{i\lambda}-\Phi_2 e^{2i\lambda}-\ldots-\Phi_p e^{pi\lambda})} \,|1-e^{-i\lambda}|^{-2d},$$

$f(\lambda)$ represents the spectral density of the ARFIMA(p,d,q) process defined by $[1-\Phi(L)](1-L)^d y_j = [1+\Theta(L)]\,\varepsilon_j$ with ε_j ~> white noise$(0,\sigma_\varepsilon^2)$ (see Brockwell and Davis (1991)).

To estimate the parameters of the FIGARCH process by this method, we only need to write it in the ARFIMA form so as to use the corresponding spectral density given in Eq.(45). The ARFIMA representation of the FIGARCH process is defined in Eq.(39).

The FIEGARCH process can also be turned into an ARFIMA process as follows

$$Ø(L)(1-L)^{d_0}(\ln\sigma_{\varepsilon,j}^2 - \omega) = [1+\alpha(L)]\, g(e_{j-1})\,. \qquad (46)$$

But this expression is sightly different from that of Eq.(4) because of the presence of the term $g(e_{j-1})$ for which the coefficients of e_{j-1} and $|e_{j-1}|$ cannot be estimated by the Whittle function since they do not appear in the spectral density function given in Eq.(45).

On the contrary to the FIEGARCH process, the LMSV model does not involve the term $g(e_{j-1})$. Breidt, Crato and de Lima (1998) show that the spectral density for this process is written as

$$f(\lambda) = \frac{\sigma_\eta^2}{2\pi} \frac{|\varphi(e^{-i\lambda})|^2}{|\theta(e^{-i\lambda})|^2} \,|1-e^{-i\lambda}|^{-2d} + \frac{\sigma_\xi^2}{2\pi} \qquad (47)$$

because $\log(x_j^2)$ of Eq.(20) can be expressed as

$$\log(x_j^2) = [\,\log(\sigma^2) + E[\log(\tilde{\varepsilon}_j^2)]\,] + v_j + [\,\log(\tilde{\varepsilon}_j^2) - E[\log(\tilde{\varepsilon}_j^2)]\,] = \mu + v_j + \xi_j \qquad (48)$$

where $v_j = \frac{[1-\varphi(L)]\,(1-L)^{-d}}{[1-\theta(L)]}\eta_j$ with η_j ~> iid$(0,\sigma_\eta^2)$ and $[\,\log(\tilde{\varepsilon}_j^2) - E[\log(\tilde{\varepsilon}_j^2)]\,]$ ~> iid$(0,\sigma_\xi^2)$.

Therefore in this paper and for the discrete time case, we will only consider the estimation of the ARFIMA-FIGARCH and ARFIMA-LMSV processes by the frequency-domain likelihood method.

In the continuous time, the fractional-mean-reverting-FIGARCH or the fractional-mean-reverting-LMSV diffusion processes can also be estimated by the present approach. However, the algorithm steps for estimation are here a little different from those we have described for the discrete time long memory processes. The method we employ is stemming from Comte (1996)'s work. The algorithm starts computing the residuals $\hat{z}_t$ of the fractional diffusion process as follows :

$$\hat{z}_t = \left(T_t - \hat{T}_t^m\right) - (1-\hat{a})\left(T_{t-1} - \hat{T}_{t-1}^m\right) \qquad (49)$$

where $\hat{a}$ refers to the initial value provided by the econometrician.

Next, a serie $\hat{Z}_t$ is calculated which is defined as

$$\hat{Z}_t = \hat{z}_t^{(\hat{d})} = \sum_{ph<t} \frac{(t-ph)^{-\hat{d}}}{\Gamma(1-\hat{d})} \Delta\hat{z}_{ph} \tag{50}$$

where $\hat{d}$ stands for the given initial value which can be estimated by the Geweke and Porter-Hudak (1983)'s approach discussed in part 6.2.

Thereafter, the serie $\hat{Z}_t$ is squared and used to compute the spectral likelihood function in Eq.(43). The estimates of $\kappa = (a, d, d_0, \Psi_l, \beta_l, \sigma_\varepsilon)$ for the fractional-mean-reverting-FIGARCH process and $\upsilon = (a, d, d_0, \theta_l, \varphi_l, \sigma_\eta, \sigma_\xi)$ for the fractional-mean-reverting-LMSV model are obtained by maximizing Eq.(43).

We now examine the New York daily average temperature. We show by applying the tests presented below that this serie contains a long memory in the mean and also in the volatility.

6. Evidence for long memory in temperatures

To check whether there is a long memory in a time serie, several tests were proposed. We content ourself here to use the so-called Lo (1991)'s test and the Geweke and Porter-Hudak (1983)'s approach which are simple to implement.

6.1 Lo (1991)'s modified R/S test

The R/S statistic (or rescaled range statistic) is due to Hurst (1951). It is calculated as follows

$$R/S = \frac{1}{\left[\frac{1}{n}\sum_{j=1}^{n}(X_j - \overline{X})^2\right]^{1/2}} \times \left[\max_{1\le k\le n} \sum_{j=1}^{k}(X_j - \overline{X}) - \min_{1\le k\le n} \sum_{j=1}^{k}(X_j - \overline{X})\right] \tag{51}$$

where n is the number of observations and $\overline{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$.

We obtain the Hurst exponent by the following formula

$$H \sim \frac{\log(R/S)}{\log(n)} \text{ with } 0<H<1. \tag{52}$$

Lo (1991) points out that the R/S statistic misbehaved in the presence of a short memory. It tends to be high and makes us conclude in favour of a long memory although the time serie contains only a short memory. Lo (1991) suggests a modified R/S statistic which is sensitive to the long memory but not to the short memory as well as a significativity test for R/S for which the null hypothesis is the presence of a short memory.

This statistic is defined as

$$R/S(q) = \frac{1}{S(q)} \times \left[\max_{1 \le k \le n} \sum_{j=1}^{k} (X_j - \overline{X}) - \min_{1 \le k \le n} \sum_{j=1}^{k} (X_j - \overline{X}) \right], \qquad (53)$$

$$S(q) = \frac{1}{n} \sum_{j=1}^{n} (X_j - \overline{X})^2 + \frac{2}{n} \sum_{j=1}^{q} \omega_j(q) \left[\sum_{i=j+1}^{n} (X_i - \overline{X}) (X_{i-j} - \overline{X}) \right], \qquad (54)$$

$$\omega_j(q) = 1 - \frac{j}{q+1} \;, \; q<n , \qquad (55)$$

$$q = \text{integer}[k_n] \;,\; k_n = \left(\frac{3n}{2}\right)^{1/3} \left(\frac{2\hat{\rho}}{1-\hat{\rho}^2}\right)^{2/3} \qquad (56)$$

where $\hat{\rho}$ is the correlation coefficient for an AR(1) process.

To carry out the significativity test, we need to compute the following statistic

$$V = \frac{R/S(q)}{\sqrt{n}} \qquad (57)$$

and to compare it to the critical values provided by Lo (1991) which are 1.747 at 5% level and 1.620 at 10% level.

6.2 Geweke and Porter-Hudak (1983)'s approach

Geweke and Porter-Hudak (1983), henceforth GPH, were the first to suggest a log-periodogram regression to estimate the fractional differencing parameter d. It consists in regressing the following equation :

$$Y_k = c - d \left[2\ln(\lambda_k)\right] + U_k \qquad k = 1,...,m \qquad (58)$$

where $m = n^{0.5}$, $2\ln(\lambda_k) = \ln\left(4\sin^2\left(\frac{\lambda_k}{2}\right)\right) = \ln \mid 1- e^{-i\lambda_k} \mid^2$, $\lambda_k = \frac{2\pi k}{n} \in (-\pi,\pi)$, U_k is the noise and

$$Y_k = \ln I(\lambda_k) \qquad (59)$$

where I is the periodogram.

Let $V_k = -2\ln(\lambda_k)$, we then obtain the following estimator of d :

$$\hat{d} = (V'V)^{-1}Y'V \tag{60}$$

where $Y = (Y_1,...,Y_m)'$ et $V = (V_1,\dots V_m)'$.

They also show that the estimated parameter $\hat{d}$ is normally distributed when the sample size becomes large ($n \to \infty$). Its distribution is then given as follows

$$\hat{d} \sim> N\left(d, \pi^2\left[6\sum_{k=1}^{m}(V_k - \overline{V})^2\right]^{-1}\right) \tag{61}$$

where $\overline{V} = \frac{1}{m}\sum_{k=1}^{m} V_k$.

This result is used to construct a significativity test for $\hat{d}$.

6.3 Application of the long memory tests and estimation of the long memory processes for temperatures

We have at our disposal a sample of 4595 observations of the New York daily average temperature from January 1993 through July 2005. To apply the Lo (1991)'s test, we must check that the New York daily average temperature is stationary because a non stationary serie behaves like a long memory process, so we can obtain biased results. To make the serie stationary, we substract to it the trend c_j and the seasonal component s_j :

$$y_j = T_j - c_j - s \ , \tag{62}$$

$$y_j = T_j - (A + Bj) - C\sin(\omega j + \varphi) \tag{63}$$

where A, B, C and φ are constants, $\omega = \frac{2\pi}{365}$ and $T_j^m = A + Bj + C\sin(\omega j + \varphi)$.

By least squares, we obtain these estimates :

$$\hat{T}_j^m = 55.96 - 4.20\times10^{-5}\, j + 22.23\, \sin\left(\frac{2\pi}{365} j - 2\right). \tag{64}$$

To make certain that the serie y_j is really stationary, we apply to it the Augmented Dickey-Fuller test. The value of the statistic test for the model without trend and intercept is -25.2179 which is inferior to the 5% critical value equals to -1.9394. Hence, we conclude that the serie is stationary. We can pursue our study by the application of the long memory tests for which the results are presented in Table 1 and Table 2.

	R/S	Modified R/S
y_j	H = 0.6667	H = 0.6671 ; V = 4.0626

Table 1 : R/S test and modified R/S test (Lo (1991)'s test)

	GPH method ($m = n^{0.5}$)
y_j	$\hat{d} = 0.162$ (6.95)

Table 2 : GPH estimation of the parameter d,
the number in parentheses corresponds to the t-statistic

The tests reveal the presence of the long memory in the serie y_j. Indeed, we have H > 0.5 with V > 1.747 at 5% level and $0 < \hat{d} < ½$ with a t-statistic > 1.96 at 5% level.

To model the long memory in the serie, we estimate the ARFIMA(1,d,0) process in order to compare it later to the fractional mean-reverting diffusion process. As we mentioned before, the ARFIMA(1,d,0) process is not exactly equivalent to the fractional Ornstein-Uhlenbeck process in continuous time. We will see how different are the results produced by the two processes.

The ARFIMA process can be estimated by maximizing the Whittle function defined in Eq.(43) where the elements ε_k^2 of the periodogram are replaced by y_j. It can also be determined by the Sowell (1992)'s exact maximum likelihood method (see Lardic and Mignon (2002) and Baillie (1996) for a comparison of the methods).

Estimating the ARFIMA process by the spectral likelihood method requires starting values for the optimization problem. Lardic and Mignon (2002) suggest to calculate first the parameter d with the GPH approach or with the R/S procedure to obtain $\hat{d}$ and next to use it to compute $(1-L)^{\hat{d}} y_j = x_j$. The starting values for the autoregressive and moving average parts of the ARFIMA(p,d,q) process correspond then to the estimates of the ARMA(p,q) process given by $[1-\hat{\Phi}(L)]\, x_j = [1-\hat{\Theta}(L)]\varepsilon_j$.

By using this method and the Whittle function, we obtain this estimated ARFIMA(1,d,0) process :

$$(1\text{-}0.543L)(1\text{-}L)^{0.149}\, y_j = \hat{\varepsilon}_j \quad (65)$$

where $\hat{\varepsilon}_j$ represents the residual of the model and the t-statistic[5] of $\hat{d}$ is equal to 5.36 which is above 1.96 at 5% level. This confirms the presence of a long memory in the serie.

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(5) $t_{\hat{d}} = \frac{\hat{d}}{\sqrt{Var(\hat{d})}}$ where Var($\hat{d}$) is the variance of $\hat{d}$ which appears on the diagonal of the covariance matrix given by the inverse of the information matrix

To check whether there is a long memory in the volatility serie, we apply now the tests to the squared residuals $\hat{\varepsilon}_j^2$. We see in tables below that $\hat{\varepsilon}_j^2$ contains a long memory. Unlike the ARFIMA process, the FIGARCH process is stationary for $0 \leq d_0 < 1$ (see Baillie, Bollerslev and Mikkelsen (1996)).

	R/S	**Modified R/S**
$\hat{\varepsilon}_j^2$	H = 0.58538	H = 0.58539; V = 2.05

Table 3 : R/S test and modified R/S test (Lo (1991)'s test)

	GPH method ($m = n^{0.5}$)
$\hat{\varepsilon}_j^2$	$\hat{d}_0 = 0.146$ (6.28)

Table 4 : GPH estimation of the parameter d_0,
the number in parentheses corresponds to the t-statistic

To model the long memory in the volatility serie, we employ the FIGARCH and LMSV processes. We estimate several ARFIMA-FIGARCH and ARFIMA-LMSV processes. The results appear in Table 5 and Table 6. The selection of the 'best' model is made by using the Akaike, the Schwarz and the Hannan-Quinn information criteria which are the smallest for the model adequately reproducing the observations. They are respectively expressed as

$$AIC = -2\left(\frac{LL}{n}\right) + \frac{2(p_0+q_0)}{n} , \tag{66}$$

$$SC = -2\left(\frac{LL}{n}\right) + (p_0+q_0)\frac{\ln(n)}{n} , \tag{67}$$

and

$$HQ = -2\left(\frac{LL}{n}\right) + 2(p_0+q_0)\frac{\ln(\ln(n))}{n} \tag{68}$$

where LL corresponds to the log-likelihood of the model with (p_0+q_0) parameters.

Before commenting the results in Table 5 and Table 6, we recall that the FIGARCH process is subject to restrictions ensuring the positiveness of the conditional variance. In general, they are difficult to establish. Bollerslev and Mikkelsen (1996) provide the conditions for a FIGARCH($1,d_0,1$) process which is written as

$$\sigma_{\varepsilon,j}^2 = \alpha_0 + \beta_1\sigma_{\varepsilon,j-1}^2 + [1- \beta_1 L - (1-\Psi_1 L)(1-L)^{d_0}]\, \varepsilon_j^2 . \tag{69}$$

The conditions are

$$\beta_1\text{-}d_0 \le \Psi_1 \le \frac{2\text{-}d_0}{3} \quad \text{and} \quad d_0\left(\Psi_1 - \frac{1\text{-}d_0}{2}\right) \le \beta_1(\Psi_1 - \beta_1 + d_0)\ . \qquad (70)$$

Chung (1999) suggests different conditions :

$$0 \le \Psi_1 \le \beta_1 \le d_0 < 1. \qquad (71)$$

He points out that the values which satisfy these conditions may not satisfy the Bollerslev and Mikkelsen (1996)'s ones but give also a non-negative conditional variance.

In the case of the FIGARCH(1,d_0,0) process, the Bollerslev and Mikkelsen (1996)'s restriction is $\beta_1 \le d_0$ and for the FIGARCH(0 d_0,1) process, we have $d_0 + 2\Psi_1 \le 1$.

In light of the results in Table 5, we see that the positiveness constraints are all satisfied for the FIGARCH process. Moreover, the parameters d and d_0 are all significant with a t-statistic well above 1.96 at 5% level which confirms again the presence of a long memory in the mean and in the variance. The best model appears to be the ARFIMA(1,d,0)-FIGARCH(1,d_0,0) process since it has the two smallest information criteria which are in bold in Table 5.

The findings presented in Table 6 show that the coefficients of the LMSV process are not significant. However, removing them reduces neither the absolute value of the log-likelihood nor the values of the information criteria which indicates that they are in fact significant. Apparently, the t-statistics seem to be biased. This is may be due to the presence of the term ξ_j in Eq.(48). Deo and Hurvich (2001) show that the variance of the estimators of the GPH is biased in the presence of this term.

We now compare the performance for predictions of the ARFIMA(1,d,0)-FIGARCH(1,d_0,0), ARFIMA(1,d,0)-LMSV(1,d_0,1) and AR(1)-GARCH(1,1)(6) processes for the period from August 1st 2005 through March 31st 2006. We choose to include in the study the ARFIMA-LMSV(1,d_0,1) to see whether its forecasts are better than those given by the other models since there is a doubt about the non-significativity of its coefficients. The performance is measured by the RMSE (Root Mean Squared Error) and the MAE (Mean Absolute Error) criteria which must be the smallest for the model providing the best forecasts. They are defined as

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_j \hat{\varepsilon}_j^2} \qquad (72)$$

and

$$\text{MAE} = \frac{1}{n}\sum_j \left|\hat{\varepsilon}_j\right| \qquad (73)$$

(6) The estimated GARCH(1,1) process is expressed as : $\hat{\sigma}^2_{\varepsilon,j} = 0.765 + 0.056\,\hat{\varepsilon}^2_{j-1} + 0.916\,\hat{\sigma}^2_{\varepsilon,j\text{-}1}$. The sum of the coefficients of the ARCH(1) and GARCH(1) parts of the process is very close to one which reveals the presence of a persistent effect.

where $\hat{\varepsilon}_j$ represents the error between the prediction of the model and the observation at time j.

	ARFIMA(1,d,0)-FIGARCH(1,d_0,1)	**ARFIMA(1,d,0)-FIGARCH(1,d_0,0)**	**ARFIMA(1,d,0)-FIGARCH(0,d_0,1)**	**ARFIMA(1,d,0)-FIGARCH(0,d_0,0)**
Φ_1	0.4465 (21.28)	0.4459 (21.84)	0.4461 (21.75)	0.4427 (19.91)
d	0.1835 (9.62)	0.1830 (9.67)	0.1831 (9.72)	0.1809 (8.82)
d_0	0.1524 (10.05)	0.1708 (9.11)	0.1656 (10.17)	0.1231 (10.4824)
β_1	-0.2951 (1.20)	0.0840 (-3.41)		
Ψ_1	-0.3581 (-1.53)		-0.0785 (-3.81)	
constant	26.9152 (16.89)	26.9205 (15.87)	26.9187 (16.00)	26.7550 (26.76)
LL	-19350.9368	-19351.0096	-19351.0362	-19352.5005
AIC	8.42348	**8.42307**	8.42308	8.4233
BIC	8.42420	8.42343	8.42345	**8.4233**
HQ	8.42310	**8.42288**	8.42289	8.4233

Table 5 : Estimation of the ARFIMA-FIGARCH processes by maximizing the Whittle function.
LL refers to the log-likelihood value at the optimum and the number in parentheses corresponds to the t-statistic.

	ARFIMA(1,d,0)-LMSV(1,d_0,1)	**ARFIMA(1,d,0)-LMSV(1,d_0,0)**	**ARFIMA(1,d,0)-LMSV(0,d_0,1)**	**ARFIMA(1,d,0)-LMSV(0,d_0,0)**
Φ_1	0.5963 (435.12)	0.5962 (415.37)	0.5962 (486.50)	0.5883 (627.93)
d	0.2585 (180.06)	0.2587 (172.05)	0.2587 (201.98)	0.2664 (278.71)
d_0	0.1181 (4.02)	0.1943 (5.17)	0.1946 (4.87)	0.1645 (7.91)
θ_1	-0.0141 **(-0.03)**	-0.0935 **(-1.64)**		
φ_1	0.0541 **(-0.13)**		0.0983 **(-1.61)**	
constant	0.8120 (0.81)	0.8121 (0.81)	0.8121 (0.81)	0.8120 (0.81)
LL	-6005.7041	-6008.2831	-6008.2640	-6014.6499
AIC	**2.61489**	2.61557	2.61556	2.61791
BIC	**2.61561**	2.61594	2.61593	2.61791
HQ	**2.61451**	2.61538	2.61537	2.61791

Table 6 : Estimation of the ARFIMA-LMSV processes by maximizing the Whittle function.
LL refers to the log-likelihood value at the optimum and the number in parentheses corresponds to the t-statistic.

	AR(1)-GARCH(1,1)	ARFIMA(1,d,0)-FIGARCH(1,d_0,0)	ARFIMA(1,d,0)-LMSV(1,d_0,1)
RMSE	10.927	**10.765**	12.387
MAE	8.753	**8.644**	9.390

Table 7 : 500 simulations are run. The RMSE and MAE statistics are calculated for each of these simulations.
We represent here the mean of the obtained RMSE and MAE.
The studied period is from August 1st 2005 to March 31st 2006.

Based on the findings of the Table 7, we conclude that the ARFIMA(1,d,0)-LMSV(1,d_0,1) process does not give good predictions for the studied period whereas the ARFIMA(1,d,0)-FIGARCH(1,d_0,0) process provides better results than the AR(1)-GARCH(1,1) process. But the differences between the two processes are not great. We will analyse in part 7.3 the prices resulted from the ARFIMA(1,d,0)-FIGARCH(1,d_0,0) and AR(1)-GARCH(1,1) processes.

Therefore to calculate the actuarial weather derivative prices, we will simulate the following estimated ARFIMA(1,d,0)-FIGARCH(1,d_0,0) process :

$$\hat{T}_j = \hat{T}_j^m + 0.446\,(T_{j-1} - \hat{T}_{j-1}^m) + (1-L)^{-0.183}\,\hat{\sigma}_{\varepsilon,j-1}\,\tilde{\varepsilon}_j \quad (74)$$

with $\tilde{\varepsilon}_j \rightsquigarrow N(0,1)$,

$$\hat{T}_j^m = 55.96 - 4.20\times10^{-5}\,j + 22.23\,\sin\left(\frac{2\pi}{365}\,j - 2\right), \quad (75)$$

$$\hat{\sigma}^2_{\varepsilon,j} = 0.084\,\hat{\sigma}^2_{\varepsilon,j-1} + \hat{\varepsilon}_j^2 - 0.084\,\hat{\varepsilon}_{j-1}^2 - (1-L)^{0.171}(\hat{\varepsilon}_j^2 - 26.920) \quad (76)$$

and

$$\hat{\varepsilon}_j = (1-0.446L)(1-L)^{0.183}(T_j - \hat{T}_j^m). \quad (77)$$

To compute the financial weather derivative prices and to compare them to the actuarial ones, we need to estimate the fractional mean-reverting-FIGARCH(1,d_0,0) process. By using the frequency-domain likelihood method and by discretizing the process, we obtain the following estimated model which will be used for simulations in section 7 :

$$\hat{T}_j = \hat{T}_{j-1} + \Delta\hat{T}_j^m + 0.533\,(\hat{T}_{j-1}^m - \hat{T}_{j-1})\,\Delta j + z_j, \quad (78)$$

$$\hat{T}_j^m = 55.96 - 4.20\times10^{-5}\,j + 22.23\,\sin\left(\frac{2\pi}{365}\,j - 2\right) \quad (79)$$

where $\Delta j = 1$,

$$z_j = \sum_{k=1}^{j} \frac{(j-k+1)^{0.139}}{\Gamma(1+0.139)}\,\Delta z_k^{(0.139)}, \quad (80)$$

$$z_j^{(0.139)} \approx \hat{\sigma}_{\varepsilon,j-1}\,\Delta W_j = \hat{\sigma}_{\varepsilon,j-1}\,\tilde{\varepsilon}_j\,\sqrt{\Delta j}\ ,\ \ \tilde{\varepsilon}_j \rightsquigarrow iid(0,1), \quad (81)$$

$$\hat{\sigma}^2_{\varepsilon,j} = -0.001\,\hat{\sigma}^2_{\varepsilon,j-1} + \hat{Z}_j^2 + 0.001\,\hat{Z}_{j-1}^2 - (1-L)^{0.125}(\hat{Z}_j^2 - 34.024), \quad (82)$$

$$\hat{Z}_j = \hat{z}_j^{(0.139)} = \sum_{k=1}^{j} \frac{(j-k+1)^{-0.139}}{\Gamma(1-0.139)} \Delta\hat{z}_k \quad (83)$$

and

$$\hat{z}_j = T_j - T_{j-1} - \Delta\hat{T}_j^m - 0.533\,(\hat{T}_{j-1}^m - T_{j-1})\,\Delta j\,. \quad (84)$$

where $t_{\hat{d}} = 32.37$ and $t_{\hat{d}0} = 8.11$. We check that the variance equation satisfies the positiveness condition $\beta_1 \leq d_0$. Since $\beta_1 = -0.001$ and $d_0 = 0.125$, so it does.

Before calculating the prices, we explain in details the two pricing approaches.

7. Computation of the weather derivative prices

7.1 Financial pricing approach

The financial pricing method is based on the fact that one can create a self-financing portfolio composed of the underlying and the riskless asset which attains the value of the option at the expiration date and by the arbitrage-free principle, one sets the price of the option at time 0 equal to the cost of the portfolio at time 0. It is well-known that the price derived from this operation corresponds to the discounted expectation of the payoff of the option under the risk neutral probability measure Q which is equivalent to the real probability measure. Black and Scholes (1973) give the closed-form expression of this price in continuous time. For complex options such as the Asian options for which the prices cannot have an explicit representation, the risk-neutral density (or also called the state-price density) is inferred from the observed prices to price these contracts. But the financial pricing method does not work for the weather derivatives since the underlying is not a traded asset. No self-financing portfolio can be constituted. With the development of the organized weather markets, weather futures contracts can be regarded as substitutes for the non-traded underlying to create the self-financing portfolio. As we mentioned before, this possibility is not yet conceivable since these contracts are not liquid enough. Therefore applying directly the Black (1976)'s formula to calculate the price of the weather option on futures is not correct. However, as noted by Hamisultane (2006), information can be extracted from the quotations of the frequently traded weather futures to price the weather derivatives. The author infers the risk-neutral distribution by using the Monte-Carlo simulations as well as the market price of weather risk (which is not zero here because of the incompleteness of the market) by solving a partial differential equation. We will use in this paper these techniques to value the contracts. We first recall that in the financial framework, the weather CDD call option and the weather CDD futures prices are respectively defined at time 0 as

$$C(0,t_n)^{(7)} = e^{-rt_n}\,\delta.E^Q[\max(I_n^C - K,0) \mid F_0^H\,] = e^{-rt_n}\,\delta.\int_0^{\infty} \max(I_n^C - K,0)\; q_{t,t_n}(I_n^C)\; dI_n^C \quad (85)$$

and

(7) Benth (2003) points out that this price should be calculated with the quasi-conditional expectation and not with the classical expectation for time different from 0.

$$F(0,t_n)^{(8)} = \delta.E^Q[\, I^C_n \mid F^H_0 \,] = \delta. \int_0^{\infty} I^C_n \, q_{t,t_n}(I^C_n)\, dI^C_n \quad (86)$$

where t_n is the expiration date of the contract, r refers to the riskless interest rate, δ is the tick size, K is the strike level, F^H_0 corresponds to the information available at time 0 about the temperature which is driven by a fractional Brownian motion, E^Q denotes the expectation operator under the probability Q which is not unique here since the market is incomplete and $q_{t,t_n}(I^C_n)$ is the state-price density of the I^C_n index.

To derive the risk-neutral density from the weather futures prices, Hamisultane (2006) uses the Monte-Carlo simulations and the Jackwerth and Rubinstein (1996)'s optimization problem which is expressed as follows

$$\underset{q_j}{\text{Min}} \sum_{i=1}^{M} \left(E^Q_i[\, I^C_n \,] - F_i\right)^2 + \alpha\left(\sum_{j=1}^{N} q_j - 1\right)^2 + \alpha \sum_{j=1}^{N} \max(0, -q_j)^2 + \alpha \sum_{j=1}^{N} (q''_j)^2 \quad (87)$$

where α>0 is the penalty parameter, F_i denotes the quoted weather futures price for day i, M corresponds to the number of available quotes, we assume here that it corresponds also to the number of days in the contract, N is the number of simulations for the temperature, the simulations will be run with Eq.(78) to Eq.(84), each temperature path j is assigned a probability q_j, $E^Q_i[I^C_n]= \frac{1}{N} \sum_{j=1}^{N} I^C_{n,j}\, q_j$ refers to the theoretical price of the weather futures, $I^C_{n,j}= \sum_{i=1}^{M} \max(T_{i,j}-65,0)$ and

$$q''_j \approx \frac{q_{j-1} - 2q_j + q_{j+1}}{\Delta T^2_j} \approx q_{j-1} - 2q_j + q_{j+1} \text{ with } q_0 = q_{N+1} = 0 \,. \quad (88)$$

For $\alpha \to 0$ ($\alpha \neq 0$) , the estimated prices will be close to the observations but the solutions of the optimization problem will exhibit picks while for $\alpha \to +\infty$, the estimates will not reproduce well the observations but the solutions will form a smoothed curve. In our case, we will only favour big values for α to obtain a smoothed distribution curve.

The second possibility to compute the financial prices is to solve a PDE. Indeed, the prices given by Eq.(85) and Eq.(86) are the unique solutions of the PDE with respectively the terminal condition $C(t_n,t_n) = \delta.\max(I^C_n - K,0)$ and $F(t_n,t_n) = \delta I^C_n$. Brody, Syroka and Zervos (2002) and Benth (2003) use the fractional calculus presented in Appendix to determine the PDE in the case where the temperature obeys a mean-reverting fractional Brownian motion. Brody *et al.* (2002) and Benth (2003) provide its expression when the weather derivative is based on the cumulative temperatures and Brody *et al.* (2002) give it when the contract lies on the cumulative degree-days. This latter is written as follows (under the probability Q) when the temperature process is defined by Eq.(26) and Eq.(27) and for the weather futures price given by Eq.(86) :

(8) This price is obtained by noting that no premium is required to write a futures. Therefore, $C(0,t_n)$ is equal to zero in Eq.(85) which yields Eq.(86).

$$\frac{\partial F}{\partial t} + \left(\frac{dT_t^m}{dt} + a(T_t^m - T_t) - \lambda_t \sigma_t \right) \frac{\partial F}{\partial T} + \Psi_t \frac{\partial^2 F}{\partial T^2} + \max(T_t - 65, 0) = 0 \quad (89)$$

with terminal condition $F(t_n, t_n) = \delta I_n^C$, where λ_t stands for the market price of weather risk and

$$\Psi_t = \sigma_t \, e^{-at} \int_0^t ø_{t,s} \, \sigma_s \, e^{-as} \, ds \quad (90)$$

where $ø_{t,s} = H(2H-1)|t-s|^{2H-2}$ and H is the Hurst exponent.

Since the parameters a, T_t^m, H and σ_t which corresponds to the FIGARCH process were all estimated in the preceding section there is only one unknown in the PDE which is the market price of weather risk λ_t which has to be inferred from the weather futures quotations by solving this optimization problem

$$\underset{\lambda_t}{\mathrm{Min}} \sum_t \left(F^{theorical}(t,t_n) - F^{observed}(t,t_n) \right)^2 . \quad (91)$$

where $F^{theorical}(t,t_n)$ represents the price obtained from solving the PDE in (89) and $F^{observed}(t,t_n)$ refers to the quotation at time t.

The market price of weather risk is here different from zero since the market is incomplete for the weather derivatives. As in Pirrong and Jermakyan (2001), we allow it to depend on time t.

This PDE can be solved by using the finite difference method which consists in constructing a grid of equally spaced points and in discretizing the continuous derivatives of the PDE by using difference formulas which can be forward, backward or central difference. These formulas lead to different resolution schemes which are explicit, implicit and semi-implicit (Crank-Nicolson) methods. As in Hamisultane (2006), we use an implicit scheme to avoid oscillations which can happen in the case of the Crank-Nicolson representation. This PDE is solved in the same way as for a PDE including the diffusion term $\max(T_t-65,0) \frac{\partial F}{\partial I_n^C}$ even if it does not appear in the above equation (see Dewynne and Wilmott (1995), Randall and Tavella (2000), Hamisultane (2006)). This is because we have $I_n^C = \int_0^{t_n} \max(T_s-65,0) \, ds$ and not $\max(T_t-65,0)$ in the terminal condition. Due to the fact that I_n^C is a discrete running sum on the grid, i.e. we have $I_{n,i}^C = I_{n,i-1}^C + \max(T_i-65,0)$ for each point of time i, we have to consider the jump condition between the sampling dates i to avoid arbitrage (see Dewynne and Wilmott (1995)).

The obtained implicit scheme which is solved at each time i for the grid given by $F_{i,j}^k = F(i\Delta t, j\Delta T, k\Delta I_n^C)$ where $i = 0,\ldots,N$, $j = 0,\ldots,M$ and $k = 0,\ldots,K$ is as follows

$$F_{i+1,j}^k + \Delta t \times \max(j\Delta T - 65, 0) = \alpha_{i,j} F_{i,j-1}^k + \beta_i F_{i,j}^k + \zeta_{i,j} F_{i,j+1}^k , \quad (92)$$

$$\alpha_{i,j} = \Delta t \left(\frac{1}{2\Delta T} A_{i,j} - \frac{1}{\Delta T^2} B_i \right), \quad (93)$$

$$\beta_i = \left(1 + \frac{2\Delta t}{\Delta T^2} B_i\right), \tag{94}$$

$$\zeta_{i,j} = \Delta t \left(-\frac{1}{2\Delta T} A_{i,j} - \frac{1}{\Delta T^2} B_i\right) \tag{95}$$

where

$$A_{i,j} = \frac{\Delta T^m}{\Delta t} + a(T^m - j\Delta T) - \lambda_i \sigma_i, \tag{96}$$

$$B_i = \sigma_i \sum_{s=0}^{i} H(2H-1)|i\Delta t - s\Delta t|^{2H-2} \sigma_s e^{-a(i\Delta t + s\Delta t)} \tag{97}$$

and T^m and σ_i are calculated respectively with Eq.(79) and Eq.(82).

The terminal condition for the PDE is

$$F(T_t, I_n^C, t_n) = I_n^C. \tag{98}$$

The boundary conditions are

$$F(+\infty, I_n^C, t) = I_{n,max}^C \tag{99}$$

and

$$F(0, I_n^C, t) = 0. \tag{100}$$

Since we know the terminal and the boundary conditions and we solve the PDE backward in time, the unknowns in this system of equations given in (92) are $F^k_{i,j-1}$, $F^k_{i,j}$ and $F^k_{i,j+1}$. Once they are determined for time i, the algorithm of optimization can induce the value of λ_i by comparing the theoretical price to the observed one.

The implied state-price density and the derived market prices of risk from the sample of observations and for a given period are next used to calculate the weather derivative prices for a different period.

7.2 Actuarial pricing approach

Augros and Moréno (2002), Brix, Jewson and Ziehmann (2002) and Roustant, Laurent, Bay and Carraro (2003) propose the actuarial method to value the weather derivatives. It calculates the weather option and weather futures prices at time 0 respectively as follows

$$C(0,t_n) = \delta e^{-rt_n} (E(payoff) + \lambda\sigma_{payoff}) \tag{101}$$

and

$$F(0,t_n) = \delta (E(index) + \lambda\sigma_{index}) \tag{102}$$

where δ is the tick size, r is the riskless interest rate, t_n is the expiration date, $\lambda\sigma_{payoff}$ and $\lambda\sigma_{index}$ are called the safety loading, they denote the risk premium where λ is a real and positive number, for the sake of simplicity these authors assume that $\lambda = 0$, σ_{payoff} and σ_{index}

represent the standard deviation of the payoff and of the index and E(payoff) is the expected payoff under the real probability which is given by

$$E(\text{payoff}) = \int_R \text{payoff} \times p(I_n^C)\, dI_n^C \tag{103}$$

where $p(I_n^C)$ is the historical density of the index I_n^C.

This is Platen and West (2004) who settled the link between the actuarial and financial methods with the notion of "growth optimal portfolio"(or GOP) and brought a framework to justify the use of the actuarial approach in the case of the weather derivatives. The GOP is by definition a self-financing portfolio that maximizes the expected logarithmic utility from terminal wealth. Under certain conditions, this GOP can be considered as a numeraire portfolio which converts the variables expressed in units of this numeraire into martingales whatever the probability measure used. Therefore, they write the option price in units of the GOP at time 0 as follows

$$\hat{C}(0,t_n) = E\left(\hat{H}_{t_n} \mid \mathcal{F}_0\right) \tag{104}$$

where $S_{t_n}^{(\pi)}$ represents the GOP at time t_n, $\hat{H}_{t_n} = \dfrac{H_{t_n}}{S_{t_n}^{(\pi)}}$ is the payoff of the option in units of the GOP. The price of the option not in terms of the GOP is given by

$$C(0,t_n) = S_0^{(\pi)}\, \hat{C}(0,t_n) \tag{105}$$

and

$$C(0,t_n) = E\left(\frac{S_0^{(\pi)}}{S_{t_n}^{(\pi)}}\, H_{t_n} \mid \mathcal{F}_0\right). \tag{106}$$

By defining the discrete time Radon-Nikodym derivative as (see Long (1990) and Platen (2002))

$$\frac{dQ}{dP} = \Lambda_{t_n} = \frac{\hat{S}_{t_n}^{(0)}}{\hat{S}_0^{(0)}} = \frac{S_{t_n}^{(0)}}{S_{t_n}^{(\pi)}}\,\frac{S_0^{(\pi)}}{S_0^{(0)}} \tag{107}$$

where $\hat{S}_{t_n}^{(0)} = \dfrac{S_{t_n}^{(0)}}{S_{t_n}^{(\pi)}}$ stands for the domestic savings account in units of the GOP, they demonstrate that this price is formulated as the discounted expectation of the payoff under the probability Q , i.e.

$$C(0,t_n) = \frac{S_0^{(0)}}{S_{t_n}^{(0)}}\, E^Q\left(H_{t_n} \mid \mathcal{F}_0\right). \tag{108}$$

If the payoff of the option is independent of the GOP, they show that it can be expressed as an actuarial price, i.e.

$$C(0,t_n) = S_0^{(\pi)}\, E\left(\frac{1}{S_{t_n}^{(\pi)}}\, H_{t_n} \mid \mathcal{F}_0\right) E\left(H_{t_n} \mid \mathcal{F}_0\right) \tag{109}$$

and

$$C(0,t_n) = P(0,t_n)\, E\left(H_{t_n} \mid F_0\right) \tag{110}$$

where $P(0,t_n)= S_0^{(\pi)}\, E\left(\frac{1}{S_{t_n}^{(\pi)}}\, H_{t_n} \mid F_0\right)$ corresponds to the price of a zero coupon bond at time 0.

Approximating the GOP by the MCSI World index, they show that the weather index of Sydney is very uncorrelated with this index and therefore that the weather derivatives should be priced by the actuarial approach.

The actuarial price is very simple to compute by the Monte-Carlo simulations. The temperature is simulated several times by using Eq.(74) to Eq.(77). Next, for each day and for each draw of the temperature, the CDD is calculated. For each trajectory, we sum up the CDDs to form the index and we compute the payoff of the contract. The weather option price corresponds then to the discounted mean of the payoffs while the weather futures price refers to the mean of the indexes.

7.3 Comparison of the prices

We possess two samples of the monthly New York CDD weather futures prices with expiration date in August 2004 for one and in August 2005 for the second. The prices span from the 1st to the 31st of August 2004 and of August 2005. Since the quotations are 5 day weeks, we have respectively 22 and 23 observations for August 2004 and August 2005. From the prices of the contract expiring in August 2004, we extract by using the methods described in the previous parts the risk-neutral density illustrated in Figure 2 and the market prices of weather risk depicted in Figure 5. We also represent in Figure 3 the prices obtained from the optimization problem for a penalty parameter equals to 10 millions and by using 1000 simulations of the temperature. We can notice that the estimated prices do not reproduce well the observed ones. As we mentioned before, this is because we have chosen to take large values for the penalty parameter so as to obtain a smoothed distribution curve. With the derived risk-neutral distribution and the market prices of risk, we compute the actuarial and financial prices for the same contract but expiring in August 2005 to see whether the forecasts are close to the observations. We recall that the different models for the New York daily average temperature were estimated for the period of January 1993 through July 2005. Figure 4 and Figure 6 show respectively the forecasted actuarial and financial prices for the contract with maturity date in August 2005. We choose to display also the actuarial prices with expiration date in August 2004 so as to gauge graphically the quality of the reproduced prices stemming from the different estimated processes for the temperature. The prices resulted from the discrete time AR(1)-GARCH(1,1) and ARFIMA(1,d,0)-FIGARCH(1,d_0,0) processes are the closest to the observations, they are not very different from each other (the prices from the AR(1)-GARCH(1,1) process are a bit more smoothed), while the prices given by the continuous time fractional-mean-reverting-FIGARCH(1,d_0,0) process exhibit an important volatility and are well above the quotations. We observe that the continuous time long memory process gives results which are very different from those provided by the discrete time long memory process. We see the same results in Figure 4. In light of Figure 1 and Figure 4, we conclude that the continuous time process is not appropriate when calculating the actuarial prices. Analyzing Figure 4 and Figure 6, we remark that the inferred market prices of risk and the resolution of the PDE produce the best predictions which was also observed in

Hamisultane (2006)'s study. The prices are less volatile than those calculated with the implied risk-neutral density and Monte-Carlo simulations. The forecasted actuarial prices from the discrete time processes are very below the quotations at the beginning of the contract period. With Figure 6, we conclude that the continuous time process is not in fact appropriate when using the Monte-Carlo simulations. Figures 4, 6, 7 and 8 reveal that not considering the state-price density when pricing the weather derivatives has less serious consequences than not taking into account the market prices of weather risk.

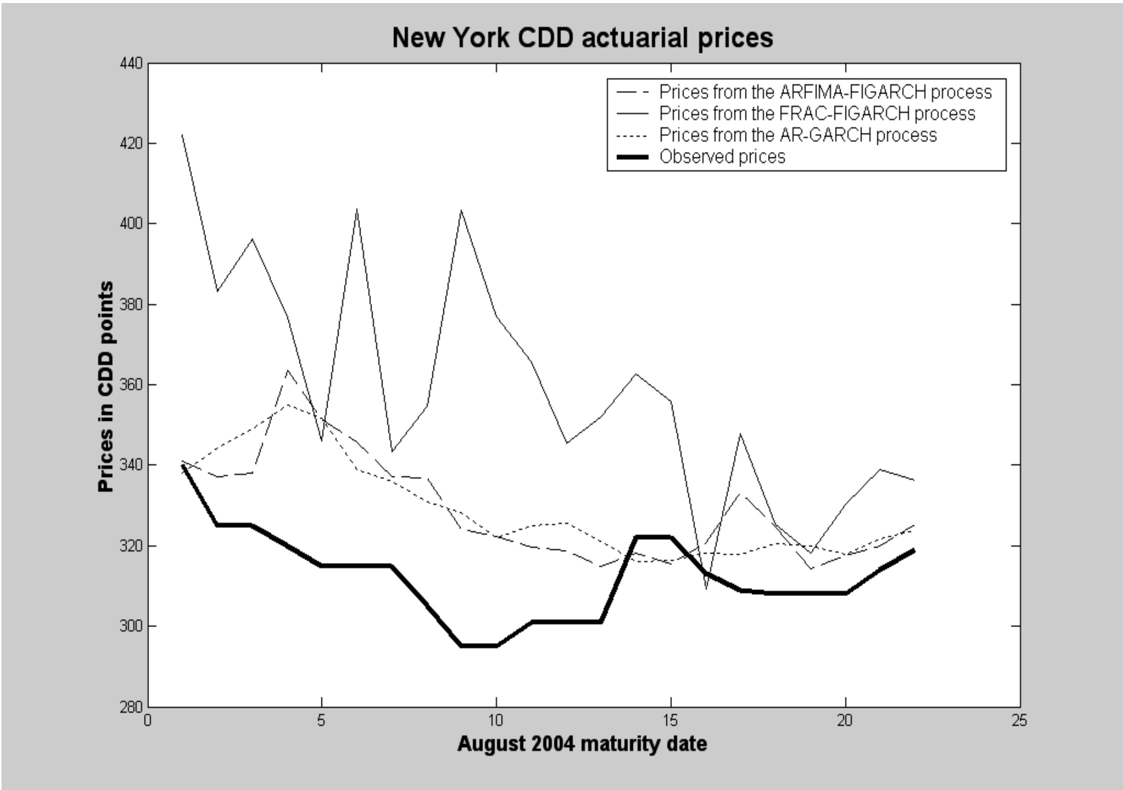

Figure 1 : Actuarial prices for the New York CDD weather futures with maturity date in August 2004. They are computed by using the different estimated processes for the New York daily average temperature : ARFIMA(1,d,0)-FIGARCH(1,d_0,0), fractional mean-reverting-FIGARCH(1,d_0,0) and AR(1)-GARCH(1,1) processes. 1000 similations of each of the processes are run , the infinite sum of the filters is truncated at 1000 and the safety loading is assumed to be zero.

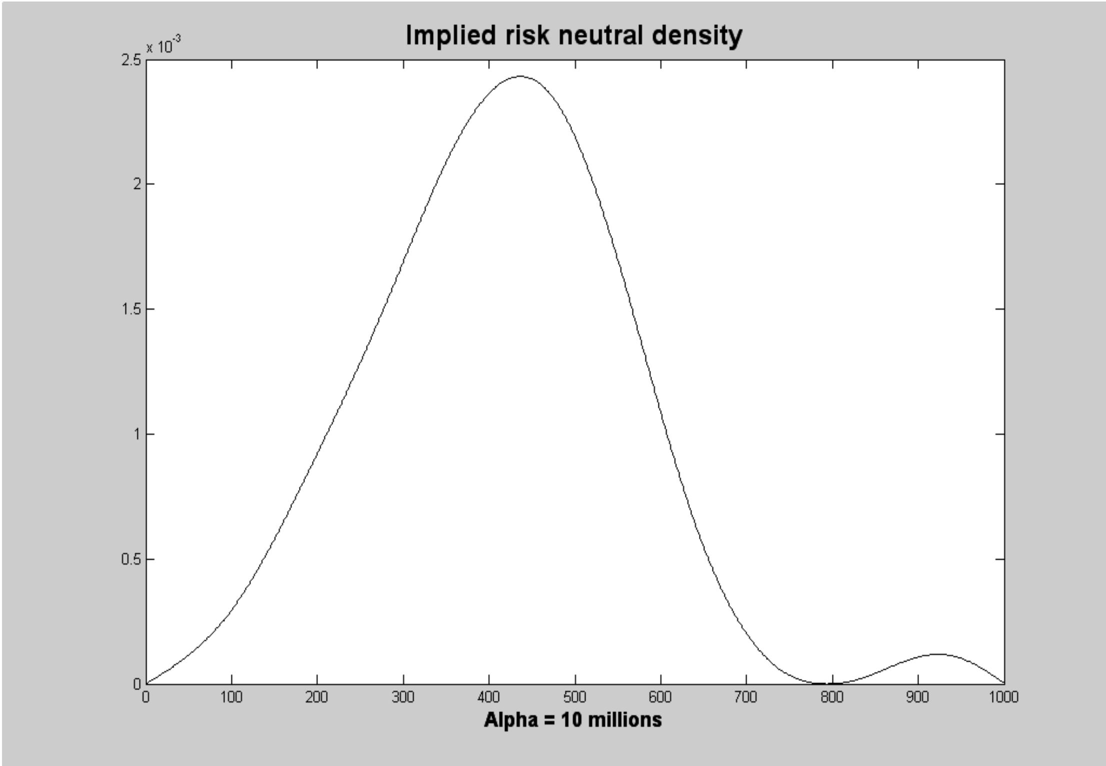

Figure 2 : The risk-neutral density is derived from the New York CDD weather futures prices with maturity date in August 2004. For its extraction, alpha=10 millions and N=1000 are used.

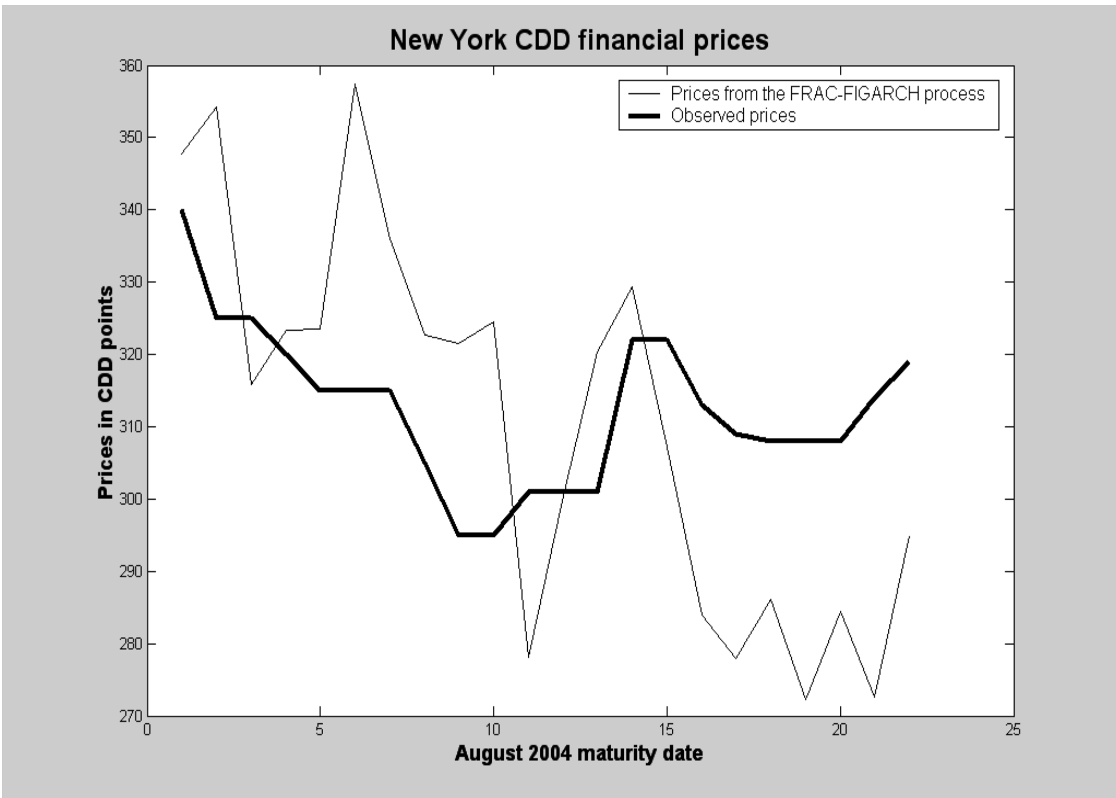

Figure 3 : Resulted financial prices from the optimization problem for alpha = 10 millions and N =1000. They correspond to the New York CDD weather futures with maturity date in August 2004.

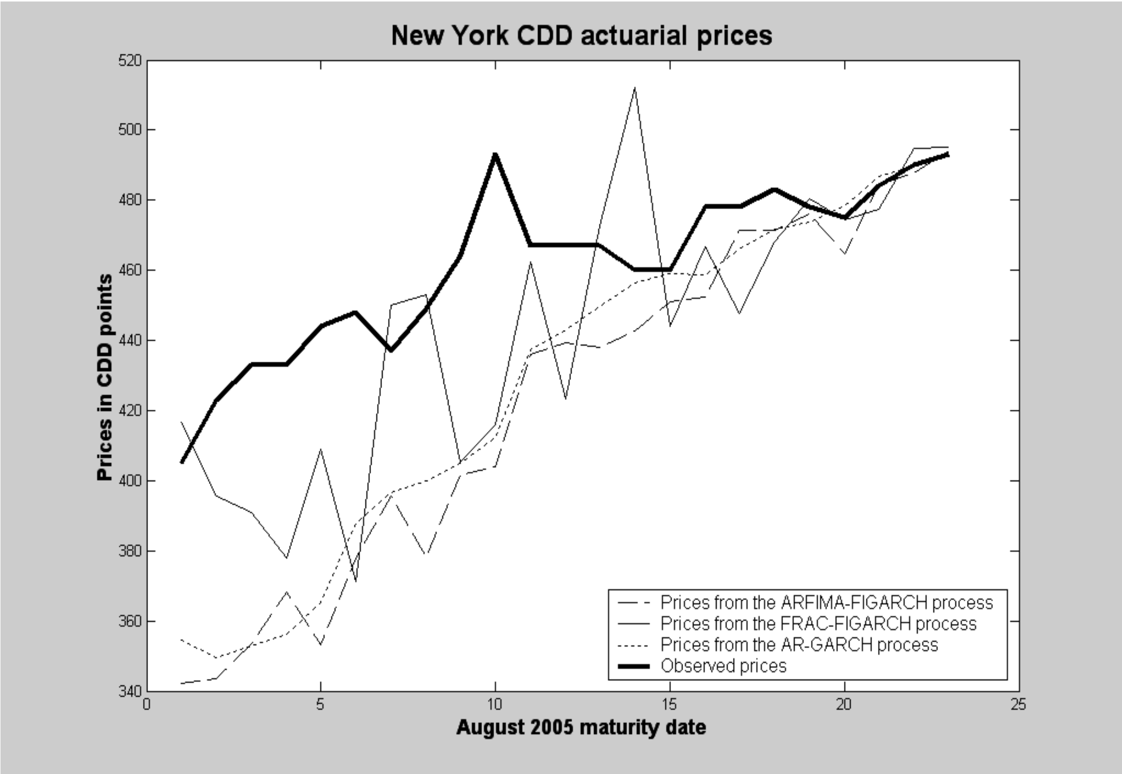

Figure 4 : Actuarial prices for the New York CDD weather futures with maturity date in August 2005. They are computed by using the different estimated processes for the New York daily average temperature : ARFIMA(1,d,0)-FIGARCH(1,d_0,0), fractional mean-reverting-FIGARCH(1,d_0,0) and AR(1)-GARCH(1,1) processes. 1000 similations of each of the processes are run, the infinite sum of the filters is truncated at 1000 and the safety loading is assumed to be zero.

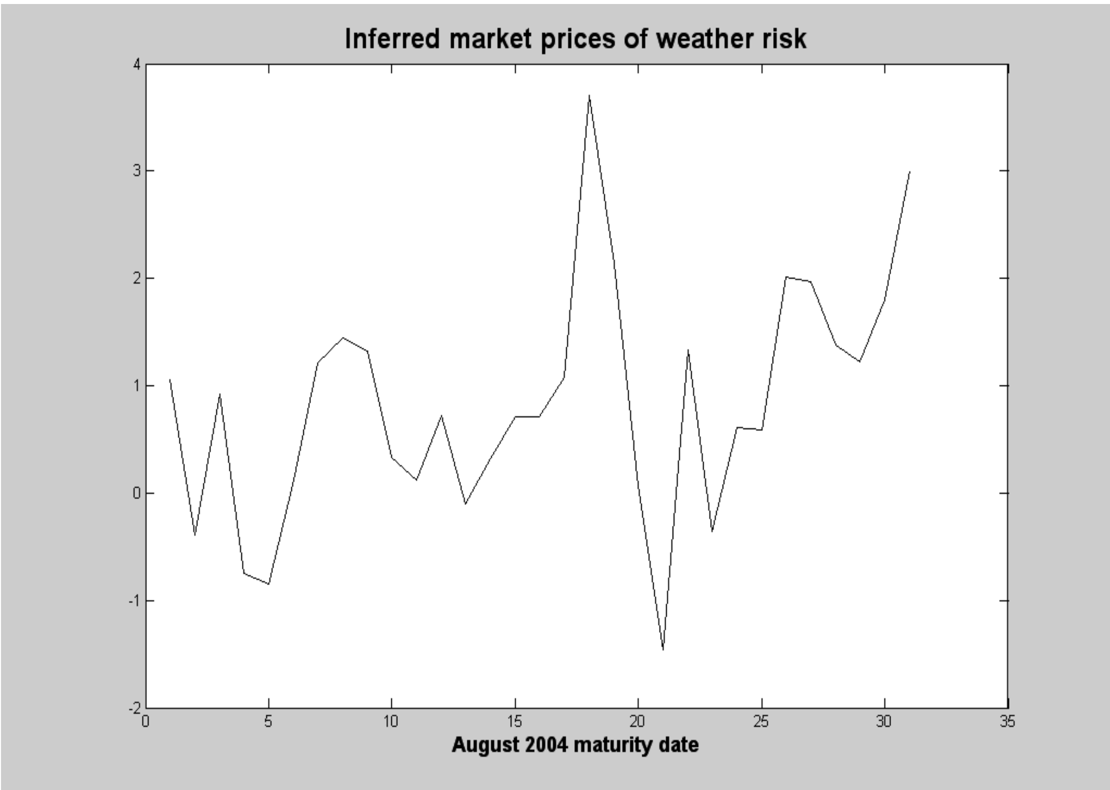

Figure 5 : The market prices of weather risk are derived from the New York CDD weather futures prices with maturity date in August 2004.

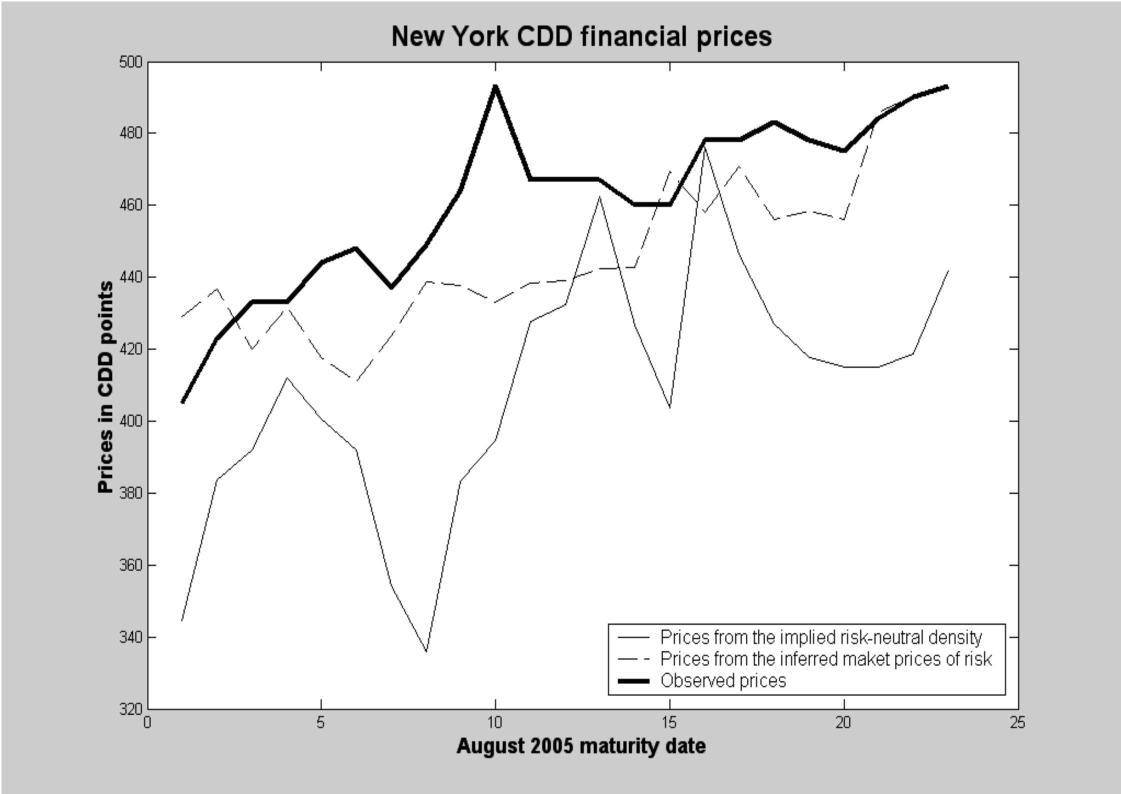

Figure 6 : Financial prices for the New York CDD weather futures with maturity date in August 2005. They are computed by using successively the implied risk-neutral density and the market prices of risk from the New York CDD weather futures expiring in August 2004.

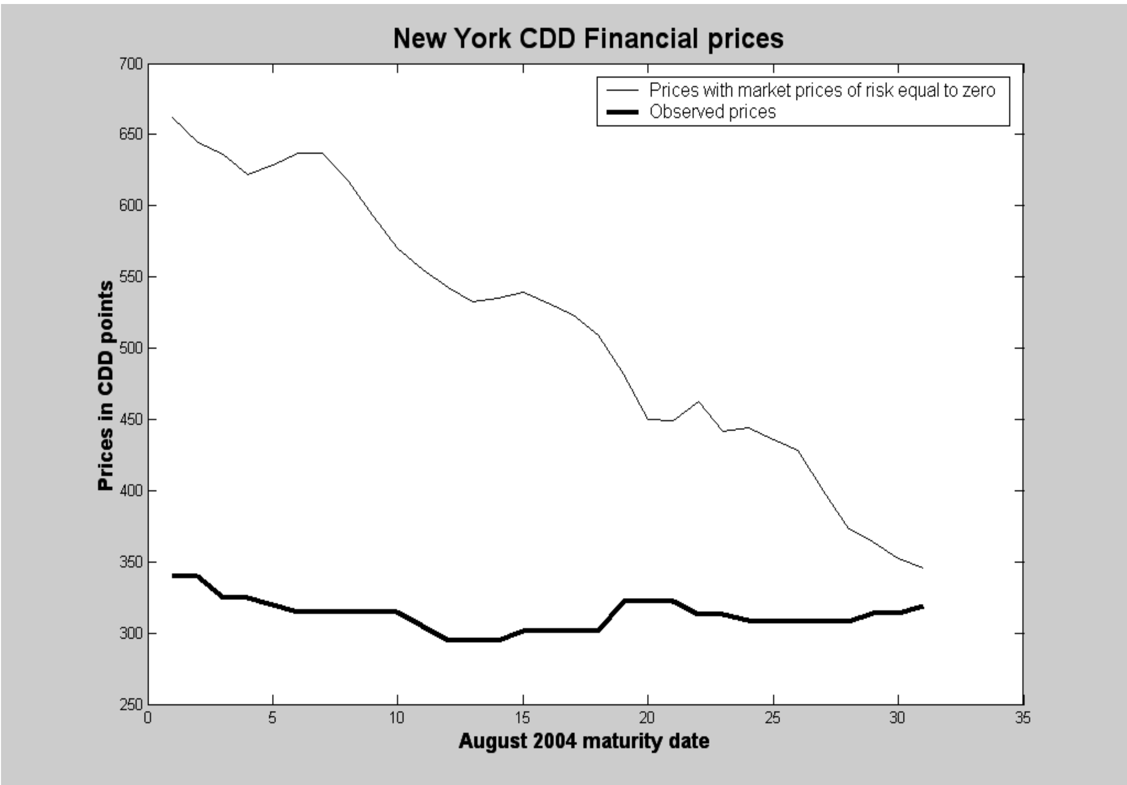

Figure 7 : Financial prices for the New York CDD weather futures with maturity date in August 2004. They are computed by solving the PDE and assuming that the market prices of weather risk are zero. The temperature process used is the fractional mean-reverting-FIGARCH(1,d_0,0) diffusion process.

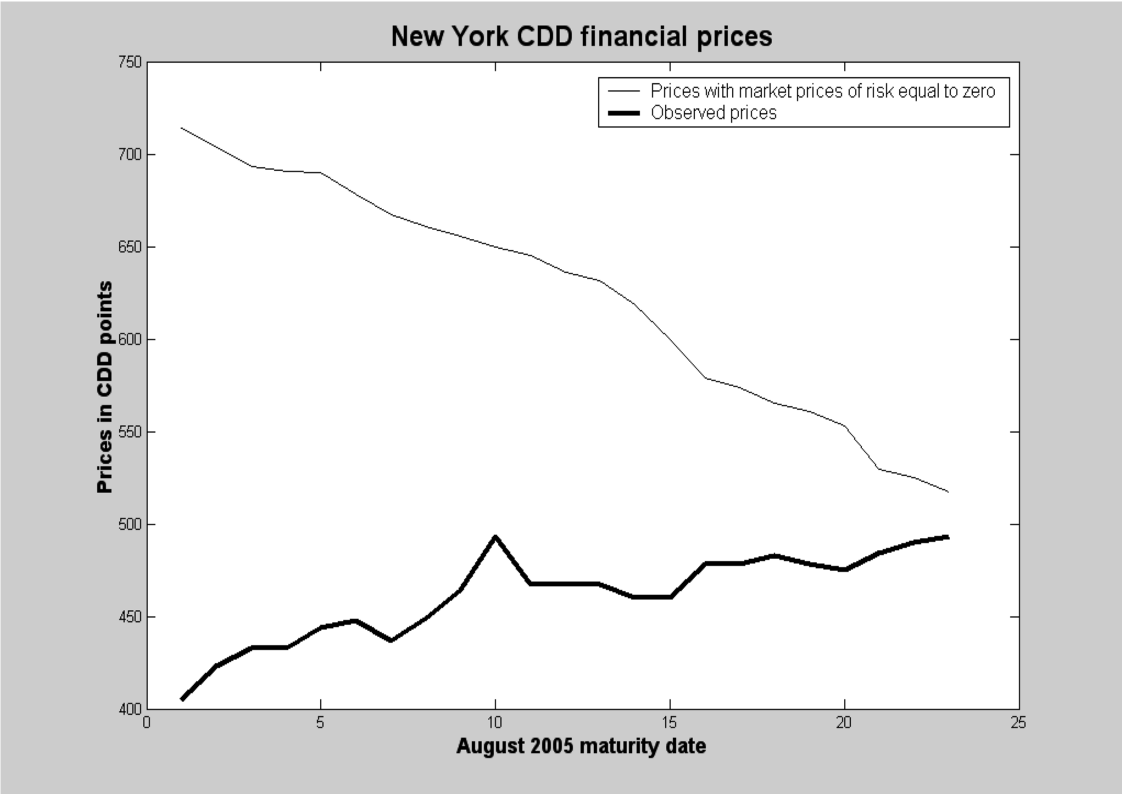

Figure 8 : Financial prices for the New York CDD weather futures with maturity date in August 2005. They are computed by solving the PDE and assuming that the market prices of weather risk are zero. The temperature process used is the fractional mean-reverting-FIGARCH(1,d_0,0) diffusion process.

8. Conclusion

In this paper, we have opposed several elements: long and short memory processes, actuarial and financial pricing approaches and at last continuous and discrete time processes. Five points have been observed. Calculating the weather derivative prices in the financial way by deriving the market prices of weather risk and solving a PDE has produced better predictions than in the actuarial way. Using the implied risk-neutral density has not given satisfactory results since these latters were very underestimated while ignoring the market prices of weather risk when solving the PDE has led to huge overestimated forecasts. Discrete time long memory and short memory processes used in the actuarial method have provided results which were quite similar. Concerning the Monte-Carlo simulations, it is not recommended to use the continuous time process since the resulted prices have appeared to be very volatile.

Appendix

Fractional Brownian motion and stochastic calculus

The fractional Brownian motion was first studied by Kolmogorov (1940). Mandelbrot and Van Ness (1968) defined it as a stochastic integral with respect to the standard Brownian motion :

$$W_t^H = \frac{1}{\Gamma(H+½)} \int_0^t (t-s)^{H-1/2}\, dW_s \qquad (111)$$

where Γ is a gamma function and W is the standard Brownian motion.

The fractional Brownian motion is a gaussian process with $E(W_t^H) = 0$, $E(W_t^H)^2 = |t|^{2H}$ and covariance

$$E(W_t^H W_s^H) = ½ \left[|t|^{2H} + |s|^{2H} - |t-s|^{2H} \right] ,\ s \le t . \qquad (112)$$

The increments of the process $(W_{t+h}^H - W_t^H)$ and $(W_{s+h}^H - W_s^H)$ are not independent. Let $s \le t$, $s+h \le t$, $t-s = nh$, the covariance $E[(W_{t+h}^H - W_t^H)(W_{s+h}^H - W_s^H)]$ is given by

$$\rho_H(n) = ½\, h^{2H}[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}] \approx h^{2H} H(2H-1)n^{2H-2} \qquad (113)$$

where $0<H<1$.

If $H = ½$, $\rho_H(n) = 0$ and W_t^H becomes the standard Brownian motion.

If $H > ½$, $\rho_H(n) > 0$, $\sum_{n=1}^{\infty} \rho_H(n) = \infty$ and W_t^H is persistent.

If $H < ½$, $\rho_H(n) < 0$, $\sum_{n=1}^{\infty} |\rho_H(n)| < \infty$ and W_t^H is anti-persistent.

For $H \neq 1/2$, W_t^H is a fractional Brownian motion. This process is not a semimartingale because of the non-independency of the increments. Therefore we cannot use the classical Itô calculus and a new definition of the stochastic integral must be found.

Lin (1995) and Dai and Heyde (1996) propose a pathwise Riemann-Stieltjes integration for the fractional Brownian motion, e.g.

$$\int_0^T g_s \, \delta W_s^H = \lim_{\Delta t_k \to 0} \sum_{k=0}^{N-1} g_{t_k} (W_{tk+1}^H - W_{tk}^H) . \tag{114}$$

But in general,

$$E\left[\int_0^T g_s \, \delta W_s^H\right] \neq 0. \tag{115}$$

Rogers (1997) shows that this integration produces arbitrage.

Duncan, Hu and Pasik-Duncan (2000) introduce a Riemann-Stieltjes integration based upon the Wick product instead of the ordinary product

$$\int_0^T g_s \, dW_s^H = \lim_{\Delta t_k \to 0} \sum_{k=0}^{N-1} g_{t_k} \lozenge (W_{tk+1}^H - W_{tk}^H) \tag{116}$$

where $\lozenge$ is the Wick product.

This integral behaves in many ways like the Itô integral with respect to the standard Brownian motion. For example, we have

$$E\left[\int_0^T g_s \, dW_s^H\right] = 0 . \tag{117}$$

Hu and Øksendal (2003) show that this integral leads to no arbitrage and that the fractional Black and Scholes market is complete.

Duncan, Hu and Pasik-Duncan (2000) give the fractional Itô formula which is stated as follows :

Theorem 1 : Assume $X_t = \int_0^t a_s dW_s^H$ where $a \in L_\phi^{1,2}$. For a function $f \in C^{1,2}(\Re_+ \times \Re)$ with bounded derivatives we have

$$\begin{aligned} f(t,X_t) = f(0,0) &+ \int_0^t \frac{\partial f}{\partial s}(s,X_s)\, ds + \int_0^t \frac{\partial f}{\partial x}(s,X_s)\, a_s \, dW_s^H \\ &+ \int_0^t \frac{\partial^2 f}{\partial x^2}(s,X_s)\, a_s \left(\int_0^s ø_{s,u}\, a_u du\right) ds \end{aligned} \tag{118}$$

where $ø_{s,t} = H(2H-1)|s-t|^{2H-2}$.

Hu and Øksendal (2003) propose a fractional version of the Girsanov theorem which is formulated in the following terms :

Theorem 2 : Let γ , θ be measurable functions with support on $[0,t_n]$, where γ is continuous and θ is the solution of the integral equation $\int_0^{t_n} \theta_s \, ø_{s,t} \, ds = \lambda_t$ for $0 \leq t \leq t_n$ .Then

$$\widetilde{W}_t^H = W_t^H + \int_0^t \lambda_s ds \tag{119}$$

is a fractional Brownian motion under the probability Q which is equivalent to the real probability P and

$$\frac{dQ}{dP} = \exp\left(-\int_0^{t_n} \theta_s dW_s^H - \frac{1}{2} \left| \theta 1_{[0,t_n]} \right|_\phi^2 \right) \tag{120}$$

which represents the Radon-Nikodym derivative.

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